

Some Computable Structure Theory of Finitely Generated Structures

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Much of this is joint work with Meng-Che “Turbo” Ho.

The rest is joint work with Melnikov, Miller, and Montalbán.

Outline

The main question:

Which classes of finitely generated structures contain complicated structures?

The particular focus will be on groups.

General outline:

- 1 Descriptions (Scott sentences) of finitely generated structures, and in particular groups, among countable structures.
- 2 A notion of universality using computable functors (or equivalently effective interpretations).
- 3 Descriptions (quasi Scott sentences) of finitely generated structures among finitely generated structures.

Scott Sentences of Finitely Generated Structures

Infinitary Logic

$\mathcal{L}_{\omega_1\omega}$ is the infinitary logic which allows countably infinite conjunctions and disjunctions.

There is a hierarchy of $\mathcal{L}_{\omega_1\omega}$ -formulas based on their quantifier complexity after putting them in normal form. Formulas are classified as either Σ_α^0 or Π_α^0 , for $\alpha < \omega_1$.

- A formula is Σ_0^0 and Π_0^0 if it is finitary quantifier-free.
- A formula is Σ_α^0 if it is a disjunction of formulas $(\exists \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is Π_β^0 for $\beta < \alpha$.
- A formula is Π_α^0 if it is a conjunction of formulas $(\forall \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is Σ_β^0 for $\beta < \alpha$.

Examples of Infinitary Formulas

Example

There is a Π_2^0 sentence which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee_{n \in \mathbb{N}} nx = 0.$$

Example

There is a Σ_1^0 formula which describes the dependence relation on triples x, y, z in a \mathbb{Q} -vector space:

$$\bigvee_{(a,b,c) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}} ax + by + cz = 0$$

Examples of Infinitary Formulas

Example

There is a Σ_3^0 sentence which says that a \mathbb{Q} -vector space has finite dimension:

$$\bigvee_{n \in \mathbb{N}} (\exists x_1, \dots, x_n) (\forall y) y \in \text{span}(x_1, \dots, x_n).$$

Example

There is a Π_3^0 sentence which says that a \mathbb{Q} -vector space has infinite dimension:

$$\bigwedge_{n \in \mathbb{N}} (\exists x_1, \dots, x_n) \text{Indep}(x_1, \dots, x_n).$$

Scott Sentences

Let \mathcal{A} be a countable structure.

Theorem (Scott)

There is an $\mathcal{L}_{\omega_1\omega}$ -sentence φ such that:

$$\mathcal{B} \text{ countable, } \mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

φ is a *Scott sentence* of \mathcal{A} .

Example

$(\omega, <)$ has a Π_3^0 Scott sentence consisting of the Π_2^0 axioms for infinite linear orders together with:

$$\forall y_0 \bigvee_{n \in \omega} \exists y_n < \dots < y_1 < y_0 [\forall z (z > y_0) \vee (z = y_0) \vee (z = y_1) \vee \dots \vee (z = y_n)].$$

An Upper Bound on the Complexity of Finitely Generated Structures

Theorem (Knight-Saraph)

Every finitely generated structure has a Σ_3^0 Scott sentence.

Often there is a simpler Scott sentence.

A Scott Sentence for the Integers

Example

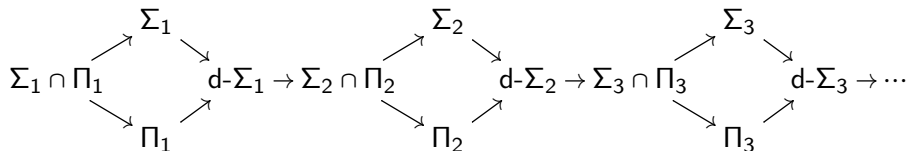
A Scott sentence for the group \mathbb{Z} consists of:

- the axioms for torsion-free abelian groups,
- for any two elements, there is an element which generates both,
- there is a non-zero element with no proper divisors:

$$(\exists g \neq 0) \bigwedge_{n \geq 2} (\forall h) [nh \neq g].$$

$d\text{-}\Sigma_2^0$ Sentences

φ is $d\text{-}\Sigma_2^0$ if it is a conjunction of a Σ_2^0 formula and a Π_2^0 formula.



Theorem (Miller)

Let \mathcal{A} be a countable structure. If \mathcal{A} has a Σ_3^0 Scott sentence, and also has a Π_3^0 Scott sentence, then \mathcal{A} has a $d\text{-}\Sigma_2^0$ Scott sentence.

A Scott Sentence for the Integers

Example

A Scott sentence for the group \mathbb{Z} consists of:

- the axioms for torsion-free abelian groups,
- for any two elements, there is an element which generates both,
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This is a $d\text{-}\Sigma_2^0$ Scott sentence.

A Scott Sentence for the Free Group

Example (CHKLMMMQR)

A Scott sentence for the free group \mathbb{F}_2 on two elements consists of:

- the group axioms,
- every finite set of elements is generated by a 2-tuple,
- there is a 2-tuple \bar{x} with no non-trivial relations such that for every 2-tuple \bar{y} , \bar{x} cannot be expressed as an “imprimitive” tuple of words in \bar{y} .

This is a $d\text{-}\Sigma_2^0$ Scott sentence.

$d\text{-}\Sigma_2^0$ Scott Sentences for Many Groups

Theorem (Knight-Saraph, CHKLMMMQR, Ho)

The following groups all have $d\text{-}\Sigma_2^0$ Scott sentences:

- *abelian groups,*
- *free groups,*
- *nilpotent groups,*
- *polycyclic groups,*
- *lamplighter groups,*
- *Baumslag-Solitar groups $BS(1, n)$.*

Question

Does every finitely generated group have a $d\text{-}\Sigma_2^0$ Scott sentence?

Characterizing the Structures with $d\text{-}\Sigma_2^0$ Scott Sentences

The first step is to understand when a finitely generated structure has a $d\text{-}\Sigma_2^0$ Scott sentence.

Theorem (A. Miller, HT-Ho, Alvir-Knight-McCoy)

Let \mathcal{A} be a finitely generated structure. The following are equivalent:

- *\mathcal{A} has a Π_3^0 Scott sentence.*
- *\mathcal{A} has a $d\text{-}\Sigma_2^0$ Scott sentence.*
- *\mathcal{A} is the only model of its Σ_2^0 theory.*
- *some generating tuple of \mathcal{A} is defined by a Π_1^0 formula.*
- *every generating tuple of \mathcal{A} is defined by a Π_1^0 formula.*
- *\mathcal{A} does not contain a copy of itself as a proper Σ_1^0 -elementary substructure.*

Proof, First Direction

Suppose that \mathcal{A} does not contain a copy of itself as a proper Σ_1^0 -elementary substructure.

Let p be the \forall -type of a generating tuple for \mathcal{A} .

We can write down a d - Σ_2^0 Scott sentence for \mathcal{A} :

- there is a tuple \bar{x} satisfying p , and
- for all tuples \bar{x} satisfying p and for all y , y is in the substructure generated by \bar{x} .

Proof, Second Direction

Now suppose that \mathcal{A} does contain a copy of itself as a proper Σ_1^0 -elementary substructure.

Take the union of the chain

$$\mathcal{A} <_{\Sigma_1^0} \mathcal{A} <_{\Sigma_1^0} \mathcal{A} <_{\Sigma_1^0} \cdots <_{\Sigma_1^0} \mathcal{A}^*.$$

Then \mathcal{A}^* has the same Σ_2^0 theory as \mathcal{A} , but is not finitely generated.

In particular, \mathcal{A} does not have a d- Σ_2^0 Scott sentence.

A Complicated Group

Theorem (HT-Ho)

There is a computable finitely generated group G which does not have a d - Σ_2^0 Scott sentence.

The construction of G uses small cancellation theory and HNN extensions.

Theorem (HT-Ho)

There is a computable finitely generated ring $\mathbb{Z}[G]$ which does not have a d - Σ_2^0 Scott sentence.

This is just the group ring of the previous group.

No Complicated Fields

Theorem (HT-Ho)

Every finitely generated field has a d - Σ_2^0 Scott sentence.

Proof idea:

An embedding of a finitely generated field F into itself makes F an algebraic extension of itself.

Algebraic extensions cannot be Σ_1^0 -elementary.

Open Questions

Question

Does every finitely presented group have a $d\text{-}\Sigma_2^0$ Scott sentence?

Question

Does every commutative ring have a $d\text{-}\Sigma_2^0$ Scott sentence?

Question

Does every integral domain have a $d\text{-}\Sigma_2^0$ Scott sentence?

Computable Functors, Effective Interpretations, and Universality

Computable Dimension

Definition

The computable dimension of a computable structure \mathcal{A} is the number of computable copies up to computable isomorphism.

Theorem (Goncharov; Goncharov and Dzgoev; Metakides and Nerode; Nurtazin; LaRoche; Remmel)

All structures in each of the following classes have computable dimension 1 or ω :

- *algebraically closed fields,*
- *real closed fields,*
- *torsion-free abelian groups,*
- *linear orderings,*
- *Boolean algebras.*

Finite Computable Dimension > 1

Theorem (Goncharov)

For each $n > 0$ there is a computable structure with computable dimension n .

Theorem (Goncharov; Goncharov, Molokov, and Romanovskii; Kudinov)

For each $n > 0$ there are structures with computable dimension n in each of the following classes:

- *graphs,*
- *lattices,*
- *partial orderings,*
- *2-step nilpotent groups,*
- *integral domains.*

Hirschfeldt, Khoussainov, Shore, and Slinko in *Degree Spectra and Computable Dimensions in Algebraic Structures*, 2000:

Whenever a structure with a particularly interesting computability-theoretic property is found, it is natural to ask whether similar examples can be found within well-known classes of algebraic structures, such as groups, rings, lattices, and so forth... The codings we present are general enough to be viewed as establishing that the theories mentioned above are computably complete in the sense that, for a wide range of computability-theoretic nonstructure type properties, if there are any examples of structures with such properties then there are such examples that are models of each of these theories.

Universal Classes

Theorem (Hirschfeldt, Khousseinov, Shore, Slinko)

Each of the classes

- *undirected graphs,*
- *partial orderings,*
- *lattices,*
- *integral domains,*
- *commutative semigroups, and*
- *2-step nilpotent groups.*

is complete with respect to

- *degree spectra of nontrivial structures,*
- *effective dimensions,*
- *degree spectra of relations,*
- *degrees of categoricity,*
- *Scott ranks,*
- *categoricity spectra, ...*

A Better Definition of Universality

The problem: We can always add more properties to this list.

Solution one (Miller, Poonen, Schoutens, Shlapentokh): Use computable category theory.

Theorem (Miller, Poonen, Schoutens, Shlapentokh)

There is a computable equivalence of categories between graph and fields.

Solution two (Montalbán): Use effective bi-interpretations.

Theorem (Montalbán)

If \mathcal{A} and \mathcal{B} are bi-interpretable, then they are essentially the same from the point of view of computable structure theory. In particular, the complexity of their optimal Scott sentences are the same.

These two solutions are equivalent.

Effective Interpretations

Let $\mathcal{A} = (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \dots)$ where $P_i^{\mathcal{A}} \subseteq A^{a(i)}$.

Definition

\mathcal{A} is *effectively interpretable* in \mathcal{B} if there exist a uniformly computable Δ_1^0 -definable relations $(\text{Dom}_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, R_1, \dots)$ such that

- (1) $\text{Dom}_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$,
- (2) \sim is an equivalence relation on $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$,
- (3) $R_i \subseteq (B^{<\omega})^{a(i)}$ is closed under \sim within $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$,

and a function $f_{\mathcal{A}}^{\mathcal{B}}: \text{Dom}_{\mathcal{A}}^{\mathcal{B}} \rightarrow \mathcal{A}$ which induces an isomorphism:

$$(\text{Dom}_{\mathcal{A}}^{\mathcal{B}} / \sim; R_0 / \sim, R_1 / \sim, \dots) \cong (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \dots).$$

Computable Functors

Definition

$\text{Iso}(\mathcal{A})$ is the category of copies of \mathcal{A} with domain ω . The morphisms are isomorphisms between copies of \mathcal{A} .

Recall: a functor F from $\text{Iso}(\mathcal{A})$ to $\text{Iso}(\mathcal{B})$

- (1) assigns to each copy $\widehat{\mathcal{A}}$ in $\text{Iso}(\mathcal{A})$ a structure $F(\widehat{\mathcal{A}})$ in $\text{Iso}(\mathcal{B})$,
- (2) assigns to each isomorphism $f: \widehat{\mathcal{A}} \rightarrow \widetilde{\mathcal{A}}$ in $\text{Iso}(\mathcal{A})$ an isomorphism $F(f): F(\widehat{\mathcal{A}}) \rightarrow F(\widetilde{\mathcal{A}})$ in $\text{Iso}(\mathcal{B})$.

Definition

F is *computable* if there are computable operators Φ and Φ_* such that

- (1) for every $\widehat{\mathcal{A}} \in \text{Iso}(\mathcal{A})$, $\Phi^{D(\widehat{\mathcal{A}})}$ is the atomic diagram of $F(\widehat{\mathcal{A}})$,
- (2) for every isomorphism $f: \widehat{\mathcal{A}} \rightarrow \widetilde{\mathcal{A}}$, $F(f) = \Phi_*^{D(\widehat{\mathcal{A}}) \oplus f \oplus D(\widetilde{\mathcal{A}})}$.

Equivalence

An effective interpretation of \mathcal{A} in \mathcal{B} induces a computable functor from \mathcal{B} to \mathcal{A} .

Theorem (HT-Melnikov-Miller-Montalbán)

Effective interpretations of \mathcal{A} in \mathcal{B} are in correspondence with computable functors from \mathcal{B} to \mathcal{A} .

By “in correspondence” we mean that every computable functor is effectively isomorphic to one induced by an effective interpretation.

Equivalence

Theorem (HT-Melnikov-Miller-Montalbán)

Effective bi-interpretations between \mathcal{A} and \mathcal{B} are in correspondence with effective equivalences of categories between \mathcal{A} and \mathcal{B} .

Theorem (HT-Miller-Montalbán)

Non-effective bi-interpretations between \mathcal{A} and \mathcal{B} are in correspondence with Borel equivalences of categories between \mathcal{A} and \mathcal{B} and with continuous isomorphisms between the automorphism groups of \mathcal{A} and \mathcal{B} .

A Definition of Universality

Definition

A class \mathcal{C} of structures is universal if each structure \mathcal{A} is uniformly effectively bi-interpretable with a structure in \mathcal{C} .

Theorem

Each of the following classes is universal:

- *undirected graphs,*
- *partial orderings,*
- *lattices, and*
- *fields,*

and, after naming finitely many constants,

- *integral domains,*
- *commutative semigroups, and*
- *2-step nilpotent groups.*

Classes Which Are Not Universal

Theorem

Each of the following classes is not universal:

- *algebraically closed fields,*
- *real closed fields,*
- *abelian groups,*
- *linear orderings,*
- *Boolean algebras.*

Proof: In these classes, the computable dimension can only be 1 or ω .

Universality for Finitely Generated Structures

What about for finitely generated structures? These are never going to be universal. So we have to restrict our attention to finitely generated structures.

Definition

Let \mathcal{C} be a class of finitely-generated structures. \mathcal{C} is universal among finitely generated structures if every finitely generated structure is uniformly effectively bi-interpretable with one in \mathcal{C} .

Example

Fields are not universal among finitely generated structures.

Finitely Generated Groups Are Universal

Theorem (HT-Ho)

Finitely generated groups are universal among finitely generated structures (after naming some constants).

Moreover, the orbits of the constants are Σ_1^0 definable.

So now, instead of constructing a finitely generated group with some property, we can construct a finitely generated structure in whatever language we like.

Quasi Scott Sentences

Quasi Scott Sentences

When we constructed a group with no $d\text{-}\Sigma_2^0$ Scott sentence before, we used an infinitely generated group with the same Σ_2^0 theory. What happens if we ask for a description of a finitely generated group within the class of finitely generated groups?

Definition

We say that a sentence φ is a quasi Scott sentence for a finitely generated structure \mathcal{A} if \mathcal{A} is the unique finitely generated structure satisfying φ .

Fact (HT-Ho)

Each finitely generated structure has a Π_3^0 quasi Scott sentence.

Let p be the atomic type of a generating tuple of \mathcal{A} . The Π_3^0 quasi Scott sentence for \mathcal{A} says that every tuple is generated by a tuple of type p .

A Partial Characterization

When does a finitely generated structure have a simpler description?

It turns out that things are more complicated than they were before.

Theorem (HT-Ho)

Let \mathcal{A} be a finitely generated structure. The following are equivalent:

- *The Σ_2^0 theory of \mathcal{A} has more than one finitely generated model.*
- *There is a finitely generated structure \mathcal{B} not isomorphic to \mathcal{A} such that $\mathcal{A} <_{\Sigma_1^0} \mathcal{B}$ and $\mathcal{B} <_{\Sigma_1^0} \mathcal{A}$.*

The Σ_2^0 theory of \mathcal{A} contains both the Σ_2^0 and the Π_2^0 sentences true of \mathcal{A} .

A Complicated Structure

Theorem (HT-Ho)

There is a finitely generated structure whose Σ_2^0 theory has more than one finitely generated model. This structure has no $d\text{-}\Sigma_2^0$ quasi Scott sentence.

In particular, it is possible to have both a Σ_3^0 and a Π_3^0 quasi Scott sentence without having a $d\text{-}\Sigma_2^0$ quasi Scott sentence.

Corollary (HT-Ho)

There is a finitely generated group whose Σ_2^0 theory has more than one finitely generated model. This group has no $d\text{-}\Sigma_2^0$ quasi Scott sentence.

$d\text{-}\Sigma_2^0$ Quasi Scott Sentences

We do not have a complete characterization of structures with $d\text{-}\Sigma_2^0$ quasi Scott sentences.

One can still often write one down by hand.

Simple Quasi Scott Sentence But No Simple Scott Sentence

Theorem (HT-Ho)

There is a finitely generated structure which has no $d\text{-}\Sigma_2^0$ Scott sentence, but has a $d\text{-}\Sigma_2^0$ quasi Scott sentence.

We build a finitely generated structure which is a Σ_1^0 -elementary substructure of itself, but not of any other finitely generated structure.

Corollary (HT-Ho)

There is a finitely generated group which has no $d\text{-}\Sigma_2^0$ Scott sentence, but has a $d\text{-}\Sigma_2^0$ quasi Scott sentence.

Open Questions

Question

Characterize the finitely generated structures which are the only finitely-generated models of their Σ_2^0 theory, but which do not have a d - Σ_2^0 quasi Scott sentence?

Question

Are there any such models?