Independence in Computable Algebra

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Introduction

The standard computable presentation of the infinite dimensional $\mathbb{Q}$-vector-space has a computable basis. In the 1960's Mal'cev noticed that there is another computable presentation with no computable basis.

Many other algebraic structures have a notion of "independence" generalizing linear independence in vector spaces and algebraic independence in fields.

A *pregeometry* is a natural formalization of an independence relation. There is a corresponding notion of *basis*.

Example One: Torsion-free Abelian Groups

Consider $\mathbb{Z}$-linear independence on abelian groups.

Theorem (Nurtazin 1974, Dobrica 1983)

Let $\mathcal{M}$ be a computable torsion-free abelian group of infinite dimension.

1. *There is a computable copy $\mathcal{G}$ with a computable $\mathbb{Z}$-basis.*
2. *There is a computable copy $\mathcal{B}$ with no computable $\mathbb{Z}$-basis.*
3. *$\mathcal{G}$ and $\mathcal{B}$ are Δ^0_2-isomorphic.*

Corollary (Goncharov 1982)

Let $\mathcal{M}$ and $\mathcal{N}$ be computable structures which are Δ^0_2-isomorphic but not computably isomorphic. Then they have infinitely many computable copies up to computable isomorphism.

We say that $\mathcal{M}$ has *computable dimension* ω.

Example Two: Archimedean Ordered Abelian Groups

Theorem (Goncharov, Lempp, Solomon 2003)

Let $\mathcal{M}$ be a computable archimedean ordered abelian group of infinite dimension.

1. *There is a computable copy $\mathcal{G}$ with a computable $\mathbb{Z}$-basis.*
2. *There is a computable copy $\mathcal{B}$ with no computable $\mathbb{Z}$-basis.*
3. *$\mathcal{G}$ and $\mathcal{B}$ are Δ^0_2-isomorphic.*
4. *$\mathcal{M}$ has computable dimension ω.*

The Mal'cev Property

Let $\mathcal{K}$ be a class of computable algebraic structures.

Main Question

Does every structure in $\mathcal{K}$ have:

- a computable copy with a computable basis?
- a computable copy with no computable basis?

The Mal'cev Property

Let $\mathcal{K}$ be a class of computable algebraic structures.

Definition

$\mathcal{K}$ has the *Mal'cev property* if each member $\mathcal{M}$ of $\mathcal{K}$ of infinite dimension has

- a computable presentation $\mathcal{G}$ with a computable basis
- a computable presentation $\mathcal{B}$ with no computable basis
- $\mathcal{B} \cong_{\Delta_2^0} \mathcal{G}$

Main Results

We give sufficient conditions for a class to have the Mal'cev property, and use them in new applications.

Definition of a Pregeometry

Definition

Let X be a set and $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ a function on $\mathcal{P}(X)$. We say that cl is a *pregeometry* if:

1. $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
2. $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,
3. (finite character)
$$\mathrm{cl}(A) = \bigcup_{\substack{B \text{ finite} \\ B \subseteq A}} \mathrm{cl}(B),$$
4. (exchange principle) if $a \in \mathrm{cl}(A \cup \{b\})$ and $a \notin \mathrm{cl}(A)$, then $b \in \mathrm{cl}(A \cup \{a\})$.

Properties of Pregeometries

Let (X, cl) be a pregeometry, and $A \subseteq X$.

Definition

$A \subseteq X$ is *independent* if for all $a \in A$, $a \notin \mathrm{cl}(A \backslash \{a\})$, and A is *dependent* otherwise.

B is a *basis* for X if B is independent and $X = \mathrm{cl}(B)$. Equivalently, B is a basis for X if and only if B is a maximal independent set.

X has a basis. Every basis is the same size, the *dimension* of X.

Computably Enumerable Pregeometries

Definition

A pregeometry cl on a structure $\mathcal{M}$ is relatively intrinsically computably enumerable (r.i.c.e.) if the relations

$$x \in \mathrm{cl}(y_1, \ldots, y_n)$$

are uniformly computably Σ_1 definable.

Proposition

Let $(\mathcal{M}, \mathrm{cl})$ *be a r.i.c.e. pregeometry.*

$$(\mathcal{M}, \mathrm{cl}) \text{ has a computable basis } \Leftrightarrow \mathrm{cl} \text{ is computable.}$$

Computable pregeometries have been studied by Metakides, Nerode, Downey, and Remmel.

Construction of a "nice" copy.

Construction of a "Nice" Copy

We have: a computable structure $\mathcal{M}$ with a r.i.c.e. pregeometry.

We want: $\mathcal{G} \cong_{\Delta^0_2} \mathcal{M}$ such that $\mathcal{G}$ has a computable basis.

Condition G

Definition

The *independence diagram* of $\bar{c}$ in $\mathcal{M}$ is:

$$\mathcal{I}_{\mathcal{M}}(\bar{c}) = \{\varphi(\bar{c}, \bar{x}) \text{ an existential formula} : \exists \bar{u} \text{ independent over } \bar{c} \text{ with } \mathcal{M} \models \varphi(\bar{c}, \bar{u})\}$$

Definition

Independent tuples in $\mathcal{M}$ are *locally indistinguishable* if for all $\varphi \in \mathcal{I}_{\mathcal{M}}(\bar{c})$ and $\bar{u}$ independent over $\bar{c}$, there is a tuple $\bar{v}$ with:

- $\bar{v}$ is independent over $\bar{c}$,
- $\mathcal{M} \models \varphi(\bar{c}, \bar{v})$, and
- $v_i \in \mathrm{cl}(\bar{c}, u_1, \ldots, u_i)$.

Condition G: Independent tuples are locally indistinguishable in $\mathcal{M}$ and for each $\mathcal{M}$-tuple $\bar{c}$, $\mathcal{I}_{\mathcal{M}}(\bar{c})$ is c.e. uniformly in $\bar{c}$.

Construction of a "Nice" Copy

Condition G: Independent tuples are locally indistinguishable in $\mathcal{M}$ and for each $\mathcal{M}$-tuple $\bar{c}$, $\mathcal{I}_{\mathcal{M}}(\bar{c})$ is c.e. uniformly in $\bar{c}$.

Theorem (H-T, Melnikov, Montalbán)

Let $\mathcal{M}$ be a computable structure, and let cl *be a r.i.c.e. pregeometry on $\mathcal{M}$.*

$$(\mathcal{M}, \mathrm{cl}) \textit{ has Condition G}$$

$$\Downarrow$$

$$\textit{there is } \mathcal{G} \cong_{\Delta^0_2} \mathcal{M} \textit{ with a computable basis.}$$

Construction of a “bad” copy.

Construction of a "Bad" Copy

We have: a computable structure $\mathcal{M}$ with a r.i.c.e. pregeometry.

We want: $\mathcal{B} \cong_{\Delta_2^0} \mathcal{M}$ such that $\mathcal{B}$ has no computable basis.

Condition B

Definition

We say that *dependent elements are dense in* $\mathcal{M}$ if whenever $\psi(\bar{c}, x)$ is a satisfiable existential formula, there is $b \in \mathrm{cl}(\bar{c})$ with $\mathcal{M} \models \psi(\bar{c}, b)$.

Technical note: we can assume that $\bar{c}$ always contains an independent element or two.

Condition B: Dependent elements are dense in $\mathcal{M}$.

Construction of a "Bad" Copy

Condition B: Dependent elements are dense in $\mathcal{M}$.

Theorem (H-T, Melnikov, Montalbán)

Let $\mathcal{M}$ be a computable structure, and let cl *be a r.i.c.e. pregeometry upon $\mathcal{M}$. Suppose that the* cl*-dimension of $\mathcal{M}$ is infinite.*

$$(\mathcal{M}, \mathrm{cl}) \text{ has Condition B}$$

$$\Downarrow$$

$$\text{there is } \mathcal{B} \cong_{\Delta^0_2} \mathcal{M} \text{ with no computable basis.}$$

Mal'cev Property

Theorem (H-T, Melnikov, Montalbán)

Let $\mathcal{K}$ be a class of computable structures with r.i.c.e. pregeometries.

$$\text{Structures in } \mathcal{K} \text{ have Condition G and Condition B}$$

$$\Downarrow$$

$$\mathcal{K} \text{ has the Mal'cev property.}$$

Applications.

Applications for Existing Results

We get the same results as before, but with nicer proofs which separate the algebra and combinatorics from the computability.

Recall that the following structures have the Mal'cev property:

- vector spaces over an infinite field with linear independence [Mal'cev]
- algebraically closed fields with algebraic independence [Folklore]
- torsion-free abelian groups with $\mathbb{Z}$-linear independence [Nurtazin, Dobrica]
- archimedean ordered abelian groups with $\mathbb{Z}$-linear independence [Goncharov, Lempp, Solomon]

New Applications

We also have some new applications:

Theorem (H-T, Melnikov, Montalbán)

The following classes of structures have the Mal'cev property:

- *real closed fields with algebraic independence*

  (uses decidability of RCF, cell decomposition / definable Skolem functions)
- *differentially closed fields with δ-independence*

  (uses decidability of DCF_0, quantifier elimination, uniqueness of independent type)
- *difference closed fields with transformal independence*

  (uses decidability of ACFA, model completeness, uniqueness of independent type)

Thanks!