Degree Spectra of Relations on a Cone

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The main question

Setting: $\mathcal{A}$ a computable structure, and $R \subseteq A^n$ an additional relation on $\mathcal{A}$ not in the signature of $\mathcal{A}$.

Suppose that $\mathcal{A}$ is a very “nice” structure.

OR

Consider behaviour on a cone.

Which sets of degrees can be the degree spectrum of such a relation?

Conventions and basic definitions

All of our languages and structures will be countable.

Definition

A structure is computable if its atomic diagram is computable.

Definition

Let $\mathcal{A}$ be a structure and R a relation on $\mathcal{A}$. R is *invariant* if it is fixed by automorphisms of $\mathcal{A}$.

If $\mathcal{B} \cong \mathcal{A}$, we obtain a relation $R^{\mathcal{B}}$ on $\mathcal{B}$ using the invariance of R.

Degree spectra

Let $\mathcal{A}$ be a computable structure and R a relation on $\mathcal{A}$.

Definition (Harizanov)

The *degree spectrum* of R is

$$\mathrm{dgSp}(R) = \{d(R^{\mathcal{B}}) : \mathcal{B} \text{ is a computable copy of } \mathcal{A}\}$$

Pathological examples:

- (Hirschfeldt) the degrees below a given c.e. degree.
- (Harizanov) $\{0, \mathbf{d}\}$, $\mathbf{d}$ is Δ_2^0 but not a c.e. degree.
- (Hirschfeldt) $\{0, \mathbf{d}\}$, $\mathbf{d}$ is a c.e. degree.

Degree spectra of c.e. relations

Let $\mathcal{A}$ be a computable structure and R a relation on $\mathcal{A}$.

Theorem (Harizanov)

Suppose that R is computable. Suppose moreover that the property $(*)$ holds of $\mathcal{A}$ and R. Then

$$\mathrm{dgSp}(R) \neq \{\mathbf{0}\} \Rightarrow \mathrm{dgSp}(R) \supseteq \text{c.e.}$$

$(*)$ *For every $\bar{a}$, we can computably find $a \in R$ such that for all $\bar{b}$ and quantifier-free formulas $\theta(\bar{z}, x, \bar{y})$ such that $\mathcal{A} \models \theta(\bar{a}, a, \bar{b})$, there are $a' \notin R$ and $\bar{b}'$ such that $\mathcal{A} \models \theta(\bar{a}, a', \bar{b}')$.*

On a cone, the effectiveness condition holds.

Degree spectra relative to a cone

Let $\mathcal{A}$ be a computable structure and R a relation on $\mathcal{A}$.

Definition

The *degree spectrum of R below the degree* $\mathbf{d}$ is

$$\text{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \{d(R^{\mathcal{B}}) \oplus \mathbf{d} : \mathcal{B} \cong \mathcal{A} \text{ and } \mathcal{B} \leq_T \mathbf{d}\}$$

Corollary (Harizanov)

One of the following is true for all degrees $\mathbf{d}$ *on a cone:*

1. $\text{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \{\mathbf{d}\}$, *or*
2. $\text{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} \supseteq$ *degrees c.e. in and above* $\mathbf{d}$.

Relativised degree spectra

Let $\mathcal{A}$ and $\mathcal{B}$ be structures and R and S relations on $\mathcal{A}$ and $\mathcal{B}$ respectively.

For any degree $\mathbf{d}$, either $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}}$ is equal to $\mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}}$, one is strictly contained in the other, or they are incomparable. By Borel determinacy, exactly one of these happens on a cone.

Definition (Montalbán)

The degree spectrum of $(\mathcal{A}, R)$ on a cone is equal to that of $(\mathcal{B}, S)$ if we have equality on a cone, and similarly for containment and incomparability.

Two classes of degrees

Definition

A set A is d.c.e. if it is of the form $B - C$ for some c.e. sets B and C.

A set is n-c.e. if it has a computable approximation which is allowed n alternations.

We omit the definition of α-c.e.

Definition

A set A is CEA in B if A is c.e. in B and $A \geq_T B$.

A is n-CEA if there are sets $A_1, A_2, \ldots, A_n = A$ such that A_1 is c.e., A_2 is CEA in A_1, and so on.

We omit the definition of α-CEA.

Natural classes of degrees

Let Γ be a natural class of degrees which relativises. For example, Γ might be the Δ^0_α, Σ^0_α, or Π^0_α degrees. We will also be interested in the α-c.e. and α-CEA degrees we just defined.

For any of these classes Γ of degrees, there is a structure $\mathcal{A}$ and a relation R such that, for each degree $\mathbf{d}$,

$$\mathrm{dgSp}_{\leq \mathbf{d}}(\mathcal{A}, R) = \Gamma(\mathbf{d}) \oplus \mathbf{d}.$$

So we may talk, for example, about a degree spectrum being equal to the Σ_α degrees on a cone.

Main question

Harizanov's result earlier showed that degree spectra on a cone behave nicely with respect to c.e. degrees.

Corollary (Harizanov)

Any degree spectrum on a cone is either equal to Δ^0_1 or contains Σ^0_1.

Question

What are the possible degree spectra on a cone?

D.c.e. relations

Theorem (H.)

There is a computable structure $\mathcal{A}$ and relatively intrinsically d.c.e. relations R and S on $\mathcal{A}$ with the following property:

for any degree $\mathbf{d}$, $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}}$ *and* $\mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}}$ *are incomparable.*

Corollary (H.)

There are two degree spectra on a cone which are incomparable, each contained within the d.c.e. degrees and containing the c.e. degrees.

A question of Ash and Knight

Question (Ash-Knight)

Can one show (assuming some effectiveness condition) that any relation which is not intrinsically Δ^0_α realises every α-CEA degree?

Stated in terms of degree spectra on a cone, is it true that every degree spectrum on a cone is either contained in Δ^0_α, or contains α-CEA?

A question of Ash and Knight

Ash and Knight gave a result which goes towards answering this question.

Theorem (Ash-Knight)

Let $\mathcal{A}$ be a computable structure with an additional computable relation R. Suppose that R is not relatively intrinsically Δ^0_α.

Moreover, suppose that $\mathcal{A}$ is α-friendly and that for all $\bar{c}$, we can find $a \notin R$ which is α-free over $\bar{c}$.

Then for any Σ^0_α set C, there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that

$$R^{\mathcal{B}} \oplus \Delta^0_\alpha \equiv_T C \oplus \Delta^0_\alpha$$

where Δ^0_α is a Δ^0_α-complete set.

The class 2-CEA

Theorem (H.)

Let $\mathcal{A}$ be a structure and R a relation on $\mathcal{A}$. Then one of the following is true relative to all degrees on a cone:

1. $\mathrm{dgSp}(\mathcal{A}, R) \subseteq \Delta^0_2$, *or*
2. $2\text{-}CEA \subseteq \mathrm{dgSp}(\mathcal{A}, R)$.

Unresolved questions

Question

What about $\alpha > 2$?

Question

Are there more than two degree spectra on a cone which are contained within the d.c.e. degrees but strictly contain the c.e. degrees?

Question

Are degree spectra on a cone closed under join?

Thanks!