Degrees of Categoricity Above Limit Ordinals

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Abstract

A computable structure $\mathcal{A}$ has degree of categoricity $\mathbf{d}$ if $\mathbf{d}$ is exactly the degree of difficulty of computing isomorphisms between isomorphic computable copies of $\mathcal{A}$. Fokina, Kalimullin, and Miller showed that every degree d.c.e. in and above $\mathbf{0}^{(n)}$, for any $n < \omega$, and also the degree $\mathbf{0}^{(\omega)}$, are degrees of categoricity. Later, Csima, Franklin, and Shore showed that every degree $\mathbf{0}^{(\alpha)}$ for any computable ordinal α, and every degree d.c.e. in and above $\mathbf{0}^{(\alpha)}$ for any successor ordinal α, is a degree of categoricity. We show that every degree c.e. in and above $\mathbf{0}^{(\alpha)}$, for α a limit ordinal, is a degree of categoricity. We also show that every degree c.e. in and above $\mathbf{0}^{(\omega)}$ is the degree of categoricity of a prime model, making progress towards a question of Bazhenov.

1 Introduction

It has long been known that there are two computable presentations of the vector space $\mathbb{Q}^{\mathbb{N}}$, one with a computable basis, and the other without a computable basis. Certainly there can be no computable isomorphism between these presentations, as the image of the computable basis from the first presentation would be a computable basis in the second presentation. Indeed the problem of building an isomorphism between two computable presentations of $\mathbb{Q}^{\mathbb{N}}$ amounts to computing bases for each of them, since then the bases can be matched up and effectively extended to an isomorphism. To give a basis of a computable presentation of $\mathbb{Q}^{\mathbb{N}}$ requires only the ability to answer single quantifier questions: Is the next potential basis member a linear combination of what we have already included? The Turing degree of the Halting set, denoted $\mathbf{0}'$, is smart enough to answer such questions. We say that $\mathbb{Q}^{\mathbb{N}}$ is $\mathbf{0}'$-computably categorical because we can compute, using $\mathbf{0}'$, an isomorphism between any two computable presentations. More generally:

Definition 1.1. Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathcal{A}$ is $\mathbf{d}$-computably categorical if, for every computable copy $\mathcal{B}$ of $\mathcal{A}$, there is a $\mathbf{d}$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

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For many structures $\mathcal{A}$, there is a least degree $\mathbf{d}$ such that $\mathcal{A}$ is $\mathbf{d}$-computable categorical. In the case of $\mathbb{Q}^{\mathbb{N}}$, there are actually two computable presentations of $\mathbb{Q}^{\mathbb{N}}$ such that any isomorphism between the two computes $\mathbf{0}'$. So $\mathbf{0}'$ is exactly the difficulty of computing isomorphisms between copies of $\mathbb{Q}^{\mathbb{N}}$. This natural idea was formalized by Fokina, Kalimullin, and Miller [**?**].

Definition 1.2 (Fokina, Kalimullin, and Miller [**?**])**.** A Turing degree $\mathbf{d}$ is said to be the degree of categoricity of a computable structure $\mathcal{A}$ if $\mathbf{d}$ is the least degree such that $\mathcal{A}$ is $\mathbf{d}$-computably categorical.

INSERT STUFF ABOUT MULTIPLE JUMPS AND HYP HEIERARCHY. Fokina, Kalimullin, and Miller showed that every degree that can be realized as a difference of computably enumerable (d.c.e.) sets in and above $\mathbf{0}^{(n)}$, for any $n < \omega$, and also the degree $\mathbf{0}^{(\omega)}$, are degrees of categoricity. Later, Csima, Franklin, and Shore [**?**] showed that every degree $\mathbf{0}^{(\alpha)}$ for any computable ordinal α, and every degree d.c.e. in and above $\mathbf{0}^{(\alpha)}$ for any successor ordinal α, is a degree of categoricity. Csima and Ng have announced a proof that every Δ^0_2 degree is a degree of categoricity.

It is often the case when trying to prove some property $P(\alpha)$ for ordinals α that things get tricky at limit ordinals. Roughly speaking, if α is a successor ordinal and we know something must happen before α, we can safely say it has happened by $\alpha - 1$. For α a limit ordinal, in such a situation there is no canonical choice of earlier ordinal to look at. This is why the methods of [**?**] did not work above limit ordinals.

Our main result in this paper is:

Theorem 1.3. *Let α be a computable limit ordinal and $\mathbf{d}$ a degree c.e. in and above $\mathbf{0}^{(\alpha)}$. There is a computable structure with strong degree of categoricity $\mathbf{d}$.*

This fills in a gap that was missing from [**?**] above limit ordinals, making further progress towards Question 5.1 of that paper.

We have not yet explained what a *strong* degree of categoricity is. For a long time, all of the known examples had the following property: If $\mathcal{A}$ had degree of categoricity $\mathbf{d}$, then there is a copy $\mathcal{B}$ of $\mathcal{A}$ such that every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ computes $\mathbf{d}$. Thus, we can witness with just two computable copies the fact that $\mathbf{d}$ is the *least* degree such that $\mathcal{A}$ is $\mathbf{d}$-computably categorical. In this case, we say that $\mathcal{A}$ has strong degree of categoricity $\mathbf{d}$. Recently, Bazhenov, Kalimullin, and Yamaleev [**?**] have shown that there is a c.e. degree $\mathbf{d}$ and a structure $\mathcal{A}$ with degree of categoricity $\mathbf{d}$, but $\mathbf{d}$ is not a strong degree of categoricity for $\mathcal{A}$. Csima and Stephenson [**?**] have shown that there is a structure of finite computable dimension that has a degree of categoricity but no strong degree of categoricity.

INSERT STUFF ABOUT PRIME MODELS AND WHY THEY ARE AWESOME TO CONSIDER

Our second result gives progress towards a question of Bazhenov and Marchuk.

Question (Bazhenov and Marchuk [**?**])**.** What can be the degrees of categoricity of computable prime models?

A computable prime model—in fact, as Bazhenov shows, a computable homogeneous model—is always $\mathbf{0}^{(\omega+1)}$-categorical, as we can ask $\mathbf{0}^{(\omega+1)}$ if two tuples satisfy the same type. Bazhenov constructs a computable homogeneous model with degree of categoricty $\mathbf{0}^{(\omega+1)}$. The complexity here is in the structure itself, rather than in the theory. To build a prime model with degree of categoricity $\mathbf{0}^{(\omega+1)}$, the complexity must be in the theory: If $\mathcal{A}$ is a computable prime model of a theory T, then $\mathcal{A}$ is $T' \oplus \mathbf{0}^{(\omega)}$-categorical as T' can decide whether a formula is complete, and $\mathbf{0}^{(\omega)}$ can decide whether a formula holds of a tuple in $\mathcal{A}$. We build a computable prime model with degree of categoricity $\mathbf{0}^{(\omega+1)}$ (or any other degree c.e. in and above $\mathbf{0}^{(\omega)}$).

Theorem 1.4. *Let* $\mathbf{d}$ *be a degree c.e. in and above* $\mathbf{0}^{(\omega)}$. *There is a computable prime model* $\mathcal{A}$ *with strong degree of categoricity* $\mathbf{d}$.

2 Categoricity Relative to Decidable Models

As a warm-up to illustrate the methods used to prove these two theorems, we give a simple proof of a result of Goncharov [?] that for every c.e. degree $\mathbf{d}$, there is a decidable prime model with degree of categoricity $\mathbf{d}$ with respect to decidable copies. Recall that a structure is said to be decidable if its full elementary diagram is computable. In [?], Goncharov made the following definitions:

Definition 2.1. Let $\mathbf{d}$ be a Turing degree and $\mathcal{A}$ a decidable structure. Then $\mathcal{A}$ is $\mathbf{d}$-*categorical with respect to decidable copies* if for every decidable copy $\mathcal{B}$ of $\mathcal{A}$, $\mathbf{d}$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.2. Let $\mathbf{d}$ be a Turing degree and $\mathcal{A}$ a decidable structure. Then $\mathbf{d}$ is the *degree of categoricity of* $\mathcal{A}$ *with respect to decidable copies* if:

- $\mathcal{A}$ is $\mathbf{d}$-categorical with respect to decidable copies, and
- whenever $\mathcal{A}$ is $\mathbf{c}$-categorical with respect to decidable copies, $\mathbf{c} \geq \mathbf{d}$.

It is not hard to see that between any two decidable copies of a prime model, there is a $\mathbf{0}'$-computable isomorphism. Goncharov showed that any c.e. degree can be the degree of categoricity with respect to decidable copies of a prime model. We give a different proof, which we think is simpler, and which demonstrates some of the techniques that we will use later.

Theorem 2.3 (Goncharov [?, Theorem 3])**.** *Let* $\mathbf{d}$ *be a c.e. degree. Then there is a decidable prime model* $\mathcal{M}$ *which has strong degree of categoricity* $\mathbf{d}$ *with respect to decidable models.*

Proof. Let $D \in \mathbf{d}$ be a c.e. set. We will construct the structures $\mathcal{M}$ and $\mathcal{N}$. They are the disjoint union of infinitely many structures $\mathcal{M}_n$ and $\mathcal{N}_n$, with $\mathcal{M}_n$ and $\mathcal{N}_n$ picked out by unary relations R_n. The nth sort will code whether $n \in D$. Fix n. $\mathcal{M}_n$ will have infinitely many elements $(a_i)_{i\in\omega}$. There will be infinitely many unary relations $(U_\ell)_{\ell\in\omega}$ defined on $\mathcal{M}_n$ so that:

$$a_0 \in U_s \Longleftrightarrow n \in D_{\text{at } s}$$

where $n \in D_{\text{at } s}$ means that n enters D at exactly stage s, and

$$a_i \notin U_s \text{ for } i > 0 \text{ and all } s.$$

Similarly, $\mathcal{N}_n$ will have infinitely many elements $(b_i)_{i\in\omega}$ with the unary relations defined so that:

$$b_i \in U_s \Longleftrightarrow i = s \text{ and } n \in D_{\text{at } s}.$$

It is easy to see that we can build computable copies of $\mathcal{M}$ and $\mathcal{N}$. These copies are in fact decidable.

Claim 1. *$\mathcal{M}$ and $\mathcal{N}$ are decidable.*

Proof. Given a formula $\varphi(x_1, \ldots, x_n)$ with k quantifiers and $a_{i_1}, \ldots, a_{i_n} \in \mathcal{M}$, it is not hard to see that $\mathcal{M} \models \varphi(a_{i_1}, \ldots, a_{i_n})$ if and only if the finite substructure $\mathcal{M}'$ of $\mathcal{M}$ whose domain consists of $a_1, \ldots, a_{k+n+1}$ and $a_{i_1}, \ldots, a_{i_n}$ also has $\mathcal{M}' \models \varphi(a_{i_1}, \ldots, a_{i_n})$. Thus $\mathcal{M}$ is decidable.

For $\mathcal{N}$, suppose we have a formula $\varphi(x_1, \ldots, x_n)$ with at most k quantifiers and which uses only some subset of the relations $U_0, \ldots, U_k$. Let $b_{i_1}, \ldots, b_{i_n}$ be elements of $\mathcal{N}$. Then $\mathcal{N} \models \varphi(b_{i_1}, \ldots, b_{i_n})$ if and only if the finite substructure $\mathcal{N}'$ of $\mathcal{M}$ whose domain consists of $b_1, \ldots, b_{k+n+1}$ and $b_{i_1}, \ldots, b_{i_n}$ also has $\mathcal{N}' \models \varphi(b_{i_1}, \ldots, b_{i_n})$. Thus $\mathcal{N}$ is decidable. □

Claim 2. *$\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

Proof. It suffices to show that for each n, $\mathcal{M}_n$ and $\mathcal{N}_n$ are isomorphic. If $n \notin D$, then $a_i \mapsto b_i$ induces an isomorphism between $\mathcal{M}_n$ and $\mathcal{N}_n$. If $n \in D_{\text{at } s}$, then the map

$$\begin{aligned} a_0 &\mapsto b_s \\ a_{i+1} &\mapsto b_i && \text{when } i < s \\ a_i &\mapsto b_i && \text{when } i > s \end{aligned}$$

is an isomorphism between $\mathcal{M}_n$ and $\mathcal{N}_n$. □

Claim 3. *$\mathcal{M}$ and $\mathcal{N}$ are prime.*

Proof. It suffices to show that each $\mathcal{M}_n$ and $\mathcal{N}_n$ are prime, since these structures are determined inside $\mathcal{M}$ and $\mathcal{N}$ uniquely by the relation R_n. It is not hard to see that $\mathcal{M}_n$ and $\mathcal{N}_n$ are models of an $\aleph_0$-categorical theory, and hence are prime. □

Claim 4. *Any isomorphism between $\mathcal{M}$ and $\mathcal{N}$ can compute D.*

Proof. Let g be an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. For each n, let $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ be the elements in the definition of $\mathcal{M}_n$ and $\mathcal{N}_n$. Let s be such that $g(a_0) = b_s$. Then $n \in D$ if and only if $n \in D_s$. □

Claim 5. *Given a computable copy $\widetilde{\mathcal{M}}$ of $\mathcal{M}$, D can compute an isomorphism between $\mathcal{M}$ and $\widetilde{\mathcal{M}}$.*

Proof. For each n, let $\widetilde{\mathcal{M}}_n$ be the structure with domain R_n in $\widetilde{\mathcal{M}}$. It suffices to compute an isomorphism g between $\mathcal{M}_n$ and $\widetilde{\mathcal{M}}_n$ for each n. Let $(c_i)_{i\in\omega}$ be the elements of $\widetilde{\mathcal{M}}_n$. If $n \notin D$, no relation U_j holds of any of the the elements $(a_i)_{i\in\omega}$ or $(c_i)_{i\in\omega}$. So $a_i \mapsto c_i$ is an isomorphism. On the other hand, if $n \in D$, then for some unique s, $a_0 \in U_s$. We can look for c_k such that $c_k \in U_s$. Map a_0 to c_k; map each other a_i to some other c_i. □

These claims complete the proof of the theorem. □

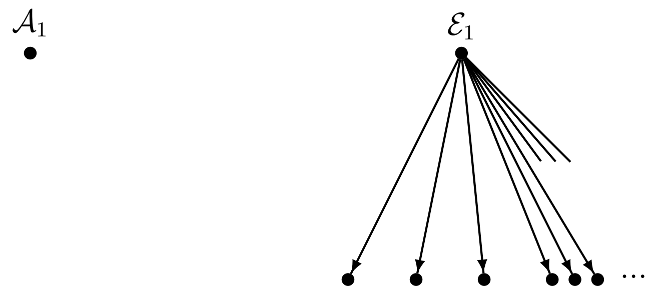

Figure 1: $\mathcal{A}_1$ and $\mathcal{E}_1$

3 Back-and-forth Trees

Fix a path through O. We will identify computable ordinals with their notation on this path. We will always first fix a limit ordinal α and work below it. Recall that one can decide effectively whether a given computable ordinal is a limit ordinal or a successor ordinal. For each limit ordinal $\beta < \alpha$, fix a fundamental sequence for β, that is, an increasing sequence of successor ordinals whose limit is β.

Hirschfeldt and White defined, for each successor ordinal β, a pair of trees $\mathcal{A}_\beta$ and $\mathcal{E}_\beta$ which can be differentiated exactly by β jumps. These trees are called *back-and-forth trees*.

Definition 3.1 ([**?**, Definition 3.1])**.** Back-and-forth trees are defined recursively in β. We view these as structures in the language of graphs with the root node distinguished.

We take $\mathcal{A}_1$ to be the tree with just a root node and no children, and we take $\mathcal{E}_1$ to be the tree where the root node has infinitely many children, none of which have children. See Fig. 1. We say that these trees have *back-and-forth rank* 1.

Suppose β is a successor ordinal. Define $\mathcal{A}_{\beta+1}$ as a root node with infinitely many children, each the root of a copy of $\mathcal{E}_\beta$, and define $\mathcal{E}_{\beta+1}$ as a root node with infinitely many children, each the root of a copy of $\mathcal{A}_\beta$, and also infinitely many other children, each the root of a copy of $\mathcal{E}_\beta$. See Fig. 2. These trees have back-and-forth rank $\beta + 1$.

Now suppose β is a non-zero limit ordinal, and let $\beta_0, \beta_1, \dots$ be a fundamental sequence of successor ordinals for β, that is, a sequence of successor ordinals below β with limit β. We first define a family of helper trees $\mathcal{L}_{\beta,k}$ where $k \in \omega \cup \{\infty\}$. Define $\mathcal{L}_{\beta,\infty}$ to consist of a root node whose children are root nodes of copies of $\mathcal{A}_{\beta_i}$, and such that each copy appears exactly once as a child. For $k \in \omega$, $\mathcal{L}_{\beta,k}$ has a root node whose children are root nodes of copies of $\mathcal{A}_{\beta_0}, \dots, \mathcal{A}_{\beta_k}, \mathcal{E}_{\beta_{k+1}}, \mathcal{E}_{\beta_{k+2}}, \dots$ where again each copy appears exactly once as a child. Such trees are shown in Fig. 3. We say these trees have back-and-forth rank β.

We can now define $\mathcal{A}_{\beta+1}$ and $\mathcal{E}_{\beta+1}$ for the non-zero limit ordinal β. For $\mathcal{A}_{\beta+1}$, we have a root node with infinitely many children, each the root node of a copy of $\mathcal{L}_{\beta,k}$ such that for each $k \in \omega$, $\mathcal{L}_{\beta,k}$ appears infinitely many times. The definition of $\mathcal{E}_{\beta+1}$ is similar, except k is drawn from $\omega \cup \{\infty\}$. See Fig. 4. These trees have back-and-forth rank $\beta + 1$.

The next two lemmas piece together the facts that we will need about the back-and-forth trees, first for arbitrary β, and second some additional properties for finite β in particular. These facts come from [**?**] and [**?**].

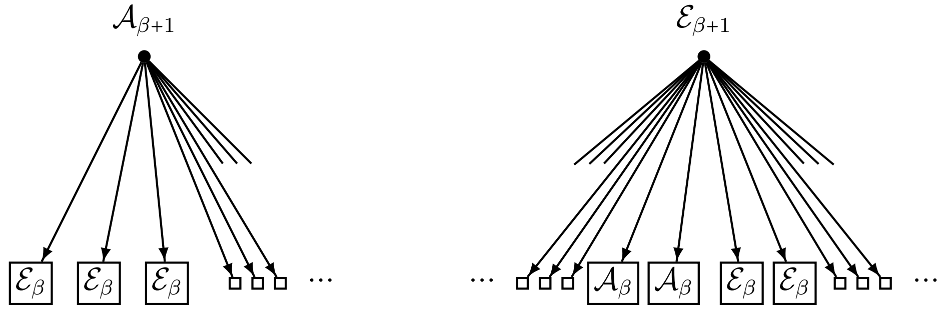

Figure 2: $\mathcal{A}_{\beta+1}$ and $\mathcal{E}_{\beta+1}$ when β is a successor ordinal.

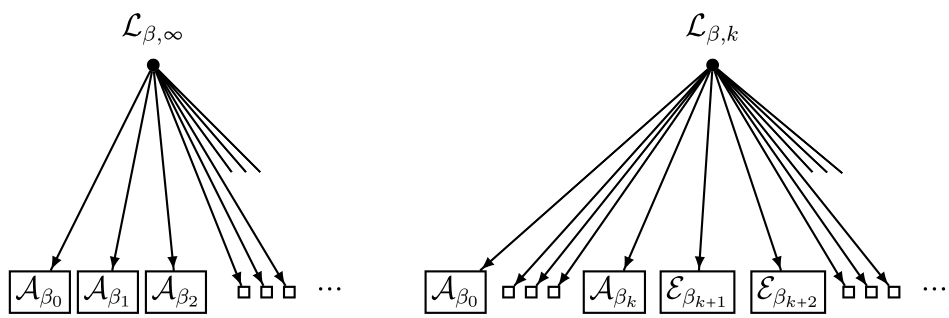

Figure 3: Helper trees $\mathcal{L}_{\beta,\infty}$ and $\mathcal{L}_{\beta,k}$ for $k \in \omega$ for the non-zero limit ordinal β.

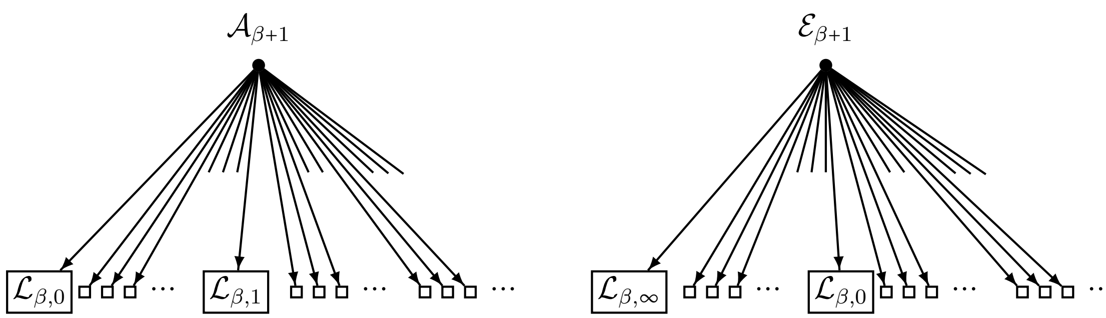

Figure 4: $\mathcal{A}_{\beta+1}$ and $\mathcal{E}_{\beta+1}$ for the non-zero limit ordinal β.

Lemma 3.2. *Let α be a computable ordinal. For a successor ordinal $\beta < \alpha$, the structures $\mathcal{A}_\beta$ and $\mathcal{E}_\beta$ satisfy the following properties:*

(1) Uniformly in β and an index for a Σ^0_β set S, there is a computable sequence of structures $\mathcal{C}_x$ such that

$$x \in S \Longleftrightarrow \mathcal{C}_x \cong \mathcal{E}_\beta \quad \text{and} \quad x \notin S \Longleftrightarrow \mathcal{C}_x \cong \mathcal{A}_\beta.$$

(2) Uniformly in β, there is a Σ^0_β sentence φ such that $\mathcal{E}_\beta \vDash \varphi$ and $\mathcal{A}_\beta \nvDash \varphi$.

(3) $\mathcal{A}_\beta$ and $\mathcal{E}_\beta$ are uniformly $\mathbf{0}^{(\beta)}$-categorical.

Proof. For (1), take $(\mathcal{C}_x)_x$ to be the computable sequence of trees given by Proposition 3.2 in [**?**].

For (2), take φ to be the sentence given by evaluating the formula guaranteed by Lemma 3.5 in [**?**] for $\mathcal{B} = \mathcal{E}_\beta$ at its own root node. The complexity of φ is the natural complexity of $\mathcal{E}_\beta$, which is Σ_β. This lemma says that for any tree $\mathcal{T}$, $\mathcal{T} \vDash \varphi$ if and only if $\mathcal{T} \cong \mathcal{E}_\beta$.

Finally, for (3), we use a result from Csima, Franklin, and Shore [**?**] about back-and-forth trees. We will consider $\mathcal{A}_\beta$; the case for $\mathcal{E}_\beta$ is identical. We have that $\mathcal{A}_\beta$ is a back-and-forth tree, and hence if $\mathcal{C}$ is a computable structure isomorphic to $\mathcal{A}_\beta$, then it is also a computable back-and-forth tree. Corollary 2.6 in [**?**] allows $\varnothing^{(\gamma)}$ to uniformly compute an isomorphism between these two trees when the back-and-forth rank of the trees is at most γ. Since the rank of $\mathcal{A}_\beta$ is β by construction, the isomorphism is uniformly computable in $\mathbf{0}^{(\beta)}$. □

Now for finite ordinals β (and writing n for β), we have some additional properties. We will state the lemma in full, including properties that were covered by the previous lemma. We think that these facts are well-known, but we do not know of a reference in print.

Lemma 3.3. *For $0 < n < \omega$, the structures $\mathcal{A}_n$ and $\mathcal{E}_n$ satisfy the properties:*

(1) Uniformly in n and an index for a Σ^0_n set S, there is a computable sequence of structures $\mathcal{C}_x$ such that

$$x \in S \Longleftrightarrow \mathcal{C}_x \cong \mathcal{E}_n \quad \text{and} \quad x \notin S \Longleftrightarrow \mathcal{C}_x \cong \mathcal{A}_n.$$

(2) For each n, there is an elementary first-order $\exists_n$ sentence φ_n, computable uniformly in n, such that $\mathcal{E}_n \vDash \varphi$ and $\mathcal{A}_n \nvDash \varphi$.

(3) $\mathcal{A}_n$ and $\mathcal{E}_n$ are prime.

(4) $\mathcal{A}_n$ and $\mathcal{E}_n$ are $\mathbf{0}^{(n)}$-categorical uniformly in n.

Proof. (1) and (4) are the same as in the previous lemma. We show using induction on n that these sequences satisfy (2) and (3) as well. It is easy to see that $\mathcal{A}_1$ and $\mathcal{E}_1$ are prime models of their theories and that they are distinguishable (in the sense of (2) in the statement of the lemma) by the existential sentence $\varphi_1 := \exists x \exists y (x \neq y)$. Assume now that $\mathcal{A}_n$ and $\mathcal{E}_n$ are prime and distinguishable by a first-order $\exists_n$ sentence φ_n (in the sense that $\mathcal{E}_n \vDash \varphi_n$ but $\mathcal{A}_n \nvDash \varphi_n$). We show that $\mathcal{A}_{n+1}$ and $\mathcal{E}_{n+1}$ are prime and distinguishable by a first-order $\exists_{n+1}$ sentence φ_{n+1}.

It is not hard to see that we can take φ_{n+1} to be the sentence $\exists x \neg\varphi_n[\leq x] \wedge$ (x is a child of the root node) where x is a new variable not appearing in φ_n and $\varphi_n[\leq x]$ is the formula obtained from φ_n by bounding every quantifier to the subtree below x. (Note that in a tree of rank n, if z is a descendant of x, i.e. there is a path from x to z, the length of the path is at most n, and so this is first-order definable and does not change the quantifier rank.) $\mathcal{E}_{n+1} \vDash \varphi_{n+1}$ but $\mathcal{A}_{n+1} \nvDash \varphi_{n+1}$.

It remains to show that $\mathcal{A}_{n+1}$ and $\mathcal{E}_{n+1}$ are prime. The same method will work for both structures. Let $\bar{a}$ be an arbitrary tuple in $\mathcal{E}_{n+1}$. We describe a formula that isolates the type of the tuple $\bar{a}$. Let $r_1, \ldots, r_k$ be the children of the root which are the roots of subtrees containing elements of $\bar{a}$; say that $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_k)$ where $\bar{a}_i$ is in the subtree below r_i. (Note that we can re-order the tuples as we like, as if the type of some permutation of $\bar{a}$ is isolated, so is $\bar{a}$.) By the induction hypothesis, we know that the subtree with root r_i is prime for every i. Hence for each $i = 1, \ldots, k$ there is a formula which isolates the type of the tuple $\bar{a}_i$ in the subtree with root r_i. There is also, for each r_i, a formula (either φ_n or $\neg\varphi_n$) which distinguishes between whether the subtree below r_i is isomorphic to $\mathcal{A}_n$ or $\mathcal{E}_n$. So we can isolate the type of $\bar{a}$ by saying that there are children $r_1, \ldots, r_k$ of the root such that $\bar{a}_i$ satisfies the formula, in the subtree below r_i, which isolates it, and by saying whether the subtree below each r_i is isomorphic to $\mathcal{A}_n$ or $\mathcal{E}_n$. □

Fokina, Kalimullin, and Miller [?] showed that there is a structure $\mathcal{A}$ with strong degree of categoricity $\mathbf{0}^{(\omega)}$. We note the well-known fact that one can also have $\mathcal{A}$ be a prime model. Our proof follows that of [?].

Theorem 3.4. *There is a computable structure $\mathcal{A}$ with strong degree of categoricity $\mathbf{0}^{(\omega)}$ such that $\mathcal{A}$ is a prime model of its theory.*

Proof sketch. The structure is just the disjoint union of infinitely many copies of each $\mathcal{E}_n$ for $n < \omega$. Theorem 3.1 of [?] shows that this has strong degree of categoricity $\mathbf{0}^{(\omega)}$, and it is not hard to see using Lemma 3.3 that this structure is prime. □

4 C.E. In And Above a Limit Ordinal

We begin this section by a short discussion of how we code a c.e. set into a structure. Consider a c.e. set C. If one knows, for each n, at what point the approximation to $C(n)$ has settled, then one can compute C. Moreover, one does not need to know exactly when C settles, but just a point after which $C(n)$ has settled. In particular, any sufficiently large function can compute C. Moreover, C itself can compute such a function. Following the terminology of Groszek and Slaman [?], we say that C has a self-modulus.

Definition 4.1 (Groszek and Slaman [?])**.** Let $F{:}\omega \to \omega$ and $X \subseteq \omega$. Then:

- F is a modulus (of computation) for X if every $G{:}\omega \to \omega$ that dominates F pointwise computes X.
- X has a self-modulus if X computes a modulus for itself.

The self-modulus of a c.e. set C is the function $f(n) = \mu s(C_s(n) = C(n))$. Groszek and Slaman showed that every Δ^0_2 or α-CEA set has a self-modulus. In fact, the self-modulus of a c.e. set has a nice form; it has a non-decreasing computable approximation.

Definition 4.2. A function $F:\omega \to \omega$ is limitwise monotonic if there is a computable approximation function $f:\omega \times \omega \to \omega$ such that, for all n,

- $F(n) = \lim_{s\to\infty} f(n,s)$.
- For all s, $f(n,s) \leq f(n,s+1)$.

In fact, it is well-known and an easy exercise to show that the sets of c.e. degree are exactly those with limitwise monotonic self-moduli. These remarks also relativize.

The next lemma encodes a limitwise monotonic function into the isomorphisms of copies of a computable structure. Any isomorphism dominates the limitwise monotonic function; but it does not seem to be the case that dominating the limitwise monotonic function is sufficient to compute isomorphisms.

Lemma 4.3. *Let α be a computable limit ordinal. Let $f:\omega \to \omega$ be limitwise monotonic relative to $\mathbf{0}^{(\alpha)}$. There is a structure with computable copies $\mathcal{M}$ and $\mathcal{N}$ such that:*

(1) Every isomorphism between $\mathcal{M}$ and $\mathcal{N}$ computes a function which dominates f.

(2) $f \oplus \mathbf{0}^{(\alpha)}$ computes an isomorphism between any two computable copies of $\mathcal{M}$ and $\mathcal{N}$.

Proof. Let Φ be a computable operator such that $f(n) = \lim_{s\to\infty} \Phi^{\varnothing^{(\alpha)}}(n,s)$ and this is monotonic in s. Write $\varnothing^{(\alpha)} = \bigoplus_{\gamma<\alpha} \varnothing^{(\gamma)}$ for successor ordinals $\gamma < \alpha$. By convention, for $\beta < \alpha$, we say that $\Phi^{\varnothing^{(\beta)}}(n,s)$ converges if the computation $\Phi^{\varnothing^{(\alpha)}}(n,s)$ halts, but the only part of the oracle $\varnothing^{(\alpha)} = \bigoplus_{\gamma<\alpha} \varnothing^{(\gamma)}$ that is read during the computation is that part with $\gamma \leq \beta$. So if $\Phi^{\varnothing^{(\beta)}}(n,s) = m$ then $\Phi^{\varnothing^{(\alpha)}}(n,s) = m$, and because α is a limit ordinal, if $\Phi^{\varnothing^{(\alpha)}}(n,s) = m$ then $\Phi^{\varnothing^{(\beta)}}(n,s) = m$ for some successor ordinal $\beta < \alpha$.

Let $(\mathcal{A}_\beta)_{\beta<\alpha}$ and $(\mathcal{E}_\beta)_{\beta<\alpha}$ be as in Lemma 3.2. We will construct the structures $\mathcal{M}$ and $\mathcal{N}$. They are the disjoint union of infinitely many structures $\mathcal{M}_n$ and $\mathcal{N}_n$, with $\mathcal{M}_n$ and $\mathcal{N}_n$ picked out by unary relations R_n. The nth sort will code the value of $f(n)$.

Fix n. $\mathcal{M}_n$ will have infinitely many elements $(a_i)_{i\in\omega}$ satisfying a unary relation S. Each of these elements will be attached to, for each successor ordinal $\beta < \alpha$, a "box" $\mathcal{M}_{i,\beta}$ which contains within it a copy of either $\mathcal{A}_\beta$ or $\mathcal{E}_\beta$; each of the boxes are disjoint. By this we mean that there are binary relations T_β such that $T_\beta(a_i, x)$ holds for exactly those $x \in \mathcal{M}_{i,\beta}$. $\mathcal{M}_{i,\beta}$ will be a structure in the language of Lemma 3.2 and will be defined so that:

(1) $\mathcal{M}_{0,\beta} \cong \mathcal{A}_\beta$ for all β.

(2) $\mathcal{M}_{i,\beta} \cong \mathcal{E}_\beta$, $i \geq 1$, if there is s such that $\Phi^{\varnothing^{(\beta)}}(n,s) \geq i$.

(3) $\mathcal{M}_{i,\beta} \cong \mathcal{A}_\beta$, $i \geq 1$, otherwise.

Note that the condition in (2) is Σ^0_β and so we can build such a structure $\mathcal{M}_n$ computably.

Similarly, $\mathcal{N}_n$ will have infinitely many elements $(b_i)_{i\in\omega}$, each of which is attached to, for each $\beta < \alpha$, a box $\mathcal{N}_{i,\beta}$ which contains within it:

(1) $\mathcal{N}_{i,\beta} \cong \mathcal{E}_\beta$ if there is s such that $\Phi^{\varnothing^{(\beta)}}(n,s) > i$.

(2) $\mathcal{N}_{i,\beta} \cong \mathcal{A}_\beta$ otherwise.

Again, the condition in (1) is Σ^0_β and so we can build such a structure $\mathcal{N}_n$ computably.

Claim 1. *Fix n.*

(1) For each $j < f(n)$, there is $\beta < \alpha$ such that:

- *for $\gamma < \beta$, $\mathcal{M}_{j+1,\gamma} \cong \mathcal{N}_{j,\gamma} \cong \mathcal{A}_\gamma$,*
- *for $\gamma \geq \beta$, $\mathcal{M}_{j+1,\gamma} \cong \mathcal{N}_{j,\gamma} \cong \mathcal{E}_\gamma$,*

(2) For each $j \geq f(n)$ and $\beta < \alpha$, $\mathcal{M}_{j+1,\beta} \cong \mathcal{N}_{j,\beta} \cong \mathcal{M}_{0,\beta} \cong \mathcal{A}_\beta$.

Proof. For (1), it is clear from the definitions of $\mathcal{M}_{j+1,\beta}$ and $\mathcal{N}_{j,\beta}$ that for all $\beta < \alpha$, $\mathcal{M}_{j+1,\beta} \cong \mathcal{N}_{j,\beta}$. Since $j < f(n)$, there is s such that $\Phi^{\varnothing^{(\alpha)}}(n,s) = f(n) > j$. In particular, there must be some $\beta < \alpha$ such that there is s with $\Phi^{\varnothing^{(\beta)}}(n,s) > j$. Let β be the least such ordinal. Then for all $\gamma \geq \beta$, there is s such that $\Phi^{\varnothing^{(\beta)}}(n,s) > j$, and so $\mathcal{M}_{j+1,\gamma} \cong \mathcal{N}_{j,\gamma} \cong \mathcal{E}_\gamma$. By choice of β, for $\gamma < \beta$, there is no s such that $\Phi^{\varnothing^{(\beta)}}(n,s) > j$, and so $\mathcal{M}_{j+1,\gamma} \cong \mathcal{N}_{j,\gamma} \cong \mathcal{A}_\gamma$.

For (2), it is clear that $\mathcal{M}_{j+1,\beta} \cong \mathcal{N}_{j,\beta}$ for each $j \geq f(n)$ and each $\beta < \alpha$, and it is also clear that $\mathcal{M}_{0,\beta} \cong \mathcal{A}_\beta$ for each $\beta < \alpha$. If $j \geq f(n)$, then since Φ is limitwise monotonic approximation to f, $\Phi^{\varnothing^{(\beta)}}(n,s) \leq f(n) \leq j$ for all s and β. Thus $\mathcal{N}_{j,\beta} \cong \mathcal{A}_\beta$ for all β. □

Claim 2. *$\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

Proof. It suffices to show that for each n, $\mathcal{M}_n$ and $\mathcal{N}_n$ are isomorphic. Fix n. Using Claim 1, we see that the map

$$\begin{aligned} a_0 &\mapsto b_{f(n)} \\ a_{i+1} &\mapsto b_i && \text{when } 0 < i \leq f(n) \\ a_i &\mapsto b_i && \text{when } i > f(n) \end{aligned}$$

extends to an isomorphism between $\mathcal{M}_n$ and $\mathcal{N}_n$. □

Claim 3. *Any isomorphism between $\mathcal{M}$ and $\mathcal{N}$ can compute a function which dominates f.*

Proof. Let g be an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. We will compute, using g, a function $\hat{g}$ which dominates f. For each n, define $\hat{g}(n)$ as follows. Let $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ be the elements in the definition of $\mathcal{M}_n$ and $\mathcal{N}_n$. Then $\hat{g}(n)$ is the number satisfying $g(a_0) = b_{\hat{g}(n)}$.

To see that $\hat{g}(n) \geq f(n)$, we use Claim 1. For each $\beta < \alpha$, $\mathcal{M}_{0,\beta} \cong \mathcal{A}_\beta$, but if $j < f(n)$, there is $\beta < \alpha$ such that $\mathcal{N}_{j,\beta} \cong \mathcal{E}_\beta$. Thus no isomorphism can map a_0 to b_j for $j < f(n)$, and so $\hat{g}(n) \geq f(n)$. □

Claim 4. *Given a computable copy $\widetilde{\mathcal{N}}$ of $\mathcal{N}$, $f \oplus \mathbf{0}^{(\alpha)}$ can compute an isomorphism between $\mathcal{N}$ and $\widetilde{\mathcal{N}}$.*

It is more convenient for the proof to consider $\mathcal{N}$ rather than $\mathcal{M}$ in this claim, but as they are isomorphic it does not matter which we choose.

Proof. For each n, let $\widetilde{\mathcal{N}}_n$ be the structure with domain R_n in $\widetilde{\mathcal{N}}$. It suffices to compute an isomorphism g between $\mathcal{N}_n$ and $\widetilde{\mathcal{N}}_n$ for each n. Inside of $\widetilde{\mathcal{N}}_n$, let $(c_i)_{i\in\omega}$ list the elements x satisfying $S(x)$. For each c_i, let $\widetilde{\mathcal{N}}_{i,\beta}$ be the tree whose domain consists of the elements y satisfying $T_\beta(c_i, y)$. To begin, we will define g on $(b_i)_{i\in\omega} \subseteq \mathcal{N}_n$. Compute $f(n)$. Using $\mathbf{0}^{(\alpha)}$, look for $f(n)$ elements c_i such that, for some $\beta < \alpha$, $\widetilde{\mathcal{N}}_{i,\beta} \cong \mathcal{E}_\beta$. This search is computable relative to $\mathbf{0}^{(\alpha)}$ by Lemma 3.2 (2), and by Claim 1 we know that there are exactly $f(n)$ such elements and so the search will terminate after finding every such element. Rearranging $(c_i)_{i\in\omega}$, we may assume that these elements are $c_0, \ldots, c_{f(n)-1}$.

Now, for each $k < f(n)$, find the least β_k such that $\mathcal{N}_{k,\beta_k} \cong \mathcal{E}_{\beta_k}$, and the least γ_k such that $\widetilde{\mathcal{N}}_{k,\gamma_k} \cong \mathcal{E}_{\gamma_k}$. Again, this is computable in $\mathbf{0}^{(\alpha)}$ by Lemma 3.2 (2). Note that we must ask $\mathbf{0}^{(\alpha)}$ to determine what β_k and γ_k are least. The sets $\{\beta_0, \ldots, \beta_{f(n)-1}\}$ and $\{\gamma_0, \ldots, \gamma_{f(n)-1}\}$ must be identical including multiplicity (but possibly in a different order) as $\widetilde{\mathcal{N}}_n$ and $\mathcal{N}_n$ are isomorphic. So by rearranging $(c_i)_{i\in\omega}$ once again we may assume that $\beta_k = \gamma_k$ for each $k < f(n)$.

We have now rearranged the list $(c_i)_{i\in\omega}$ so that for each i and $\beta < \alpha$, $\mathcal{N}_{i,\beta} \cong \widetilde{\mathcal{N}}_{i,\beta}$. Define g so that $g(a_i) = c_i$. For each i and $\beta < \alpha$, $\mathcal{N}_{i,\beta} \cong \widetilde{\mathcal{N}}_{i,\beta}$ are isomorphic to either $\mathcal{A}_\beta$ or $\mathcal{E}_\beta$, which are uniformly $\mathbf{0}^{(\beta)}$-categorical (Lemma 3.2 (3)), and we can compute using $\mathbf{0}^{(\alpha)}$ which case we are in. So we can define g on $\mathcal{N}_{i,\beta}$ to be an isomorphism to $\widetilde{\mathcal{N}}_{i,\beta}$. Thus g is an isomorphism from $\mathcal{N}_n$ to $\widetilde{\mathcal{N}}_n$. □

These claims complete the proof of the theorem. □

Using this lemma, and taking the limitwise monotonic function to be the self-modulus of a c.e. set, it is not hard to prove our main theorem.

Theorem 1.3. *Let α be a computable limit ordinal and $\mathbf{d}$ a degree c.e. in and above $\mathbf{0}^{(\alpha)}$. There is a computable structure with strong degree of categoricity $\mathbf{d}$.*

Proof. Fix α and let $D \in \mathbf{d}$ be a set c.e. in and above $\mathbf{0}^{(\alpha)}$. Since D is c.e. in and above $\mathbf{0}^{(\alpha)}$, it has a self-modulus f that is limitwise monotonic relative to $\mathbf{0}^{(\alpha)}$. Consider the structure $\mathcal{M}$ constructed in Lemma 4.3 for this f. We will enrich this structure slightly to produce a new structure $\mathcal{S}$. Let $\mathcal{S}_\alpha$ be the computable structure with strong degree of categoricity $\mathbf{0}^{(\alpha)}$ constructed in Theorem 3.1 of Csima, Franklin and Shore [**?**]. The new structure $\mathcal{S}$ consists of $\mathcal{M}$ and a disjoint copy of $\mathcal{S}_\alpha$, and a new unary relation R such that $R(x)$ holds exactly when x belongs to the copy of $\mathcal{S}_\alpha$. We claim that $\mathcal{S}$ has strong degree of categoricity $\mathbf{d}$.

First, suppose that $\mathcal{T}$ is some other computable copy of $\mathcal{S}$. We will show that there is a $\mathbf{d}$-computable isomorphism between $\mathcal{S}$ and $\mathcal{T}$. Using the relation R, we may identify the component of $\mathcal{T}$ isomorphic to $\mathcal{S}_\alpha$. Since $\mathcal{S}_a$ has (strong) degree of categoricity $\mathbf{0}^{(\alpha)} \leq \mathbf{d}$, we can $\mathbf{d}$-computably find an isomorphism between the copies of $\mathcal{S}_\alpha$ in $\mathcal{S}$ and $\mathcal{T}$. We can also identify the component isomorphic to $\mathcal{M}$ in each structure. By choice of $\mathcal{M}$, any two such copies have an isomorphism between them computable in $f \oplus \mathbf{0}^{(\alpha)}$, and D can compute this self-modulus f. Hence $\mathbf{d}$ can computably produce such an isomorphism, since it can compute $f \oplus \mathbf{0}^{(\alpha)}$. Gluing these two isomorphisms together gives us the result.

Since $\mathcal{S}_\alpha$ has strong degree of categoricity $\mathbf{0}^{(\alpha)}$, there is a computable copy $\hat{\mathcal{S}}_\alpha$ of $\mathcal{S}_\alpha$ such that every isomorphism between the two computes $\mathbf{0}^{(\alpha)}$. Let $\widetilde{\mathcal{S}}$ be a computable copy of $\mathcal{S}$ built in the following way. Rather than using the "standard" copy $\mathcal{S}_\alpha$, use the "hard" copy $\hat{\mathcal{S}}_\alpha$ of $\mathcal{S}_\alpha$. Additionally, rather than using $\mathcal{M}$, instead use $\mathcal{N}$ as built in Lemma 4.3. Any isomorphism between $\mathcal{S}_\alpha$ and $\hat{\mathcal{S}}_\alpha$ computes $\mathbf{0}^{(\alpha)}$, and any isomorphism between $\mathcal{M}$ and $\mathcal{N}$ must compute a function that dominates f. Let g be any isomorphism between $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. Then by using R, we can restrict g to an isomorphism between $\mathcal{S}_\alpha$ and $\hat{\mathcal{S}}_\alpha$ and hence g can compute $\mathbf{0}^{(\alpha)}$. Since g can also be restricted to an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, it must compute a function dominating f. But f is a modulus for D computable in $\mathbf{0}^{(\alpha)}$, and hence g must be able to compute D since it can compute $\mathbf{0}^{(\alpha)}$ and a function dominating f. Hence g can compute $\mathbf{d}$. □

We now turn to prime models, working above $\mathbf{0}^{(\omega)}$. Essentially, our work here is to check that in taking $\alpha = \omega$ in the previous theorem and lemma, the construction results in a prime model.

Lemma 4.4. *Let $f{:}\,\omega \to \omega$ be limitwise monotonic relative to $\mathbf{0}^{(\omega)}$. There is a prime model with two computable copies $\mathcal{M}$ and $\mathcal{N}$ such that:*

(1) Every isomorphism between $\mathcal{M}$ and $\mathcal{N}$ computes a function which dominates f.

(2) $f \oplus \mathbf{0}^{(\omega)}$ computes an isomorphism between any two computable copies of $\mathcal{M}$ and $\mathcal{N}$.

Proof. The construction is exactly the same as that of Lemma 4.3 with $\alpha = \omega$. We refer to the structures $\mathcal{A}_\beta$ and $\mathcal{E}_\beta$ of Lemma 3.2 as $\mathcal{A}_n$ and $\mathcal{E}_n$, $n < \omega$, but of course these are the same. It remains to argue, using the properties from Lemma 3.3 which hold only for the structures $\mathcal{A}_n$ and $\mathcal{E}_n$ with n finite, that the resulting structure $\mathcal{N}$ is prime.

Recall that $\mathcal{N}$ is the disjoint union of structures $\mathcal{N}_n$, each of which satisfies the relation R_n. So it suffices to show that the structures $\mathcal{N}_n$ are prime. $\mathcal{N}_n$ was defined as follows: there were infinitely many elements $(b_i)_{i\in\omega}$ (satisfying the unary relation S), each of which is attached to (by binary relations T_m), for each $m < \omega$, a box $\mathcal{N}_{i,m}$ which contains within it:

(1) $\mathcal{N}_{i,m} \cong \mathcal{E}_m$ if there is s such that $\Phi^{\varnothing^{(m)}}(n,s) > i$.

(2) $\mathcal{N}_{i,m} \cong \mathcal{A}_m$ otherwise.

By Claim 1 of Lemma 4.3, for each i, either $i < f(n)$ and there is some $m_i < \omega$ such that:

- for $\ell < m_i$, $\mathcal{N}_{i,\ell} \cong \mathcal{A}_\ell$,
- for $\ell \geq m_i$, $\mathcal{N}_{i,\ell} \cong \mathcal{E}_\ell$,

or $i \geq f(n)$ and for all $m < \omega$, $\mathcal{N}_{i,m} \cong \mathcal{A}_m$. Note that the sequence $\{m_i\}_{i<f(n)}$ is non-decreasing.

By Lemma 3.3 (2), for $i < f(n)$, the automorphism orbit of b_i is determined by the first-order formula with free variable x which expresses that S holds of x, that the structure with domain $T_{m_i}(x,\cdot)$ satisfies φ_{m_i} (and so is isomorphic to $\mathcal{E}_{m_i}$), and that the structure with domain $T_{m_i-1}(x,\cdot)$ satisfies $\neg\varphi_{m_i-1}$ (and so is isomorphic to $\mathcal{A}_{m_i-1}$). For $i \geq f(n)$, the automorphism orbit of b_i is determined by the first-order sentence with free variable x which expresses that S holds of x, and that the structure with domain $T_{m_{f(n)-1}}(x,\cdot)$ satisfies $\neg\varphi_{m_{f(n)-1}}$ (and so is isomorphic to $\mathcal{A}_{m_{f(n)-1}}$).

Fix a tuple $\bar{c}$ from $\mathcal{N}_n$. We will give a first-order formula defining the orbit of $\bar{c}$. We may assume that whenever $\bar{c}$ contains an element of $\mathcal{N}_{i,m}$, $\bar{c}$ contains b_i as well. We can break the tuple $\bar{c}$ up into finitely many elements $b_{i_1}, \ldots, b_{i_k}$ and finitely many tuples $\bar{c}_{i,m}$ from $\mathcal{N}_{i,m}$. The orbit of $\bar{c}$ is determined by the orbits of $b_{i_1}, \ldots, b_{i_k}$ (each of which is determined by a first-order formula as described in the previous paragraph), the fact that $T_m(b_i, y)$ holds for any $y \in \bar{c}_{i,m}$, and the orbits of each of the tuples $\bar{c}_{i,m}$ within $\mathcal{N}_{i,m}$. The latter orbits are first-order definable by Lemma 3.3 (3). □

Theorem 1.4. *Let* $\mathbf{d}$ *be a degree c.e. in and above* $\mathbf{0}^{(\omega)}$*. There is a computable prime model* $\mathcal{A}$ *with strong degree of categoricity* $\mathbf{d}$*.*

Proof. The construction of such a model is similar to Theorem 1.3, except we replace $\mathcal{M}$ and $\mathcal{N}$ from Lemma 4.3 with those $\mathcal{M}$ and $\mathcal{N}$ from Lemma 4.4 (which are actually the same structures, if $\alpha = \omega$), and we also replace the "easy" and "hard" copies of $\mathcal{S}_\alpha$ with copies of the structure from Theorem 3.4 such that any isomorphism between them computes $\mathbf{0}^{(\omega)}$. The same argument from Theorem 1.3 shows that this new structure has strong degree of categoricity $\mathbf{d}$. It remains to show that such models are prime; they are the disjoint union of prime structures, distinguishable by the relation R, and hence must be prime themselves. □

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