THE UPWARD CLOSURE OF A PERFECT THIN CLASS

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Abstract. There is a perfect thin Π^0_1 class whose upward closure in the Turing degrees has full measure (and indeed contains every 2-random degree.) Thus, in the Muchnik lattice of Π^0_1 classes, the degree of 2-random reals is comparable with the degree of some perfect thin class. This solves a question of Simpson [15].

1. Introduction

Our concern in this paper is with computably bounded Π^0_1 classes. Without loss of generality, we consider these as being subclasses of Cantor space 2^ω. In particular we shall be concerned with the Turing degrees of members of perfect thin Π^0_1 classes. Recall that a Π^0_1 class $\mathcal{P}$ is called *thin* if $\mathcal{P}$ is infinite and for all Π^0_1 subclasses $\hat{\mathcal{P}}$ of $\mathcal{P}$, there is a clopen $\mathcal{U}$ with $\hat{\mathcal{P}} = \mathcal{P} \cap \mathcal{U}$. A thin class is perfect (contains no isolated points) if and only if it has no computable members. Thin classes were essentially introduced by Martin and Pour-El [13] under duality, and have come to occupy a central area in the study of Π^0_1 classes, as can be found in [1, 2, 3, 4], which will serve as background material for thin Π^0_1 classes.

The perfect thin classes are an extremely interesting subclass of the thin classes, in that they form an orbit in the automorphism group of the lattice of Π^0_1 classes, and form an invariant class for the "array noncomputable degrees" as found in Cholak, Coles, Downey and Herrmann [5]. Thus they are analogues of the maximal sets in the lattice of computably enumerable sets.

The motivation of the present paper comes from Simpson's paper [15], where he relates randomness considerations to his study of an extension of the c.e. degrees (Simpson [16]). Given two classes of reals $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is *Muchnik* reducible to $\mathcal{B}$ if every element of $\mathcal{B}$ computes some element of $\mathcal{A}$. This transitive relation gives rise to a degree structure: the Muchnik (or *weak* degrees), which turns out to be a distributive lattice (indeed, isomorphic to the lattice of upwards closed sets of Turing degrees under inclusion, union and intersection.) Of particular interest is the lattice $\mathscr{P}_w$ of Muchnik degrees of Π^0_1 classes, whose greatest element is $\mathbf{pa}$, the degree of the class of completions of Peano Arithmetic, and least element is $\mathbf{0}$, the degree of classes which contain computable elements. Simpson showed that the mapping $\deg_T W \mapsto \deg_w\{W\} \wedge \mathbf{pa}$ is an embedding of the c.e. degrees $\mathscr{R}$ into $\mathscr{P}_w$.

Simpson found that in $\mathscr{P}_w$ there are natural intermediate degrees. For example, if $\mathbf{r}_1$ denotes the degree of the class of 1-random reals and $\mathbf{r}_2$ is the degree of the

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class of 2-random reals[1], then Simpson showed [16, Theorem 3.6] that $\mathbf{r}_1 \in \mathscr{P}_w$, $\mathbf{r}_2 \wedge \mathbf{pa} \in \mathscr{P}_w$, and

$$\mathbf{0} < \mathbf{r}_1 < \mathbf{r}_2 \wedge \mathbf{pa} < \mathbf{pa}.$$

Now, Simpson [15, Theorem 9.15] proved that if $\mathbf{a}$ is the Muchnik degree of a thin perfect Π^0_1 class, then $\mathbf{a}$ is incomparable with $\mathbf{r_1}$. It is natural to ask whether $\mathbf{r}_2 \wedge \mathbf{pa}$ is also incomparable with the degrees of perfect thin classes. It turns out that for a Π^0_1 class $\mathcal{P}$, $\deg_w \mathcal{P} \leqslant \mathbf{r}_2$ iff the Lebesgue measure of the upward closure of the Turing degrees of the elements of $\mathcal{P}$ is 1 (by a 0-1 law, a measurable set of Turing degrees has measure either 0 or 1). This implies that the degrees of some perfect thin classes are incomparable with $\mathbf{r}_2 \wedge \mathbf{pa}$. In particular, the existence of a perfect thin separating class follows from Martin and Pour-El [13]; the upward closure in the Turing degrees of such a class must have measure 0 by Jockush and Soare [10, Theorem 5.3]. Is this true of every perfect thin class? Simpson [15] proved that the Lebesgue measure of thin Π^0_1 classes is always 0. On the other hand, we prove that the upward closure of a perfect thin class can have measure 1, proving that some perfect thin classes have degrees comparable with $\mathbf{r}_2 \wedge \mathbf{pa}$.

Theorem 1.1. *There is a perfect, thin Π^0_1 class $\mathcal{P}$ such that the set of reals which compute elements of $\mathcal{P}$ has full measure.*

Our proof uses the idea of "risking measure", which goes back to Paris [14], Martin (see [8]) and Kurtz [11], as well as an effective 0-1 law, as discussed in Downey and Hirschfeldt [8].

Notation is standard and follows Soare [17].

2. The Proof

We build a Π^0_1 class $\mathcal{P}$ by defining a computable tree $P \subset 2^{<\omega}$. At stage s of the construction we define P_s, the s^{th} level of the tree. In general, for all $\sigma \in P_s$, we would include both $\sigma 0$ and $\sigma 1$ in P_{s+1}, unless we decide to *terminate* σ in which case we include neither.

Additionally, we build a Π^0_2 class $\mathcal{C}$ and a procedure for computing an element of $\mathcal{P}$ from any element of $\mathcal{C}$. By the effective 0-1 law, in order to guarantee that the class of reals which compute elements of $\mathcal{P}$ has full measure, it is sufficient to ensure that $\mathcal{C}$ has *positive* Lebesgue measure. For notational convenience, we prefer to build $\mathcal{C}$ not as a subclass of Cantor space 2^ω but rather as a subclass of the Euclidean interval $[0, 1)$. We note that after removing countable sets (the binary rationals on one side, and the finite and cofinite sequences on the other), there is an effective measure-preserving isomorphism between Cantor space and the interval $[0, 1)$, and so our construction also implies the result for Cantor space.

To define the (effective, hence continuous) mapping from $\mathcal{C}$ to $\mathcal{P}$, we map rational intervals to strings. We thus define a partial computable function Γ which maps sub-intervals of $[0, 1)$ with rational endpoints to strings in P; Γ is *consistent* (like a Turing functional) in that if I, J are intervals in the domain of Γ and $I \subset J$, then $\Gamma(I)$ is a string that extends $\Gamma(J)$. At stage s, we ensure that Γ is *onto* P_s.

[1] Recall that a real A is *n-random* iff $A \notin \bigcap_{n\in\omega} U_n$ for every effective sequence of Σ^0_n classes $\{U_n\}_{n\in\omega}$ with the measure of U_n bounded by 2^{-n}. See, for instance, Downey and Hirschfeldt [8], or Li-Vitanyi [12] for more background here.

To define the reduction, for any $x \in [0,1)$, we let

$$\Gamma^x = \bigcup \{\Gamma(I) \,:\, I \in \operatorname{dom}\Gamma \text{ and } x \in I\}$$

and let

$$\mathcal{C} = \{x \in [0,1) \,:\, \Gamma^x \in 2^\omega\}.$$

Since $\mathcal{P}$ is closed and the range of Γ is in P, we get that if $x \in \mathcal{C}$ then Γ^x, which is clearly computable from x, is in $\mathcal{P}$.

Let $T_0, T_1, \ldots$ be an effective enumeration of all Π^0_1 subtrees of $2^{<\omega}$ (so $T_e = \bigcap_s T_{e,s}$ where $T_{e,s}$ are uniformly computable trees.) To ensure that $\mathcal{P}$ is thin, we must meet the requirements

R_e**:** If $[T_e] \subseteq \mathcal{P}$ then there is some clopen set $\mathcal{U}$ such that $[T_e] = \mathcal{P} \cap \mathcal{U}$.

Recall for a moment the standard way of meeting these requirements, for example, R_0. We wait for a stage s at which we discover that there is some string $\sigma \in P_s - T_{0,s}$. We then terminate all other strings of length s (so we ensure that $\mathcal{P} \subset [\sigma]$.) Since $[T_0] \cap [\sigma] = \emptyset$, we get that if $[T_0] \subseteq \mathcal{P}$ then $[T_0] = \emptyset$ and so we can pick $\mathcal{U} = \emptyset$ to witness R_0.

For our purposes, the problem with this approach outlined above is that when σ is found, too much measure is permanently thrown out of $\mathcal{C}$. We need to be able to control the size of the set that we fail on if we hope for $\mathcal{C}$ to have positive measure. If we follow this naïve approach, Γ will end up being the identity function and $\mathcal{C}$ will equal $\mathcal{P}$, and hence have measure 0 since the measure of a thin class is always 0. Thus our idea will be to modify the approach above by risking measure.

What we will do is the following. We note that if we knew σ *in advance*, then we could define $\Gamma([0,1)) = \sigma$, and so even though the measure of $\mathcal{P}$ is small, all reals can still compute elements of $\mathcal{P}$. However, we cannot know whether such a σ will ever occur in $P_s - T_{0,s}$. The idea is to set aside some measure to test this hypothesis. This is an amount of measure that the requirement R_0 is allowed to waste (by removing it from $\mathcal{C}$). If we ensure that the total amount of measure risked by all requirements is smaller than 1, then we will have ensured that $\mathcal{C}$ is not null.

Suppose for simplicity that R_0 is allowed to risk a measure of $1/2$. We would then divide the domain $[0,1)$ into two intervals: say $I_0 = [0,1/2)$ and $I_1 = [1/2,1)$. To begin with, we leave I_0 out of the domain of Γ and only define Γ on I_1 and its subintervals (according to the action done for weaker requirements). If no string σ ever occurs in $P_s - T_{0,s}$, then $\mathcal{C} \subset I_1$, but R_0 is met vacuously: if $[T_0] \subseteq \mathcal{P}$ then $[T_0] = \mathcal{P}$ and so we can let $\mathcal{U} = 2^\omega$ witness R_0. If, on the other hand, we find some $\sigma \in P_s - T_{0,s}$, then we define $\Gamma(I_0) = \sigma$ and stop defining Γ on subintervals of I_1 (in fact, we only define it on subintervals of I_0 from now on.) As in the naïve strategy, we trim P to ensure that $\mathcal{P} \subset [\sigma]$, so R_0 is met and has only wasted I_1.

We need to modify this strategy if we are allowed to waste less than $1/2$ of the full measure. Suppose, for example, that R_0 is allowed to waste measure $1/3$. At first, we divide the domain into $I_0 = [0,1/3)$ and $I_1 = [1/3,1)$. We colour I_0 *red* (reserved for future action) and I_1 *blue* (free for weaker requirements, for now, but may be claimed later). We only define Γ on I_1 and its subintervals, while we wait for a stage s at which we find some $\sigma \in P_s - T_{0,s}$.

If there is such a stage s, we define $\Gamma(I_0) = \sigma$. On $[\sigma]$ we won R_0 and so I_0 and its subintervals are free for weaker requirements. Every interval J such that $\Gamma(J) = \sigma$ (including of course I_0) is coloured *white* (positively processed). For every other string $\sigma' \in P_s$, since Γ is onto P_s, there are intervals J such that $\Gamma(J) = \sigma'$. On each such interval J we can play the 1/2-module relative to $[\sigma']$: we break J into two subintervals J_0 and J_1 of equal length, colour J_0 red and J_1 blue. We allow Γ to be defined on subintervals of J_1 but not of J_0 (so we define $\Gamma(J_1) = \Gamma(J) = \sigma'$.) If at a later stage t we discover some string τ *extending* σ' which is in $P_t - T_{0,t}$, we define $\Gamma(J_0) = \tau$, colour J_0 white, and this time colour J_1 *black* (permanently removed from $\mathcal{C}$), never allow Γ to be defined on subintervals of J_1, and ensure that $\mathcal{P} \cap [\sigma'] \subset [\tau]$ by terminating all strings in P_t which extend σ' and are distinct from τ.

The reason that we can play the 1/2-module is that we have already ensured that R_0 passes at least 1/3 measure (the white intervals) to weaker requirements. It then risks 1/2 of the other 2/3, namely, not more than the 1/3 which it is allowed.

In all eventualities, R_0 is met. Assume that $[T_0] \subseteq \mathcal{P}$. If no initial stage s is found, then $[T_0] = \mathcal{P}$ and we can let $\mathcal{U} = 2^\omega$. Otherwise, for each $\sigma' \in P_s$ different from σ, if no stage t as above is found, then $\mathcal{P} \cap [\sigma'] = [T_0] \cap [\sigma']$, and so we can let $\mathcal{U}$ be the union of those $[\sigma']$ for which no stage t is found (and no subintervals coloured white and black).

In general, a $1/k$-module for R_0 will have $k-1$ iterations: first colouring an interval of length $1/k$ red (and the rest blue), waiting for a string σ as above, and if one is found, then the red turns white, the blue is broken into small subintervals on which the $1/(k-1)$-module is played with red and blue colours, until we get to play the 1/2-module. Only the 1/2-module is allowed to terminate any strings from P (and colour intervals black). At each stage of the process, no more than $1/k$-much measure is risked by R_0.

There are two more issues we need to discuss to complete the proof: how weaker requirements are affected and behave, and how to ensure that $\mathcal{P}$ is perfect.

The key to the solution of both issues is that each requirement acts finitely many times, that is, this is a *finite injury* construction. This is why every requirement acts on the basis of the belief that all stronger requirements have already ceased all action. If some requirement acts on some interval, then it initialises the actions of weaker requirements on any subinterval. This means that all the colourings done by the initialised requirement are removed, in essence bringing back into $\mathcal{C}$ intervals that the initialised requirement may have previously coloured red. We cannot, however, remove black markings, since they are mapped to strings already terminated on P and so the corresponding black intervals cannot be returned to $\mathcal{C}$. To compensate, we also keep the white intervals, since they represent a definite win for the corresponding requirement.[2]

Take R_1. Whenever it starts (after each stage at which R_0 acts and initialises R_1) it will start with the 1/8-module (in general, R_e will be allowed to waste, say, $2^{-(e+2)}$-much measure, which will ensure that $\mu(\mathcal{C}) \geqslant 1/2$).

When R_1 starts at stage s, it is given a collection of (pairwise disjoint) intervals which R_0 marked as either blue or white. These intervals are mapped onto P_s. (To verify this, note that whenever an interval J_0 is marked red, another subinterval J_1 of its superinterval J is marked blue and is mapped by Γ to the same string.) Now R_1 plays the 1/8-module separately on each of these intervals, as described in

[2]Another possible approach is to remove the white markings and further restrict the amount of measure R_e can spend, say to 2^{-s} where s is the stage at which the initialisation occurs.

the second stage of the $1/k$-module for R_0: each such interval J is partitioned into subintervals J_0 and J_1 (the first of length $\mu(J)/8$); J_0 is coloured red by R_1 and J_1 is coloured blue by J_1 (and mapped to $\Gamma(J)$); J_1 is available for requirement R_2. A search for a string τ extending $\Gamma(J)$ and in $P_t - T_{1,t}$ commences; the rest is the same. The total measure risked by R_1 at any stage is $1/8$ of what is passed to it by R_0, which is of course no more that $1/8$ of the total measure.

To ensure that $\mathcal{P}$ is perfect, we need to ensure that if $\sigma \in P$ is extendible (i.e. $\mathcal{P} \cap [\sigma] \neq \emptyset$), then there are incomparable extendible nodes on P extending σ. Note, however, that if R_1 starts action at stage s and is not injured after that stage, then *every* node in P_s is extendible. This is because if J is mapped to σ at stage s, then some subclass of J of positive measure remains in $\mathcal{C}$ (and is mapped to $\mathcal{P} \cap [\sigma]$). To ensure splitting, we only need to modify the construction as follows: each time R_e is initialised, before starting its module, we split every interval J as in the instructions to two subintervals J' and J'', extend Γ by mapping $\Gamma(J') = \Gamma(J)0$ and $\Gamma(J'') = \Gamma(J)1$ and starting the module for R_e only in the next stage (starting with J' and J'' instead of J), thus ensuring that if indeed R_e is not injured later, then both $\Gamma(J)0$ and $\Gamma(J)1$ are extendible on P. This is of course done for all R_e, $e \geqslant 1$.

3. The formal details

We construct a partial computable mapping Γ from intervals $[a, b) \subseteq [0, 1)$ with rational endpoints to $2^{<\omega}$. At stage s, we decide which intervals are mapped to strings of length s. We let $P_s = 2^s \cap \operatorname{range} \Gamma$. At any given moment, let G_s be the set of minimal intervals in $\Gamma^{-1} P_s$ [note that we can have intervals $J' \supsetneq J$ both in $\Gamma^{-1} P_s$; in this case, because of consistency, we'd have $\Gamma(J') = \Gamma(J)$.]

For all $e < \omega$ we also enumerate sets $\texttt{White}_e$ and $\texttt{Black}_e$ of intervals. [These sets will actually be uniformly computable.] We let $\texttt{Black} = \bigcup_e \texttt{Black}_e$ and $\texttt{White} = \bigcup_e \texttt{White}_e$.

We also approximate a d.c.e. set A_e of *pairs* of intervals. These sets are partitioned into subsets $A_{e,k}$ (for $2 \leqslant k \leqslant 2^{e+2}$). We let $\texttt{Red}_e$ be the domain of A_e (the projection of A_e onto the first coordinate) and $\texttt{Blue}_e$ be the image of A_e (the projection of A_e onto the second coordinate). We define A, $\texttt{Red}$, $\texttt{Blue}$ as expected.

The $1/k$-*module* for R_e on interval I has two parts. It is *started* as follows:

> Partition I into two subintervals I_0 and I_1 such that the length of I_0 is $1/k$ the length of I. Enumerate (I_0, I_1) into $A_{e,k}$. Define $\Gamma(I_1) = \Gamma(I)$.

To *release* the module (associated with $(I_0, I_1) \in A_{e,k}$) at stage s using a string σ, we do the following:

(1) Extract (I_0, I_1) from A_e.
(2) Define $\Gamma(I_0) = \sigma$.
(3) For all $(J_0, J_1) \in A$ such that $J_0 \cup J_1 \subseteq I_1$, remove the pair (J_0, J_1) from A and define $\Gamma(J_0)$ to be some string in P_s which extends $\Gamma(J_1)$.
(4) For all $J \subseteq I_0 \cup I_1$ in G_s such that $\Gamma(J) = \sigma$ (including I_0, and possibly intervals such as J_0 from (3)), enumerate J into $\texttt{White}_e$.
(5) For all $J \subseteq I_1$ in G_s such that $\Gamma(J) \neq \sigma$ (including possibly intervals such as J_0 from (3)),
    (a) If $k = 2$, enumerate J into $\texttt{Black}_e$.
    (b) If $k > 2$, start the $1/(k-1)$-module for R_e on J.

Construction. At stage 0, we let $\Gamma([0,1)) = \langle\rangle$ and start the 1/4-module for R_0 on the interval $[0,1)$.

Let $s > 0$. Stage s has three phases:

(1) Define P_s by letting, for all $J \in G_{s-1} \setminus \texttt{Black}$, $\Gamma(J_i) = \Gamma(J)i$, where $i < 2$ and J_0, J_1 is a partition of J into two subintervals (say of equal length).
(2) Search for pairs (J_0, J_1) which for some $e < s$ we have $(J_0, J_1) \in A_e$ and there is some $\sigma \supseteq \Gamma(J_1)$ in P_s which is not on $T_{e,s}$, and such that $J_0 \cup J_1$ is *maximal* (by containment) with respect to this property. For each such pair, release the module associated with (J_0, J_1) using σ.
(3) For $I \in G_s$ such that no module on $J \supseteq I$ was released at the second phase, find some $J \supset I$ in $(\texttt{Blue}_e \cup \texttt{White}_e) \cap G_{s-1}$ for some e, find the least $e' > e$ such that there is no $J' \supseteq J$ in $\texttt{White}_{e'}$, and start the $1/2^{e'+2}$-module for $R_{e'}$ on I.

Verification. We consider $\operatorname{dom}\Gamma$ as a partial ordering, ordered by reverse inclusion. Let Γ_s be Γ, as it is defined at the *end* of stage s. Also let $\texttt{Red}_{e,s}$, A_s, $A_{e,s}$, $A_{e,k,s}$ etc., denote the approximated sets at the end of stage s.

Note that indeed, at stage s, we only map intervals (by Γ) to strings of length s. Thus the value of G_s is fixed after the end of stage s, and $s \mapsto P_s$ is a computable function.

Lemma 3.1.

(1) $\operatorname{dom}\Gamma_s$ *is a tree: for all* $I \in \operatorname{dom}\Gamma_s$, $\{J \in \operatorname{dom}\Gamma_s : J \supseteq I\}$ *is finite and linearly ordered by* $\supseteq$. *Indeed, if* $I, J \in \operatorname{dom}\Gamma$ *are not comparable then they are disjoint.*
(2) G_s *is a set of leaves of* $\operatorname{dom}\Gamma_s$.
(3) If $I_0 \in \texttt{Red}_s$ *then there is no* $J \subseteq I$ *in* $\operatorname{dom}\Gamma_s$.

Proof. By induction on s. Suppose the lemma holds for s; we prove it for $s+1$. There are two main points:

(i) If $(I_0, I_1) \in A_s$, $(J_0, J_1) \in A_s$ are distinct and $I_0 \cup I_1$, $J_0 \cup J_1$ are not disjoint, then either $J_0 \cup J_1 \subseteq I_1$ or $I_0 \cup I_1 \subseteq J_1$. This is because all of $I_0 \cup I_1$, I_1, $J_0 \cup J_1$, J_1 are in $\operatorname{dom}\Gamma_s$ and because no subinterval of J_0 or I_0 is in $\operatorname{dom}\Gamma_s$, I_1 is the unique immediate successor of $I_0 \cup I_1$ in $\operatorname{dom}\Gamma_s$ (and similarly for the J's).
(ii) $\operatorname{dom}\Gamma_{s+1}$ is an *end-extension* of $\operatorname{dom}\Gamma_s$; if $I \in \operatorname{dom}\Gamma_{s+1} - \operatorname{dom}\Gamma_s$ then one of two holds:
    (a) either there is some $J \in G_s$ such that $J \subset I$; or
    (b) $I \in \texttt{Red}_s \setminus \texttt{Red}_{s+1}$.

This (together with the splitting of intervals which occurs at stage $s+1$) ensures that (1) holds for $s+1$; (2) is immediate. For (3), say $(I_0, I_1) \in A_{s+1}$. If $(I_0, I_1) \in A_s$ then by induction, no subinterval of I_0 is in $\operatorname{dom}\Gamma_s$, and none are added at stage $s+1$ (or we'd remove the pair from A). Certainly if (I_0, I_1) is added to A at stage $s+1$ then $I_0 \cup I_1$ is in G_{s+1} when the module is started, so is a leaf of $\operatorname{dom}\Gamma$ at the time, and no subinterval of I_0 is added to $\operatorname{dom}\Gamma_{s+1}$ after the module is started. □

Note that as mentioned, we only place strings in $\texttt{Black}$ or $\texttt{White}$ at stage s if they are in G_s. Thus both sets are computable.

Lemma 3.2. *For every* $I \in \texttt{Blue}_s$ *there is some* $J \subseteq I$ *in* $G_s \setminus \texttt{Black}$.

Proof. By induction on s; assume this holds at the end of stage s. Let $I \in \texttt{Blue}_{e,s+1}$. Of course if I is added to $\texttt{Blue}_e$ during stage $s+1$ then we can take $J = I$, so we assume that $I \in \texttt{Blue}_{e,s}$. By induction, there is some $J \subseteq I$ in $G_s \setminus \texttt{Black}$. At phase one of stage $s+1$, J is split into subintervals in G_{s+1}. We're done unless they are all coloured black during phase two of stage $s+1$. But if this happens, since I is not removed from $\texttt{Blue}$ at stage $s+1$, there must be some (J_0, J_1) for which a module is terminated at stage $s+1$ and $J_0 \cup J_1 \subseteq I$. Then we can take $J = J_0$. □

We say that requirement R_e *acts* at stage s if a module for R_e is either started or released at stage s. We say that R_e acts *on* J at stage s if a module for R_e is started on J at stage s, or if a module on J for R_e is released at stage s.

Lemma 3.3. $G_s \subseteq \texttt{Blue}_s \cup \texttt{Black}_s \cup \texttt{White}_s$.

Proof. By induction on s. Suppose that the lemma holds for $s-1$. Let $I \in G_s$. There are two cases:

- Some R_e acts at stage s on some $J \supset I$. Then J is enumerated at stage s into $\texttt{Black}_e \cup \texttt{White}_e \cup \texttt{Blue}_e$.
- Otherwise there is some $I' \supset I$ in G_{s-1}. By induction, $I' \in \texttt{Blue}_e \cup \texttt{White}_e$ for some e. Then at stage s, I is enumerated into $\texttt{Blue}_{e'}$ for some $e' > e$.

□

As a corollary of the lemma and its proof, we get:

Corollary 3.4. *The instructions of the construction can always be carried out.*

Proof. There are two points to verify.

(1) Step (3) of the release of a module can always be performed. This is guaranteed by Lemma 3.2.
(2) For every $I \in G_s$ treated in phase three (so no requirement acted on some $J \supseteq I$ at phase two of stage s), there is (in fact a unique) $J \supset I$ in $G_{s-1} \cap (\texttt{Blue}_s \cup \texttt{White}_s)$. This is guaranteed by Lemma 3.3.

□

Corollary 3.5. Γ *is monotone: if* $I, J \in \operatorname{dom}\Gamma$ *and* $I \subset J$ *then* $\Gamma(I) \supseteq \Gamma(J)$.

Proof. Follows from the instructions, once we realise that every extension of $\operatorname{dom}\Gamma$ is an end-extension. □

For all $I \in G_s$, let e_I be the unique $e \leqslant s$ which acts during stage s on some $J \supset I$.

Lemma 3.6. *For all* $I \in G_s$*, for all* $e' \leqslant e_I$*, there is some* $J \supseteq I$ *in* $\texttt{Blue}_{e',s} \cup \texttt{White}_{e',s}$.

Proof. By induction on s. At stage 0, R_0 acts on $[0,1) = I_0 \cup I_1$ and $G_0 = \texttt{Blue}_{0,0} = \{I_1\}$.

Assume the lemma holds for stage $s-1$, and let $I \in G_s$. Again, there are two cases.

- If some R_e acts during phase two of stage s on some $J \supset I$ then $e_I = e$. Say $J = J_0 \cup J_1$ and $(J_0, J_1) \in A_{e,s-1} \setminus A_{e,s}$. Then if $I \subset I_1$ then $I \in \texttt{Black}_{e,s} \cup \texttt{Blue}_{e,s}$ and if $I \subseteq J_0$ then $I = J_0 \in \texttt{White}_{e,s}$.

For $e' < e$, let $t < s$ be the stage at which (J_0, J_1) was enumerated into A_e. Then $J_1 \in G_t$ and so by induction, there is some $K \supset J_1$ in $\texttt{White}_{e',t} \cup \texttt{Blue}_{e',t}$. Then since J is the immediate predecessor of J_1 in $\operatorname{dom}\Gamma$, we have $K \subseteq J$, so $K \subseteq I$. Also, since (J_0, J_1) was not extracted from A_e between stages t and s, we still have $K \in \texttt{Blue}_{e',s} \cup \texttt{White}_{e',s}$ as required.

- Otherwise, some R_e acts during phase three of stage s on some $J \supset I$ (we have $e = e_I$ of course); there is some $e^* < e$ and some $J \supset I$ in $G_{s-1} \cap (\texttt{Blue}_{e^*,s-1} \cup \texttt{White}_{e^*,s-1})$. Let $e' \leqslant e$. There are four cases:
  - If $e = e'$, then we note that R_e places I into $\texttt{Blue}_{e,s}$ at stage s.
  - If $e^* < e' < e$ then by the instructions, there is some $K \supset J$ in $\texttt{White}_{e',s}$.
  - If $e' = e$, then of course we use the fact that no requirement acted on an interval containing I to see that $J \in \texttt{Blue}_{e^*,s}$.
  - If $e' < e$ then we use induction to see that there is some $K \supset J$ in $\texttt{Blue}_{e',s-1} \cup \texttt{White}_{e',s-1}$ and again the fact that no requirement acted below I shows that $K \in \texttt{Blue}_{e',s} \cup \texttt{White}_{e',s}$.

□

Lemma 3.7. *Every requirement acts only finitely many times.*

Proof. By induction on e. Suppose that at no stage $s \geqslant s^*$ does any requirement $R_{e'}$ for $e' < e$ act.

At any stage $s > s^*$, no new run of a module for R_e is started at the third phase, simply because for all $e' < e$, $R_{e'}$ doesn't act at stage $s-1$ and so no intervals in G_{s-1} are ever coloured $\texttt{White}_{e'}$ or $\texttt{Blue}_{e'}$.

So after stage s^*, the actions for R_e are well-founded: every pair of intervals in $A_{e,k}$ is possibly replaced by finitely many pairs in $A_{e,k-1}$ and so the process must halt. □

Together with Lemma 3.6, we get:

Corollary 3.8. *For every e, for almost all s, for every $I \in G_s$ there is some $J \supset I$ in $\texttt{White}_e \cup \texttt{Blue}_e$.*

Let $P = \bigcup_s P_s$. From the instructions it is clear that every string in P_{s+1} extends one in P_s, so P is a computable tree and $\mathcal{P} = [P]$ is a Π^0_1 subclass of 2^ω.

Lemma 3.9. *Every requirement R_e is met. Thus $\mathcal{P}$ is thin.*

Proof. Fix $e < \omega$. By Corollary 3.8,

$$\{[\Gamma(I)] \,:\, I \in \texttt{White}_e \cup \texttt{Blue}_e\}$$

is a finite clopen cover of $\mathcal{P}$.

If $I \in \texttt{White}_e$, then $[\Gamma(I)] \cap [T_e] = \emptyset$.

If $I \in \texttt{Blue}_e$, then since I is never removed from $\texttt{Blue}_e$, we have $\mathcal{P} \cap [\Gamma(I)] \subset [T_e]$.

So R_e is met. □

We say that an interval I which is added to $\operatorname{dom}\Gamma$ at stage s is *injured* at a stage $t > s$ if at stage t, some requirement releases a module on some $J' \supsetneq J$.

Lemma 3.10. *Suppose that $I \in \texttt{Blue} \cup \texttt{White}$ and is never injured. Then $[\Gamma(I)] \cap \mathcal{P} \neq \emptyset$.*

Proof. If $I \in \mathtt{Blue}$ then for almost all t, $I \in \mathtt{Blue}_t$. By Lemma 3.2, for almost all t there is some $J \subset I$ in G_t and so there is some $\sigma \supset \Gamma(I)$ in P_t. By compactness, $[\Gamma(I)] \cap \mathcal{P} \neq \emptyset$.

Suppose that $I \in \mathtt{White}$ and is never injured. Say I is enumerated into $\mathtt{White}$ at stage s. Then at stage $s+1$ there is some $I' \subset I$ on which a new module is begun, and that module is never injured. Then either the module is released, in which case there is some $I_1 \subsetneq I$ in $\mathtt{White}$ (which is never injured), or there is some $J \subset I$ in $\mathtt{Blue}$ (which is never removed from $\mathtt{Blue}$). In the second case, The argument for $\mathtt{Blue}$ from the first paragraph gives the desired result for I. Otherwise, we keep arguing for I_1 to get I_2, I_3, etc. Either at some point we get some $J \subset I_n \subset I$ which is permanently in $\mathtt{Blue}$, or we get a path in $\operatorname{dom}\Gamma$ which maps by Γ to a path in P extending $\Gamma(I)$. □

Lemma 3.11. *Suppose that a module is started on some interval J and J is never injured. Then $[\Gamma(J)] \cap \mathcal{P} \neq \emptyset$.*

Proof. Say $J = J_0 \cup J_1$ and $(J_0, J_1) \in A_s$. If at a later stage (J_0, J_1) is removed from A_s then by assumption this must be because the module on J is released; so $J_0 \in \mathtt{White}$, J_0 is never injured and so by Lemma 3.10, $[\Gamma(J)] \cap \mathcal{P} \supset [\Gamma(J_0)] \cap \mathcal{P} \neq \emptyset$. Otherwise, $J_1 \in \mathtt{Blue}$ and is never injured and so again by Lemma 3.10, $[\Gamma(J)] \cap \mathcal{P} = [\Gamma(J_1)] \cap \mathcal{P} \neq \emptyset$. □

Corollary 3.12. *$\mathcal{P}$ is perfect.*

Proof. Let $\sigma \in P$ be extendible. By Corollary 3.8, and the arguments preceding it, there is some $I \in \mathtt{White} \cup \mathtt{Blue}$ which is never injured such that $\Gamma(I) \supset \sigma$. Say I is enumerated into $\mathtt{White} \cup \mathtt{Blue}$ at stage s. At the first phase of stage $s+1$, I is split into two subintervals I_0 and I_1, and then at the third phase, modules are started on both. Neither interval is every injured. Thus by Lemma 3.11, both $\mathcal{P} \cap [\Gamma(I)0]$ and $\mathcal{P} \cap [\Gamma(I)1]$ are non-empty. □

Let μ be Lebesgue measure on $[0,1)$.

Suppose that a $1/k$-module is started for R_e on an interval I at a stage s. Let $t > s$ be the stage at which the module is injured (or $t = \infty$ if the module is never injured).

Let $B_I = I \cap \bigcup(\mathtt{Black}_{e,t})$, $W_I = I \cap \bigcup(\mathtt{White}_{e,t})$ and $R_I = I \cap \bigcup \mathtt{Red}_{e,t}$.

We make two calculations.

Lemma 3.13. *$k\mu(B_I \cup R_I) \leqslant \mu(I)$.*

Proof. We prove the lemma by induction on k.

Suppose that $I = I_0 \cup I_1$ is the partition for the module. First, if the module is never released (before stage t), then $B_I = \emptyset$; if $t < \infty$ then $R_I = \emptyset$ and if $t = \infty$ then $R_I = I_0$. As $\mu(I_0) = \mu(I)/k$ we have in both cases $k\mu(R_I \cup B_I) \leqslant \mu(I)$.

Suppose that the module is released at a stage $s' < t$. If $k = 2$ then $B_I \subseteq I_1$ and $R_I = \emptyset$ so (as $\mu(I_1) = \mu(I)/2$) we have $2\mu(B_I \cup R_I) \leqslant \mu(I)$.

Suppose that $k > 2$. Then at stage s', a $1/(k-1)$-module is started on several (disjoint) intervals $J \subset I_1$; let $\mathcal{J}$ be the set of such intervals. For each $J \in \mathcal{J}$, by induction, $(k-1)\mu(B_J \cup R_J) \leqslant \mu(J)$. However, $R_I = \bigcup_{J\in\mathcal{J}} R_J$ and $B_I = \bigcup_{J\in\mathcal{J}} B_J$,

$$\sum_{J\in\mathcal{J}} \mu(J) \leqslant \mu(I_1) = \frac{k-1}{k}\mu(I)$$

and so in total,

$$k\mu(B_I \cup R_I) = \frac{k}{k-1}\sum_{J\in\mathcal{J}}(k-1)\mu(B_J \cup R_J) \leqslant \frac{k}{k-1}\mu(I_1) = \mu(I)$$

as required. □

Lemma 3.14. *$k\mu(B_I) \leqslant \mu(W_I \cup B_I)$.*

Proof. Again by induction on k. Again suppose that $I = I_0 \cup I_1$ is the partition for the module. If the module is never released (before stage t), then $B_I = \emptyset$ and the inequality is immediate.

Suppose that the module is released at a stage $s' < t$. If $k = 2$ then $B_I \subseteq I_1$ and $I_0 \subseteq W_I$ and so (as $\mu(I_0) = \mu(I_1)$) we have $\mu(B_I) \leqslant \mu(W_I)$ as required.

Suppose that $k > 2$. Then at stage s', a $1/(k-1)$-module is started on several (disjoint) intervals $J \subset I_1$; let $\mathcal{J}$ be the set of such intervals. For each $J \in \mathcal{J}$, by induction, $(k-1)\mu(B_J) \leqslant \mu(W_J \cup B_J)$. Now $W_I \supseteq I_0 \cup \bigcup_{J\in\mathcal{J}} W_J$ and $B_I = \bigcup_{J\in\mathcal{J}} B_J$. Thus

$$\mu(B_I) = \sum_{J\in\mathcal{J}} \mu(B_J) \leqslant \frac{1}{k-1}\sum_{J\in\mathcal{J}} \mu(B_J \cup W_J)$$

and

$$\mu(I_0) = \frac{1}{k}\mu(I) = \frac{1}{k-1}\mu(I_1) \geqslant \frac{1}{k-1}\sum_{J\in\mathcal{J}} \mu(B_J \cup W_J)$$

so

$$k\sum_{J\in\mathcal{J}} \mu(B_J \cup W_J) \leqslant (k-1)\left(\mu(I_0) + \sum_{J\in\mathcal{J}} \mu(B_J \cup W_J)\right) \leqslant (k-1)\mu(B_I \cup W_I)$$

which all together give the desired inequality. □

Corollary 3.15. *For every e, $\mu(\mathtt{Red}_e \cup \mathtt{Black}_e) \leqslant 2^{-(e+2)}$.*

Proof. Let $\mathcal{I}_e$ be the collection of intervals I on which a $2^{-(e+2)}$-module is started for R_e. Let $\mathcal{I}'_e$ be the collection of those $I \in \mathcal{I}_e$ such that the module on I is eventually injured. For $I \in \mathcal{I}'_e$ let $\hat{I} = B_I \cup W_I$. Then

$$(\mathcal{I}_e \setminus \mathcal{I}'_e) \cup \left\{\hat{I} \,:\, I \in \mathcal{I}'_e\right\}$$

consists of pairwise disjoint subsets of $[0, 1)$, and

$$\bigcup \mathtt{Red}_e \cup \bigcup \mathtt{Black}_e = \bigcup \{B_I \cup R_I \,:\, I \in \mathcal{I}_e \setminus \mathcal{I}'_e\} \cup \bigcup \{B_I \,:\, I \in \mathcal{I}'_e\},$$

noting that if $I \in \mathcal{I}'_e$ then $B_I \subset \hat{I}$.

If $I \in \mathcal{I}_e \setminus \mathcal{I}'_e$, then Lemma 3.13 ensures that $\mu(R_I \cup B_I) \leqslant 2^{-(e+2)}\mu(I)$. If $I \in \mathcal{I}'_e$ then Lemma 3.14 ensures that $\mu(B_I) \leqslant 2^{-(e+2)}\mu(\hat{I})$. Together we get the result. □

As planned, we let, for $x \in [0, 1)$, $\Gamma^x = \bigcup\{\Gamma(I) \,:\, I \in \operatorname{dom}\Gamma \;\&\; x \in I\}$. Then for all $x \in [0, 1)$, $\Gamma^x \in 2^{\leqslant\omega}$. Let $\mathcal{C} = \{x \in [0, 1) \,:\, \Gamma^x \in 2^\omega\}$. Then $\mathcal{C}$ is a Π^0_2 subclass of $[0, 1)$, and for all $x \in \mathcal{C}$, $\Gamma^x \in \mathcal{P}$.

Lemma 3.16. $\mathcal{C} = [0, 1) \setminus (\mathtt{Red} \cup \mathtt{Black})$.

Proof. By induction on s we can show that

$$\bigcup G_s \cup \bigcup \mathtt{Red}_s \cup \bigcup \mathtt{Black}_s = [0,1).$$

The lemma follows. □

Corollary 3.17. $\mu(\mathcal{C}) \geqslant 1/2$.

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