Degrees of Categoricity Above Limit Ordinals

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Abstract

A computable structure $A$ has degree of categoricity $d$ if $d$ is exactly the degree of difficulty of computing isomorphisms between isomorphic computable copies of $A$. Fokina, Kalimullin, and Miller showed that every degree d.c.e. in and above $0^{(n)}$, for any $n < \omega$, and also the degree $0^{(\omega)}$, are degrees of categoricity. Later, Csima, Franklin, and Shore showed that every degree $0^{(\alpha)}$ for any computable ordinal $\alpha$, and every degree d.c.e. in and above $0^{(\alpha)}$ for any successor ordinal $\alpha$, is a degree of categoricity. We show that every degree c.e. in and above $0^{(\omega)}$ is the degree of categoricity of a prime model, making progress towards a question of Bazhenov.

1 Introduction

It has long been known that there are two computable presentations of the vector space $\mathbb{Q}^\mathbb{N}$, one with a computable basis, and the other without a computable basis. Certainly there can be no computable isomorphism between these presentations, as the image of the computable basis from the first presentation would be a computable basis in the second presentation. Indeed the problem of building an isomorphism between two computable presentations of $\mathbb{Q}^\mathbb{N}$ amounts to computing bases for each of them, since then the bases can be matched up and effectively extended to an isomorphism. To give a basis of a computable presentation of $\mathbb{Q}^\mathbb{N}$ requires only the ability to answer single quantifier questions: Is the next potential basis member a linear combination of what we have already included? The Turing degree of the Halting set, denoted $0'$, is smart enough to answer such questions. We say that $\mathbb{Q}^\mathbb{N}$ is $0'$-computably categorical because we can compute, using $0'$, an isomorphism between any two computable presentations. More generally:

**Definition 1.1.** Let $d$ be a Turing degree. A computable structure $A$ is $d$-computably categorical if, for every computable copy $B$ of $A$, there is a $d$-computable isomorphism between $A$ and $B$. 

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For many structures $\mathcal{A}$, there is a least degree $d$ such that $\mathcal{A}$ is $d$-computable categorical. In the case of $\mathbb{Q}^\mathbb{N}$, there are actually two computable presentations of $\mathbb{Q}^\mathbb{N}$ such that any isomorphism between the two computes $0'$. So $0'$ is exactly the difficulty of computing isomorphisms between copies of $\mathbb{Q}^\mathbb{N}$. This natural idea was formalized by Fokina, Kalimullin, and Miller [?].

**Definition 1.2** (Fokina, Kalimullin, and Miller [?]). A Turing degree $d$ is said to be the degree of categoricity of a computable structure $\mathcal{A}$ if $d$ is the least degree such that $\mathcal{A}$ is $d$-computably categorical.

Fokina, Kalimullin, and Miller showed that every degree that can be realized as a difference of computably enumerable (d.c.e.) sets in and above $0^{(n)}$, for any $n < \omega$, and also the degree $0^{(\omega)}$, are degrees of categoricity. Later, Csima, Franklin, and Shore [?] showed that every degree $0^{(\alpha)}$ for any computable ordinal $\alpha$, and every degree d.c.e. in and above $0^{(\alpha)}$ for any successor ordinal $\alpha$, is a degree of categoricity. Csima and Ng have announced a proof that every $\Delta^0_2$ degree is a degree of categoricity.

It is often the case when trying to prove some property $P(\alpha)$ for ordinals $\alpha$ that things get tricky at limit ordinals. Roughly speaking, if $\alpha$ is a successor ordinal and we know something must happen before $\alpha$, we can safely say it has happened by $\alpha - 1$. For $\alpha$ a limit ordinal, in such a situation there is no canonical choice of earlier ordinal to look at. This is why the methods of [?] did not work above limit ordinals.

Our main result in this paper is:

**Theorem 1.3.** Let $\alpha$ be a computable limit ordinal and $d$ a degree c.e. in and above $0^{(\alpha)}$. There is a computable structure with strong degree of categoricity $d$.

This fills in a gap that was missing from [?] above limit ordinals, making further progress towards Question 5.1 of that paper.

We have not yet explained what a strong degree of categoricity is. For a long time, all of the known examples had the following property: If $\mathcal{A}$ had degree of categoricity $d$, then there is a copy $\mathcal{B}$ of $\mathcal{A}$ such that every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ computes $d$. Thus, we can witness with just two computable copies the fact that $d$ is the least degree such that $\mathcal{A}$ is $d$-computably categorical. In this case, we say that $\mathcal{A}$ has strong degree of categoricity $d$. Recently, Bazhenov, Kalimullin, and Yamaleev [?] have shown that there is a c.e. degree $d$ and a structure $\mathcal{A}$ with degree of categoricity $d$, but $d$ is not a strong degree of categoricity for $\mathcal{A}$. Csima and Stephenson [?] have shown that there is a structure of finite computable dimension that has a degree of categoricity but no strong degree of categoricity.

**Question** (Bazhenov and Marchuk [?]). What can be the degrees of categoricity of computable prime models?

A computable prime model—in fact, as Bazhenov shows, a computable homogeneous model—is always $0^{(\omega+1)}$-categorical, as we can ask $0^{(\omega+1)}$ if two tuples satisfy the same type. Bazhenov constructs a computable homogeneous model with degree of categoricity $0^{(\omega+1)}$. The complexity here is in the structure itself, rather than in the
theory. To build a prime model with degree of categoricity \( 0^{(\omega+1)} \), the complexity must be in the theory: If \( A \) is a computable prime model of a theory \( T \), then \( A \) is \( T' \oplus 0^{(\omega)} \)-categorical as \( T' \) can decide whether a formula is complete, and \( 0^{(\omega)} \) can decide whether a formula holds of a tuple in \( A \). We build a computable prime model with degree of categoricity \( 0^{(\omega+1)} \) (or any other degree c.e. in and above \( 0^{(\omega)} \)).

**Theorem 1.4.** Let \( d \) be a degree c.e. in and above \( 0^{(\omega)} \). There is a computable prime model \( A \) with strong degree of categoricity \( d \).

## 2 Categoricity Relative to Decidable Models

As a warm-up to illustrate the methods used to prove these two theorems, we give a simple proof of a result of Goncharov \[?\] that for every c.e. degree \( d \), there is a decidable prime model with degree of categoricity \( d \) with respect to decidable copies. Recall that a structure is said to be decidable if its full elementary diagram is computable. In \[?]\, Goncharov made the following definitions:

**Definition 2.1.** Let \( d \) be a Turing degree and \( A \) a decidable structure. Then \( A \) is *\( d \)-categorical with respect to decidable copies* if for every decidable copy \( B \) of \( A \), \( d \) computes an isomorphism between \( A \) and \( B \).

**Definition 2.2.** Let \( d \) be a Turing degree and \( A \) a decidable structure. Then \( d \) is the *degree of categoricity of \( A \) with respect to decidable copies* if:

- \( A \) is \( d \)-categorical with respect to decidable copies, and
- whenever \( A \) is \( c \)-categorical with respect to decidable copies, \( c \geq d \).

It is not hard to see that between any two decidable copies of a prime model, there is a \( 0' \)-computable isomorphism. Goncharov showed that any c.e. degree can be the degree of categoricity with respect to decidable copies of a prime model. We give a different proof, which we think is simpler, and which demonstrates some of the techniques that we will use later.

**Theorem 2.3** (Goncharov \[?\, Theorem 3\]). Let \( d \) be a c.e. degree. Then there is a decidable prime model \( M \) which has strong degree of categoricity \( d \) with respect to decidable models.

**Proof.** Let \( D \in d \) be a c.e. set. We will construct the structures \( M \) and \( N \). They are the disjoint union of infinitely many structures \( M_n \) and \( N_n \), with \( M_n \) and \( N_n \) picked out by unary relations \( R_n \). The \( n \)th sort will code whether \( n \in D \). Fix \( n \). \( M_n \) will have infinitely many elements \((a_i)_{i \in \omega}\). There will be infinitely many unary relations \((U_\ell)_{\ell \in \omega}\) defined on \( M_n \) so that:

\[
a_0 \in U_\ell \iff n \in D_{at \ell}
\]

where \( n \in D_{at \ell} \) means that \( n \) enters \( D \) at exactly stage \( s \), and

\[
a_i \notin U_\ell \text{ for } i > 0 \text{ and all } \ell.
\]

Similarly, \( N_n \) will have infinitely many elements \((b_\ell)_{i \in \omega}\) with the unary relations defined so that:

\[
b_\ell \in U_\ell \iff i = s \text{ and } n \in D_{at \ell}.
\]

It is easy to see that we can build computable copies of \( M \) and \( N \). These copies are in fact decidable.
Claim 1. \(\mathcal{M}\) and \(\mathcal{N}\) are decidable.

Proof. Given a formula \(\varphi(x_1, \ldots, x_n)\) with \(k\) quantifiers and \(a_{i_1}, \ldots, a_{i_n} \in \mathcal{M}\), it is not hard to see that \(\mathcal{M} \models \varphi(a_{i_1}, \ldots, a_{i_n})\) if and only if the finite substructure \(\mathcal{M}'\) of \(\mathcal{M}\) whose domain consists of \(a_1, \ldots, a_{k+n+1}\) and \(a_{i_1}, \ldots, a_{i_n}\) also has \(\mathcal{M}' \models \varphi(a_{i_1}, \ldots, a_{i_n})\). Thus \(\mathcal{M}\) is decidable.

For \(\mathcal{N}\), suppose we have a formula \(\varphi(x_1, \ldots, x_n)\) with at most \(k\) quantifiers and which uses only some subset of the relations \(U_0, \ldots, U_k\). Let \(b_{i_1}, \ldots, b_{i_n}\) be elements of \(\mathcal{N}\). Then \(\mathcal{N} \models \varphi(b_{i_1}, \ldots, b_{i_n})\) if and only if the finite substructure \(\mathcal{N}'\) of \(\mathcal{M}\) whose domain consists of \(b_1, \ldots, b_{k+n+1}\) and \(b_{i_1}, \ldots, b_{i_n}\) also has \(\mathcal{N}' \models \varphi(b_{i_1}, \ldots, b_{i_n})\). Thus \(\mathcal{N}\) is decidable.

Claim 2. \(\mathcal{M}\) and \(\mathcal{N}\) are isomorphic.

Proof. It suffices to show that for each \(n\), \(\mathcal{M}_n\) and \(\mathcal{N}_n\) are isomorphic. If \(n \notin D\), then \(a_i \mapsto b_i\) induces an isomorphism between \(\mathcal{M}_n\) and \(\mathcal{N}_n\). If \(n \in D\), then the map

\[
\begin{align*}
a_0 &\mapsto b_s \\
 a_{i+1} &\mapsto b_i & \text{when } i < s \\
a_i &\mapsto b_i & \text{when } i > s
\end{align*}
\]

is an isomorphism between \(\mathcal{M}_n\) and \(\mathcal{N}_n\).

Claim 3. \(\mathcal{M}\) and \(\mathcal{N}\) are prime.

Proof. It suffices to show that each \(\mathcal{M}_n\) and \(\mathcal{N}_n\) are prime, since these structures are determined inside \(\mathcal{M}\) and \(\mathcal{N}\) uniquely by the relation \(R_n\). It is not hard to see that \(\mathcal{M}_n\) and \(\mathcal{N}_n\) are models of an \(\aleph_0\)-categorical theory, and hence are prime.

Claim 4. Any isomorphism between \(\mathcal{M}\) and \(\mathcal{N}\) can compute \(D\).

Proof. Let \(g\) be an isomorphism between \(\mathcal{M}\) and \(\mathcal{N}\). For each \(n\), let \((a_i)_{i<\omega}\) and \((b_i)_{i<\omega}\) be the elements in the definition of \(\mathcal{M}_n\) and \(\mathcal{N}_n\). Let \(s\) be such that \(g(a_0) = b_s\). Then \(n \in D\) if and only if \(n \in D_s\).

Claim 5. Given a computable copy \(\tilde{\mathcal{M}}\) of \(\mathcal{M}\), \(D\) can compute an isomorphism between \(\mathcal{M}\) and \(\tilde{\mathcal{M}}\).

Proof. For each \(n\), let \(\tilde{\mathcal{M}}_n\) be the structure with domain \(R_n\) in \(\tilde{\mathcal{M}}\). It suffices to compute an isomorphism \(g\) between \(\mathcal{M}_n\) and \(\tilde{\mathcal{M}}_n\) for each \(n\). Let \((c_i)_{i<\omega}\) be the elements of \(\tilde{\mathcal{M}}_n\). If \(n \notin D\), no relation \(U_j\) holds of any of the the elements \((a_i)_{i<\omega}\) or \((c_i)_{i<\omega}\). So \(a_i \mapsto c_i\) is an isomorphism. On the other hand, if \(n \in D\), then for some unique \(s\), \(a_0 \in U_s\). We can look for \(c_k\) such that \(c_k \in U_s\). Map \(a_0\) to \(c_k\); map each other \(a_i\) to some other \(c_i\).

These claims complete the proof of the theorem.
Fix a path through $O$. We will identify computable ordinals with their notation on this path. We will always first fix a limit ordinal $\alpha$ and work below it. Recall that one can decide effectively whether a given computable ordinal is a limit ordinal or a successor ordinal. For each limit ordinal $\beta < \alpha$, fix a fundamental sequence for $\beta$, that is, an increasing sequence of successor ordinals whose limit is $\beta$.

Hirschfeldt and White defined, for each successor ordinal $\beta$, a pair of trees $A_\beta$ and $E_\beta$ which can be differentiated exactly by $\beta$ jumps. These trees are called back-and-forth trees.

**Definition 3.1** ([?], Definition 3.1). Back-and-forth trees are defined recursively in $\beta$. We view these as structures in the language of graphs with the root node distinguished.

We take $A_1$ to be the tree with just a root node and no children, and we take $E_1$ to be the tree where the root node has infinitely many children, none of which have children. See Fig. 1. We say that these trees have back-and-forth rank 1.

Suppose $\beta$ is a successor ordinal. Define $A_{\beta+1}$ as a root node with infinitely many children, each the root of a copy of $E_\beta$, and define $E_{\beta+1}$ as a root node with infinitely many children, each the root of a copy of $A_\beta$, and also infinitely many other children, each the root of a copy of $E_\beta$. See Fig. 2. These trees have back-and-forth rank $\beta + 1$.

Now suppose $\beta$ is a non-zero limit ordinal, and let $\beta_0, \beta_1, \ldots$ be a fundamental sequence of successor ordinals for $\beta$, that is, a sequence of successor ordinals below $\beta$ with limit $\beta$. We first define a family of helper trees $L_{\beta,k}$ where $k \in \omega \cup \{\infty\}$. Define $L_{\beta,\infty}$ to consist of a root node whose children are root nodes of copies of $A_\beta$, and such that each copy appears exactly once as a child. For $k \in \omega$, $L_{\beta,k}$ has a root node whose children are root nodes of copies of $A_{\beta_0}, \ldots, A_{\beta_k}, E_{\beta_{k+1}}, E_{\beta_{k+2}}, \ldots$ where again each copy appears exactly once as a child. Such trees are shown in Fig. 3. We say these trees have back-and-forth rank $\beta$.

We can now define $A_{\beta+1}$ and $E_{\beta+1}$ for the non-zero limit ordinal $\beta$. For $A_{\beta+1}$, we have a root node with infinitely many children, each the root node of a copy of $L_{\beta,k}$ such that for each $k \in \omega$, $L_{\beta,k}$ appears infinitely many times. The definition of $E_{\beta+1}$ is similar, except $k$ is drawn from $\omega \cup \{\infty\}$. See Fig. 4. These trees have back-and-forth rank $\beta + 1$.

The next two lemmas piece together the facts that we will need about the back-and-forth trees, first for arbitrary $\beta$, and second some additional properties for finite $\beta$ in particular. These facts come from [?] and [?].
Figure 2: $A_{\beta+1}$ and $E_{\beta+1}$ when $\beta$ is a successor ordinal.

Figure 3: Helper trees $L_{\beta,0}$, $L_{\beta,1}$, $L_{\beta,\infty}$ for $k \in \omega$ for the non-zero limit ordinal $\beta$.

Figure 4: $A_{\beta+1}$ and $E_{\beta+1}$ for the non-zero limit ordinal $\beta$. 
Lemma 3.2. Let $\alpha$ be a computable ordinal. For a successor ordinal $\beta < \alpha$, the structures $A_\beta$ and $E_\beta$ satisfy the following properties:

1. Uniformly in $\beta$ and an index for a $\Sigma^0_\beta$ set $S$, there is a computable sequence of structures $C_x$ such that
   
   $$ x \in S \iff C_x \cong E_\beta \quad \text{and} \quad x \notin S \iff C_x \cong A_\beta. $$

2. Uniformly in $\beta$, there is a $\Sigma^0_\beta$ sentence $\varphi$ such that $E_\beta \models \varphi$ and $A_\beta \not\models \varphi$.

3. $A_\beta$ and $E_\beta$ are uniformly $0^{(\beta)}$-categorical.

Proof. For (1), take $(C_x)_x$ to be the computable sequence of trees given by Proposition 3.2 in [?].

   For (2), take $\varphi$ to be the sentence given by evaluating the formula guaranteed by Lemma 3.5 in [?] for $B = E_\beta$ at its own root node. The complexity of $\varphi$ is the natural complexity of $E_\beta$, which is $\Sigma^0_\beta$. This lemma says that for any tree $T$, $T \cong E_\beta$ if and only if $T \models \varphi$.

Finally, for (3), we use a result from Csima, Franklin, and Shore [?] about back-and-forth trees. We will consider $A_\beta$; the case for $E_\beta$ is identical. We have that $A_\beta$ is a back-and-forth tree, and hence if $C$ is a computable structure isomorphic to $A_\beta$, then it is also a computable back-and-forth tree. Corollary 2.6 in [?] allows $\phi^{(\gamma)}$ to uniformly compute an isomorphism between these two trees when the back-and-forth rank of the trees is at most $\gamma$. Since the rank of $A_\beta$ is $\beta$ by construction, the isomorphism is uniformly computable in $0^{(\beta)}$. \qed

Now for finite ordinals $\beta$ (and writing $n$ for $\beta$), we have some additional properties. We will state the lemma in full, including properties that were covered by the previous lemma. We think that these facts are well-known, but we do not know of a reference in print.

Lemma 3.3. For $0 < n < \omega$, the structures $A_n$ and $E_n$ satisfy the properties:

1. Uniformly in $n$ and an index for a $\Sigma^0_n$ set $S$, there is a computable sequence of structures $C_x$ such that
   
   $$ x \in S \iff C_x \cong E_n \quad \text{and} \quad x \notin S \iff C_x \cong A_n. $$

2. For each $n$, there is an elementary first-order $\exists_n$ sentence $\varphi_n$, computable uniformly in $n$, such that $E_n \models \varphi_n$ and $A_n \not\models \varphi_n$.

3. $A_n$ and $E_n$ are prime.

4. $A_n$ and $E_n$ are $0^{(n)}$-categorical uniformly in $n$.

Proof. (1) and (4) are the same as in the previous lemma. We show using induction on $n$ that these sequences satisfy (2) and (3) as well. It is easy to see that $A_1$ and $E_1$ are prime models of their theories and that they are distinguishable (in the sense of (2) in the statement of the lemma) by the existential sentence $\varphi_1 := \exists x \exists y (x \neq y)$. Assume now that $A_n$ and $E_n$ are prime and distinguishable by a first-order $\exists_n$ sentence $\varphi_n$ (in the sense that $E_n \models \varphi_n$ but $A_n \not\models \varphi_n$). We show that $A_{n+1}$ and $E_{n+1}$ are prime and distinguishable by a first-order $\exists_{n+1}$ sentence $\varphi_{n+1}$.  

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It is not hard to see that we can take \( \varphi_{n+1} \) to be the sentence \( \exists x \neg \varphi_n[\leq x] \land (x \text{ is a child of the root node}) \) where \( x \) is a new variable not appearing in \( \varphi_n \) and \( \varphi_n[\leq x] \) is the formula obtained from \( \varphi_n \) by bounding every quantifier to the subtree below \( x \). (Note that in a tree of rank \( n \), if \( z \) is a descendant of \( x \), i.e., there is a path from \( x \) to \( z \), the length of the path is at most \( n \), and so this is first-order definable and does not change the quantifier rank.) \( \mathcal{E}_{n+1} \models \varphi_{n+1} \) but \( \mathcal{A}_{n+1} \not\models \varphi_{n+1} \).

It remains to show that \( \mathcal{A}_{n+1} \) and \( \mathcal{E}_{n+1} \) are prime. The same method will work for both structures. Let \( \bar{a} \) be an arbitrary tuple in \( \mathcal{E}_{n+1} \). We describe a formula that isolates the type of the tuple \( \bar{a} \). Let \( r_1, \ldots, r_k \) be the children of the root which are the roots of subtrees containing elements of \( \bar{a} \); say that \( \bar{a} = (\bar{a}_1, \ldots, \bar{a}_k) \) where \( \bar{a}_i \) is in the subtree below \( r_i \). (Note that we can re-order the tuples as we like, as if the type of some permutation of \( \bar{a} \) is isolated, so is \( \bar{a} \).) By the induction hypothesis, we know that the subtree with root \( r_i \) is prime for every \( i \). Hence for each \( i = 1, \ldots, k \) there is a formula which isolates the type of the tuple \( \bar{a}_i \) in the subtree with root \( r_i \). There is also, for each \( r_i \), a formula (either \( \varphi_n \) or \( \neg \varphi_n \)) which distinguishes between whether the subtree below \( r_i \) is isomorphic to \( \mathcal{A}_n \) or \( \mathcal{E}_n \). So we can isolate the type of \( \bar{a} \) by saying that there are children \( r_1, \ldots, r_k \) of the root such that \( \bar{a}_i \) satisfies the formula, in the subtree below \( r_i \), which isolates it, and by saying whether the subtree below each \( r_i \) is isomorphic to \( \mathcal{A}_n \) or \( \mathcal{E}_n \).

Fokina, Kalimullin, and Miller [2] showed that there is a structure \( \mathcal{A} \) with strong degree of categoricity \( 0^{(\omega)} \). We note the well-known fact that one can also have \( \mathcal{A} \) be a prime model. Our proof follows that of [2].

**Theorem 3.4.** There is a computable structure \( \mathcal{A} \) with strong degree of categoricity \( 0^{(\omega)} \) such that \( \mathcal{A} \) is a prime model of its theory.

**Proof sketch.** The structure is just the disjoint union of infinitely many copies of each \( \mathcal{E}_n \) for \( n < \omega \). Theorem 3.1 of [2] shows that this has strong degree of categoricity \( 0^{(\omega)} \), and it is not hard to see using Lemma 3.3 that this structure is prime.

## 4 C.E. In And Above a Limit Ordinal

We begin this section by a short discussion of how we code a c.e. set into a structure. Consider a c.e. set \( C \). If one knows, for each \( n \), at what point the approximation to \( C(n) \) has settled, then one can compute \( C \). Moreover, one does not need to know exactly when \( C \) settles, but just a point after which \( C(n) \) has settled. In particular, any sufficiently large function can compute \( C \). Moreover, \( C \) itself can compute such a function. Following the terminology of Groszek and Slaman [2], we say that \( C \) has a self-modulus.

**Definition 4.1** (Groszek and Slaman [2]). Let \( F: \omega \to \omega \) and \( X \subseteq \omega \). Then:

- \( F \) is a modulus (of computation) for \( X \) if every \( G: \omega \to \omega \) that dominates \( F \) pointwise computes \( X \).
- \( X \) has a self-modulus if \( X \) computes a modulus for itself.

The self-modulus of a c.e. set \( C \) is the function \( f(n) = \mu s(C_s(n) = C(n)) \). Groszek and Slaman showed that every \( \Delta^0_2 \) or \( \alpha \)-CEA set has a self-modulus. In fact, the
self-modulus of a c.e. set has a nice form; it has a non-decreasing computable approximation.

**Definition 4.2.** A function $F : \omega \to \omega$ is limitwise monotonic if there is a computable approximation function $f : \omega \times \omega \to \omega$ such that, for all $n$,

- $F(n) = \lim_{s \to \infty} f(n, s)$.
- For all $s$, $f(n, s) \leq f(n, s + 1)$.

In fact, it is well-known and an easy exercise to show that the sets of c.e. degree are exactly those with limitwise monotonic self-moduli. These remarks also relativize.

The next lemma encodes a limitwise monotonic function into the isomorphisms of copies of a computable structure. Any isomorphism dominates the limitwise monotonic function; but it does not seem to be the case that dominating the limitwise monotonic function is sufficient to compute isomorphisms.

**Lemma 4.3.** Let $\alpha$ be a computable limit ordinal. Let $f : \omega \to \omega$ be limitwise monotonic relative to $0^{(\alpha)}$. There is a structure with computable copies $M$ and $N$ such that:

1. Every isomorphism between $M$ and $N$ computes a function which dominates $f$.
2. $f \oplus 0^{(\alpha)}$ computes an isomorphism between any two computable copies of $M$ and $N$.

**Proof.** Let $\Phi$ be a computable operator such that $f(n) = \lim_{s \to \infty} \Phi^{(\alpha)}(n, s)$ and this is monotonic in $s$. Write $\mathcal{O}^{(\alpha)} = \bigoplus_{\gamma \prec \alpha} \mathcal{O}^{(\gamma)}$ for successor ordinals $\gamma \prec \alpha$. By convention, for $\beta \prec \alpha$, we say that $\Phi^{(\beta)}(n, s)$ converges if the computation $\Phi^{(\alpha)}(n, s)$ halts, but the only part of the oracle $\mathcal{O}^{(\alpha)} = \bigoplus_{\gamma \prec \alpha} \mathcal{O}^{(\gamma)}$ that is read during the computation is that part with $\gamma \leq \beta$. So if $\Phi^{(\beta)}(n, s) = m$ then $\Phi^{(\alpha)}(n, s) = m$, and because $\alpha$ is a limit ordinal, if $\Phi^{(\alpha)}(n, s) = m$ then $\Phi^{(\beta)}(n, s) = m$ for some successor ordinal $\beta \prec \alpha$.

Let $(A_{\beta})_{\beta \prec \alpha}$ and $(E_{\beta})_{\beta \prec \alpha}$ be as in Lemma 3.2. We will construct the structures $M$ and $N$. They are the disjoint union of infinitely many structures $M_n$ and $N_n$, with $M_n$ and $N_n$ picked out by unary relations $R_n$. The $n$th sort will code the value of $f(n)$.

Fix $n$. $M_n$ will have infinitely many elements $(a_i)_{i \in \omega}$ satisfying a unary relation $S$. Each of these elements will be attached to, for each successor ordinal $\beta < \alpha$, a “box" $M_{i,\beta}$ which contains within it a copy of either $A_{\beta}$ or $E_{\beta}$; each of the boxes are disjoint. By this we mean that there are binary relations $T_\beta$ such that $T_\beta(a_i, x)$ holds for exactly those $x \in M_{i,\beta}$. $M_{i,\beta}$ will be a structure in the language of Lemma 3.2 and will be defined so that:

1. $M_{0,\beta} \cong A_{\beta}$ for all $\beta$.
2. $M_{i,\beta} \cong E_{\beta}$, $i \geq 1$, if there is $s$ such that $\Phi^{(\beta)}(n, s) \geq i$.
3. $M_{i,\beta} \cong A_{\beta}$, $i \geq 1$, otherwise.

Note that the condition in (2) is $\Sigma^0_\beta$ and so we can build such a structure $M_n$ computably.

Similarly, $N_n$ will have infinitely many elements $(b_i)_{i \in \omega}$, each of which is attached to, for each $\beta < \alpha$, a box $N_{i,\beta}$ which contains within it:

1. $N_{i,\beta} \cong E_{\beta}$ if there is $s$ such that $\Phi^{(\beta)}(n, s) > i$. 

(2) \( \mathcal{N}_{i, \beta} \cong \mathcal{A}_{\beta} \) otherwise.

Again, the condition in (1) is \( \Sigma^0_3 \) and so we can build such a structure \( \mathcal{N}_n \) computably.

**Claim 1.** Fix \( n \).

1. For each \( j < f(n) \), there is \( \beta < \alpha \) such that:
   - for \( \gamma < \beta \), \( \mathcal{M}_{j+1, \gamma} \cong \mathcal{N}_{j, \gamma} \cong \mathcal{A}_{\gamma} \),
   - for \( \gamma \geq \beta \), \( \mathcal{M}_{j+1, \gamma} \cong \mathcal{N}_{j, \gamma} \cong \mathcal{E}_{\gamma} \),
2. For each \( j \geq f(n) \) and \( \beta < \alpha \), \( \mathcal{M}_{j+1, \beta} \cong \mathcal{N}_{j, \beta} \cong \mathcal{M}_{0, \beta} \cong \mathcal{A}_{\beta} \).

**Proof.** For (1), it is clear from the definitions of \( \mathcal{M}_{j+1, \beta} \) and \( \mathcal{N}_{j, \beta} \) that for all \( \beta < \alpha \), \( \mathcal{M}_{j+1, \beta} \cong \mathcal{N}_{j, \beta} \). Since \( j < f(n) \), there is \( s \) such that \( \Phi^{(\alpha)}(n, s) = f(n) > j \). In particular, there must be some \( \beta < \alpha \) such that there is \( s \) with \( \Phi^{(\alpha)}(n, s) > j \). Let \( \beta \) be the least such ordinal. Then for all \( \gamma \geq \beta \), there is \( s \) such that \( \Phi^{(\alpha)}(n, s) > j \), and so \( \mathcal{M}_{j+1, \gamma} \cong \mathcal{N}_{j, \gamma} \cong \mathcal{E}_{\gamma} \). By choice of \( \beta \), for \( \gamma < \beta \), there is no \( s \) such that \( \Phi^{(\alpha)}(n, s) > j \), and so \( \mathcal{M}_{j+1, \gamma} \cong \mathcal{N}_{j, \gamma} \cong \mathcal{A}_{\gamma} \).

For (2), it is clear that \( \mathcal{M}_{j+1, \beta} \cong \mathcal{N}_{j, \beta} \) for each \( j \geq f(n) \) and each \( \beta < \alpha \), and it is also clear that \( \mathcal{M}_{0, \beta} \cong \mathcal{A}_{\beta} \) for each \( \beta < \alpha \). If \( j \geq f(n) \), then since \( \Phi \) is limitwise monotonic approximation to \( f \), \( \Phi^{(\alpha)}(n, s) \leq f(n) \leq j \) for all \( s \) and \( \beta \). Thus \( \mathcal{N}_{j, \beta} \cong \mathcal{A}_{\beta} \) for all \( \beta \).

**Claim 2.** \( \mathcal{M} \) and \( \mathcal{N} \) are isomorphic.

**Proof.** It suffices to show that for each \( n \), \( \mathcal{M}_n \) and \( \mathcal{N}_n \) are isomorphic. Fix \( n \). Using Claim 1, we see that the map

\[
\begin{align*}
a_0 &\mapsto b_{f(n)} \\
a_{i+1} &\mapsto b_i \\
a_i &\mapsto b_i
\end{align*}
\]

when \( 0 < i \leq f(n) \)
when \( i > f(n) \)

extends to an isomorphism between \( \mathcal{M}_n \) and \( \mathcal{N}_n \).

**Claim 3.** Any isomorphism between \( \mathcal{M} \) and \( \mathcal{N} \) can compute a function which dominates \( f \).

**Proof.** Let \( g \) be an isomorphism between \( \mathcal{M} \) and \( \mathcal{N} \). We will compute, using \( g \), a function \( \hat{g} \) which dominates \( f \). For each \( n \), define \( \hat{g}(n) \) as follows. Let \( (a_i)_{i \in \omega} \) and \( (b_i)_{i \in \omega} \) be the elements in the definition of \( \mathcal{M}_n \) and \( \mathcal{N}_n \). Then \( \hat{g}(n) \) is the number satisfying \( g(a_0) = b_{\hat{g}(n)} \).

To see that \( \hat{g}(n) \geq f(n) \), we use Claim 1. For each \( \beta < \alpha \), \( \mathcal{M}_{0, \beta} \cong \mathcal{A}_{\beta} \); but if \( j < f(n) \), there is \( \beta < \alpha \) such that \( \mathcal{N}_{j, \beta} \cong \mathcal{E}_{\beta} \). Thus no isomorphism can map \( a_0 \) to \( b_j \) for \( j < f(n) \), and so \( \hat{g}(n) \geq f(n) \).

**Claim 4.** Given a computable copy \( \mathcal{N} \) of \( \mathcal{N} \), \( f \oplus \Phi^{(\alpha)} \) can compute an isomorphism between \( \mathcal{N} \) and \( \mathcal{N} \).

It is more convenient for the proof to consider \( \mathcal{N} \) rather than \( \mathcal{M} \) in this claim, but as they are isomorphic it does not matter which we choose.
Proof. For each $n$, let $\tilde{N}_n$ be the structure with domain $R_n$ in $\tilde{N}$. It suffices to compute an isomorphism $g$ between $N_n$ and $\tilde{N}_n$ for each $n$. Inside of $\tilde{N}_n$, let $(c_i)_{i \in \omega}$ list the elements $x$ satisfying $S(x)$. For each $c_i$, let $\tilde{N}_{i,\beta}$ be the tree whose domain consists of the elements $y$ satisfying $T_\beta(c_i, y)$. To begin, we will define $g$ on $(b_i)_{i \in \omega} \subseteq N_n$. Compute $f(n)$. Using $0^{(\alpha)}$, look for $f(n)$ elements $c_i$ such that, for some $\beta < \alpha$, $\tilde{N}_{i,\beta} \equiv E_\beta$. This search is computable relative to $0^{(\alpha)}$ by Lemma 3.2 (2), and by Claim 1 we know that there are exactly $f(n)$ such elements and so the search will terminate after finding every such element. Rearranging $(c_i)_{i \in \omega}$, we may assume that these elements are $c_0, \ldots, c_{f(n)-1}$.

Now, for each $k < f(n)$, find the least $\beta_k$ such that $\tilde{N}_{k,\beta_k} \equiv E_{\beta_k}$, and the least $\gamma_k$ such that $\tilde{N}_{k,\gamma_k} \equiv E_{\gamma_k}$. Again, this is computable in $0^{(\alpha)}$ by Lemma 3.2 (2). Note that we must ask $0^{(\alpha)}$ to determine what $\beta_k$ and $\gamma_k$ are least. The sets $\{\beta_0, \ldots, \beta_{f(n)-1}\}$ and $\{\gamma_0, \ldots, \gamma_{f(n)-1}\}$ must be identical including multiplicity (but possibly in a different order) as $\tilde{N}_n$ and $N_n$ are isomorphic. So by rearranging $(c_i)_{i \in \omega}$ once again we may assume that $\beta_k = \gamma_k$ for each $k < f(n)$.

We have now rearranged the list $(c_i)_{i \in \omega}$ so that for each $i$ and $\beta < \alpha$, $\tilde{N}_{i,\beta} \equiv \tilde{N}_{i,\beta}$. Define $g$ so that $g(a_i) = c_i$. For each $i$ and $\beta < \alpha$, $\tilde{N}_{i,\beta} \equiv \tilde{N}_{i,\beta}$ are isomorphic to either $A_\beta$ or $E_\beta$, which are uniformly $0^{(\beta)}$-$\alpha$-categorical (Lemma 3.2 (3)), and we can compute using $0^{(\alpha)}$ which case we are in. So we can define $g$ on $\tilde{N}_{i,\beta}$ to be an isomorphism to $\tilde{N}_{i,\beta}$. Thus $g$ is an isomorphism from $N_n$ to $\tilde{N}_n$.

These claims complete the proof of the theorem.

Using this lemma, and taking the limitwise monotonic function to be the self-modulus of a c.e. set, it is not hard to prove our main theorem.

**Theorem 1.3.** Let $\alpha$ be a computable limit ordinal and $d$ a degree c.e. in and above $0^{(\alpha)}$. There is a computable structure with strong degree of categoricity $d$.

**Proof.** Fix $\alpha$ and let $D \in d$ be a set c.e. in and above $0^{(\alpha)}$. Since $D$ is c.e. in and above $0^{(\alpha)}$, it has a self-modulus $f$ that is limitwise monotonic relative to $0^{(\alpha)}$. Consider the structure $M$ constructed in Lemma 4.3 for this $f$. We will enrich this structure slightly to produce a new structure $S$. Let $S_\alpha$ be the computable structure with strong degree of categoricity $0^{(\alpha)}$ constructed in Theorem 3.1 of Csima, Franklin and Shore [?]. The new structure $S$ consists of $M$ and a disjoint copy of $S_\alpha$, and a new unary relation $R$ such that $R(x)$ holds exactly when $x$ belongs to the copy of $S_\alpha$. We claim that $S$ has strong degree of categoricity $d$.

First, suppose that $T$ is some other computable copy of $S$. We will show that there is a $d$-computable isomorphism between $S$ and $T$. Using the relation $R$, we may identify the component of $T$ isomorphic to $S_\alpha$. Since $S_\alpha$ has (strong) degree of categoricity $0^{(\alpha)} \leq d$, we can $d$-computably find an isomorphism between the copies of $S_\alpha$ in $S$ and $T$. We can also identify the component isomorphic to $M$ in each structure. By choice of $M$, any two such copies have an isomorphism between them computable in $f \oplus 0^{(\alpha)}$, and $D$ can compute this self-modulus $f$. Hence $d$ can computably produce such an isomorphism, since it can compute $f \oplus 0^{(\alpha)}$. Gluing these two isomorphisms together gives us the result.

Since $S_\alpha$ has strong degree of categoricity $0^{(\alpha)}$, there is a computable copy $\tilde{S}_\alpha$ of $S_\alpha$ such that every isomorphism between the two computes $0^{(\alpha)}$. Let $\tilde{S}$ be a computable copy of $S$ built in the following way. Rather than using the “standard” copy $S_\alpha$,
use the “hard” copy $\hat{\alpha}$ of $\alpha$. Additionally, rather than using $\mathcal{M}$, instead use $\mathcal{N}$ as built in Lemma 4.3. Any isomorphism between $\mathcal{S}_\alpha$ and $\hat{\alpha}$ computes $\mathbf{0}^{(\alpha)}$, and any isomorphism between $\mathcal{M}$ and $\mathcal{N}$ must compute a function that dominates $f$. Let $g$ be any isomorphism between $\mathcal{S}$ and $\hat{\alpha}$. Then by using $R$, we can restrict $g$ to an isomorphism between $\mathcal{S}_\alpha$ and $\hat{\alpha}$ and hence $g$ can compute $\mathbf{0}^{(\alpha)}$. Since $g$ can also be restricted to an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, it must compute a function dominating $f$. But $f$ is a modulus for $D$ computable in $\mathbf{0}^{(\alpha)}$, and hence $g$ must be able to compute $D$ since it can compute $\mathbf{0}^{(\alpha)}$ and a function dominating $f$. Hence $g$ can compute $d$. \hfill $\square$

We now turn to prime models, working above $\mathbf{0}^{(\omega)}$. Essentially, our work here is to check that in taking $\alpha = \omega$ in the previous theorem and lemma, the construction results in a prime model.

**Lemma 4.4.** Let $f: \omega \to \omega$ be limitwise monotonic relative to $\mathbf{0}^{(\omega)}$. There is a prime model with two computable copies $\mathcal{M}$ and $\mathcal{N}$ such that:

1. Every isomorphism between $\mathcal{M}$ and $\mathcal{N}$ computes a function which dominates $f$.
2. $f \oplus \mathbf{0}^{(\omega)}$ computes an isomorphism between any two computable copies of $\mathcal{M}$ and $\mathcal{N}$.

**Proof.** The construction is exactly the same as that of Lemma 4.3 with $\alpha = \omega$. We refer to the structures $\mathcal{A}_\beta$ and $\mathcal{E}_\beta$ of Lemma 3.2 as $\mathcal{A}_n$ and $\mathcal{E}_n$, $n < \omega$, but of course these are the same. It remains to argue, using the properties from Lemma 3.3 which hold only for the structures $\mathcal{A}_n$ and $\mathcal{E}_n$ with $n$ finite, that the resulting structure $\mathcal{N}$ is prime.

Recall that $\mathcal{N}$ is the disjoint union of structures $\mathcal{N}_n$, each of which satisfies the relation $R_n$. So it suffices to show that the structures $\mathcal{N}_n$ are prime. $\mathcal{N}_n$ was defined as follows: there were infinitely many elements $(b_i)_{i \in \omega}$ (satisfying the unary relation $S$), each of which is attached to (by binary relations $T_m$), for each $m < \omega$, a box $\mathcal{N}_{i,m}$ which contains within it:

1. $\mathcal{N}_{i,m} \simeq \mathcal{E}_m$ if there is $s$ such that $\Phi^{(m)}(n, s) > i$.
2. $\mathcal{N}_{i,m} \simeq \mathcal{A}_m$ otherwise.

By Claim 1 of Lemma 4.3, for each $i$, either $i < f(n)$ and there is some $m_i < \omega$ such that:

- for $\ell < m_i$, $\mathcal{N}_{i,\ell} \simeq \mathcal{A}_\ell$,
- for $\ell \geq m_i$, $\mathcal{N}_{i,\ell} \simeq \mathcal{E}_\ell$,

or $i \geq f(n)$ and for all $m < \omega$, $\mathcal{N}_{i,m} \simeq \mathcal{A}_m$. Note that the sequence $\{m_i\}_{i < f(n)}$ is non-decreasing.

By Lemma 3.3 (2), for $i < f(n)$, the automorphism orbit of $b_i$ is determined by the first-order formula with free variable $x$ which expresses that $S$ holds of $x$, that the structure with domain $T_m(x, \cdot)$ satisfies $\varphi_{m_\ell}$ (and so is isomorphic to $\mathcal{E}_{m_\ell}$), and that the structure with domain $T_{m_{\ell-1}}(x, \cdot)$ satisfies $\neg \varphi_{m_{\ell-1}}$ (and so is isomorphic to $\mathcal{A}_{m_{\ell-1}}$). For $i \geq f(n)$, the automorphism orbit of $b_i$ is determined by the first-order sentence with free variable $x$ which expresses that $S$ holds of $x$, and that the structure with domain $T_{m_f(n)-1}(x, \cdot)$ satisfies $\neg \varphi_{m_{f(n)-1}}$ (and so is isomorphic to $\mathcal{A}_{m_{f(n)-1}}$).
Fix a tuple \( \bar{c} \) from \( N_n \). We will give a first-order formula defining the orbit of \( \bar{c} \). We may assume that whenever \( \bar{c} \) contains an element of \( N_{i,m} \), \( \bar{c} \) contains \( b_i \) as well. We can break the tuple \( \bar{c} \) up into finitely many elements \( b_{i_1}, \ldots, b_{i_k} \) and finitely many tuples \( \bar{c}_{i,m} \) from \( N_{i,m} \). The orbit of \( \bar{c} \) is determined by the orbits of \( b_{i_1}, \ldots, b_{i_k} \) (each of which is determined by a first-order formula as described in the previous paragraph), the fact that \( T_m(b_i, y) \) holds for any \( y \in \bar{c}_{i,m} \), and the orbits of each of the tuples \( \bar{c}_{i,m} \) within \( N_{i,m} \). The latter orbits are first-order definable by Lemma 3.3 (3).

**Theorem 1.4.** Let \( d \) be a degree c.e. in and above \( 0^{(\omega)} \). There is a computable prime model \( \mathcal{A} \) with strong degree of categoricity \( d \).

**Proof.** The construction of such a model is similar to Theorem 1.3, except we replace \( M \) and \( N \) from Lemma 4.3 with those \( M \) and \( N \) from Lemma 4.4 (which are actually the same structures, if \( \alpha = \omega \)), and we also replace the “easy” and “hard” copies of \( S_\alpha \) with copies of the structure from Theorem 3.4 such that any isomorphism between them computes \( 0^{(\omega)} \). The same argument from Theorem 1.3 shows that this new structure has strong degree of categoricity \( d \). It remains to show that such models are prime; they are the disjoint union of prime structures, distinguishable by the relation \( R \), and hence must be prime themselves.

**References**


