\[ \Delta^0_2 \text{ DEGREES AND TRANSFER THEOREMS} \]

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1. The main goal of this paper is to demonstrate how weak truth table/Turing degree “transfer” techniques may be used to obtain information about the \( \Delta^0_2 \) (Turing) degrees. Such techniques have previously been applied by Ladner-Sasso \cite{13}, Stob \cite{18} and others to obtain information about \( R \), the r.e. \( T \)-degrees. The best known example of this phenomenon is Ladner and Sasso’s \cite{13} use of contiguous degrees to show that every nonzero r.e. degree has a predecessor with the anticupping property.

Let \( D \) denote the degrees, \( W \) the r.e. weak truth table (\( W \)-)degrees and \( D^W \) the weak truth table degrees. Modifying the Ladner-Sasso analysis to \( \Delta^0_2 \) degrees, we shall give a new and relatively easy proof of a result independently proved by Cooper \cite{5} and Slaman and Steel \cite{16} about structural interactions of \( R \) and \( D \):

**Theorem A.** \( \exists a, b \in R(0 < b < a \text{ and } \forall c \in D(c \cup b = a \rightarrow c = a)) \)

Such a degree \( a \) is said to have the **strong anticupping property with witness** \( b \). Actually, we get a slight improvement by constructing \( a \) with witnesses that are “downward dense” in \( R \). To prove Theorem A, we first analyse how \( D \) and \( W \) interact and then prove some results about \( D^W \) and \( W \). In particular, one result we shall establish is that every nonzero r.e. weak truth table degree has the **global anticupping property**, that is:

**Theorem B.**

\[ \forall a \in W(a \neq 0 \rightarrow \exists b \in W(0 < b < a \text{ and } \forall c \in D^W(a \leq c \cup b \rightarrow a \leq c))) \]

Theorem B also implies that the elementary theory of (for example) the weak truth table degrees below \( 0^\prime \) and the \( \Delta^0_2 \) degrees are different (since Posner and Robinson \cite{15} have shown that the nonzero \( T \)-degrees below \( 0^\prime \) all cup to \( 0^\prime \)).

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Finally, we shall give a couple of other examples to indicate some further applications of \( \Delta_2^0 \) transfer theorems. For example, we show that there exist nonzero r.e. degrees \( a \) that split in a very strong way over all lesser \( \Delta_2^0 \) degrees; namely, if \( b < a \) and \( b \in D \) then there exists an r.e. splitting \( a_1 \cup a_2 = a \) of \( a \) with \( b \cup a_1, b \cup a_2 < a \).

Our notation is standard and follows Soare [17]. \( T \)-reductions will be denoted by \( (\hat{\Phi}, \hat{\Gamma}, \ldots) \) and those with "hats" \( (\Phi, \Gamma, \ldots) \) will denote \( W \)-reductions. The recursive use corresponding to the latter will be the corresponding lower case greek letter (e.g., use \( \hat{\Phi} = \varphi \), use \( \hat{\Gamma} = \gamma \ldots \). Unless stated otherwise, we denote \( T \)-degrees by lower case boldface letters \( a, b, \ldots \). Finally all computations, etc., are bounded by \( s \) at stage \( s \).

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2. We shall first construct an r.e. degree with the strong anticupping property. To do this we modify the transfer analysis of Ladner and Sasso [13] which gave a new proof of Lachlan's result [10] that there is an r.e. degree with the anticupping property. The Ladner-Sasso analysis is summarized by the combination of (2.1) and (2.2) below.

(2.1) There exists a nonzero contiguous r.e. degree, namely a nonzero r.e. degree \( a \) consisting of a single r.e. \( W \)-degree; meaning that if \( A \) and \( B \) are r.e. and of degree \( a \), then \( A \equiv^W B \).

(2.2) \( \forall a \in W(a \neq 0 \rightarrow \exists b \in W(0 < b < a \quad \text{and} \quad \forall c \in W(c \cup b \geq a \rightarrow c \geq a)) \).

We shall replace (2.1) and (2.2) by:

(2.1)' There exists an r.e. degree \( a \neq 0 \) such that all (not necessarily r.e.) sets \( A, B \) of degree \( a \) are of the same \( W \)-degree. We call such a degree strongly contiguous.

(2.2)' (Theorem B) Every nonzero r.e. \( W \)-degree has the global anticupping property.

Then we see that—in the same way as [13]—(2.1)' and (2.2)' imply Theorem A.

We now turn to the proof of (2.1)', namely the construction of a strongly contiguous degree. For convenience, we modify the presentation of Ambos-Spies [1]. We satisfy the following requirements.

\( P_e : \bar{A} \neq W_e \)

\( N_e : \Phi_e(A) \) total (and \( (0,1) \)-valued, by convention) and \( \Gamma_e(\Phi_e(A)) = A \) implies \( A \leq^W \Phi_e(A) \).

\( \bar{N}_e : \Phi_e(A) \) total and \( \Gamma_e(\Phi_e(A)) = A \) implies \( \Phi_e(A) \leq W A \).
Here $(\Phi_e, \Gamma_e)$ denotes a standard enumeration of all pairs of $T$-reductions. Both $N_e$ and $\hat{N}_e$ are met by similar (completely compatible) techniques.

Due to the similarity of our method of satisfying $N_e$ (and $\hat{N}_e$) with the case where $\Phi_e(A)$ is r.e., it will suffice (in each case) to discuss the strategy for a single requirement, and then to leave the details of coordination of the requirements to the reader.

Let $l(e, s) = \max\{x: \forall y < x(\Gamma_e, x(\Phi_e, x(A_y), y) = A_y(y))\}$. We meet $N_e$ (and $\hat{N}_e$) by essentially the same cancellation procedure as for the case $\Phi_e(A)$ r.e. in a contiguous degree construction. The only difficulty is to see that it also works for $\Phi_e(A)$ only $\Delta^0_2$. Specifically each follower $x$ of $P_j$ for $j > e$ is equipped with a guess as to whether or not $l(e, s) \to \infty$. If a follower $x$ is guessing that $l(e, s) \to \infty$ then if

$$l(e, s) > ml(e, s) = \max\{l(e, t): t < s\}$$

we shall cancel $x$. The other key follower rules are:

(2.4) If $x$ is appointed at stage $s$ then $x = s$, and if $l(e, s) > ml(e, s)$ we give $x$ a guess that $l(e, s) \to \infty$; otherwise $x$ guesses $l(e, s) \not\to \infty$.

(2.5) If $x < y$ and $x$ and $y$ are followers and if $x$ enters $A_y$, then $x$ cancels $y$.

(2.6) If $x$ and $y$ are followers and $y > x$ (so that, by (2.4) $y$ is appointed after $x$) and $x$ is uncancelled at stage $y$, then $y$ has lower priority than $x$.

The basic idea for $N_e$ is this. For each follower $x$ following some $P_j$ for $j > e$ guessing $l(e, s) \to \infty$, we wait for the first stage when $l(e, s) > x$. At this stage (with $x$ least) we declare $x$ as $e$-confirmed and cancel all followers $y$ for $y > x$. This gives the situation in Figure 1.

![Diagram](image)

**FIG. 1**

Now the crucial points are that for this situation to occur $x$ must be guessing that $l(e, s) \to \infty$, and there are no followers left alive between $x$ and $s$. We claim that this insures that $A \leq_w \Phi_e(A)$ as follows: Let $u = \max\{u(\Phi_e, s(A_y), y)\leq x\}$. To determine whether $x \in A$ compute the least stage $t > s$ with $l(e, t) > ml(e, t)$ and

$$\Phi_e, x(A_y)[u] = \Phi_e(A)[u]$$

(Notice here we are not asking that $\forall t' > t(\Phi_e, x(A_y)[u] = \Phi_e(A)[u]$ as
would occur in the r.e. case). We claim that \( x \in A \iff x \in A_r \). There are two cases to consider.

**Case 1.** \( \Phi_{e,s}(A_s)[u] = \Phi_{e,s}(A_r)[u] \). In this case the situation of Figure 1 is unchanged and because \( u \) measures a use function it must be that \( A[x] = A_s[x] = A_r[x] \).

**Case 2.** Otherwise. Since there were no numbers \( z \) alive at stage \( s \) with \( x \leq z < s \), by (2.4) the only way this can occur is if some follower \( y \leq x \) enters \( A - A_r \). By (2.5) such a follower either cancels \( x \) or equals \( x \). In either case \( x \in A \iff x \in A_r \).

As with the case where \( \Phi_e(A) \) is r.e., the cancellation/confirmation procedure implemented for \( N_e \) also meets \( \hat{N} \). To see this, we must show that the cancellation of numbers between \( x \) and \( s \) in Figure 1 also allows \( A \) to w-compute \( \Phi_e(A) \). Let \( z \) be given. To compute whether \( z \in \Phi_e(A) \) first find the least stage \( s_1 \) where \( l(e, s_1) > z \) and \( l(e, s_1) > ml(e, s_1) \). Now \( A \) can only change (allowing \( \Phi_{e,s_1}(A_{s_1}) \) to change) due to the entry of followers. At stage \( s_1 \), the only such followers \( g \) left alive must be guessing that \( l(e, s) \to \infty \).

By the way we appoint followers (2.4), if no follower \( < s_1 \) enters \( A \) after stage \( s_1 \) it must be that

\[
\Phi_{e,s_1}(A_{s_1}; z) = \Phi_e(A; z).
\]

If \( \Phi_{e,s_1}(A_{s_1}; z) \) is to change, it follows that some follower \( g \) alive at stage \( s_1 \) must enter \( A - A_{s_1} \). Suppose \( g_1 \) is the first such, and \( g_1 \) enters at stage \( t \). Let \( s_2 \) be the least stage \( > t \) with \( l(e, s_2) > ml(e, s_2) \). Let \( \hat{g}_1 \) be the least follower that enters at any stage \( t' \) with \( t \leq t' < s_2 \). Then \( \hat{g}_1 \leq g_1 \) and \( \hat{g}_1 \) was present at stage \( s_1 \) (by (2.4)).

The crucial observation is:

(2.8) There are no followers \( x \) left alive with \( \hat{g}_1 \leq x < s_2 \) at stage \( s_2 \).

To see this first observe that by (2.5), when \( \hat{g}_1 \) enters \( A \)—say at stage \( \hat{t} \)—it must cancel all followers \( p \) with \( \hat{g}_1 \leq p \leq \hat{t} \). By choice of \( s_2 \) as the least stage \( > t \) with \( l(e, s_2) > ml(e, s_2) \), any follower \( q \) appointed after stage \( \hat{t} \) but before stage \( s_2 \) must be guessing that \( l(e, s) \to \infty \) (by (2.4)). But then we automatically cancel such \( q \) at stage \( s_2 \). These observations give (2.8).

Now, we see that after stage \( s_2 \) either \( A_{s_2}[\hat{g}_1] = A[\hat{g}_1] \) and so by (2.8), \( A_{s_2}[s_2 - 1] = A[s_2 - 1] \) implying that \( \Phi_{e,s_2}(A_{s_2}; z) = \Phi_e(A; z) \) or some number \( \leq \hat{g}_1 \) must enter \( A \) after stage \( s_2 \).

In the latter case, repeating the above process, we eventually arrive at a \( \hat{g}_2 \) and \( s_3 \) (say) etc. Combining all the above ideas, we get to our desired w-reduction: To compute \( \Phi_e(A; z) \), find the least stage \( \hat{s} > s \), with

\[
l(e, \hat{s}) > ml(e, \hat{s}) \quad \text{and} \quad A_s[s_1] = A[s_1].
\]
Then it must be that \( \Phi_{e,s}(A_s; z) = \Phi_{e}(A; z) \) since the only followers below \( u(\Phi_{e,s}(A_s; z)) \) alive at stage \( s \) were already present at stage \( s_t \).

The remaining details of the full construction are to organize the above strategies with the usual \( \pi \)-guessing tree. Should the reader be unfamiliar with this, we refer him to [1] for further details.

We now turn to the proof of Theorem B.

(2.3)' Every nonzero r.e. \( W \)-degree has the global anticupping property.

Proof. Let \( A \) be a given r.e. nonrecursive set. We construct a coinfinite r.e. set \( B = \bigcup_{s} B_s \) in stages to satisfy the following.

\( P_e \): \( |W_e| = \infty \) implies \( W_e \cap B \neq \emptyset \)

\( N_e \): If \( C \) is any set and \( \hat{\Delta}_e(B \oplus C) = A \) then \( A \leq_W C \).

We remind the reader that here \( \hat{\Delta}_e \) denotes the \( e \)-th \( W \)-reduction with use \( \gamma_e \).

This particular result gives a nice demonstration of the way some results for \( D_W \) can be obtained using techniques not applicable in the r.e. case. The reader should note that in this construction we cannot know \( C \) since there may be \( 2^{N_0} \) possibilities. The key point, though, is that no matter which \( C \) pertains the use \( \gamma_e \) is the same. There are several ways to satisfy condition \( N_e \) above, but it seems easiest to use a construction similar to one of Ladner and Sasso [13]. We shall use an "almost monotone" restraint \( r(e, s) \) which only drops when the "\( A \)-side" changes. To do this, we define a marking function \( \alpha(e, s) \) as follows: Let \( \alpha, \tau, \ldots \) denote strings. Define \( \alpha(e, 0) = 0 \). Set \( \alpha(e, s + 1) \) as the least \( x \) such that one of the following holds:

(i) \( \alpha(e, s) \) and \( x \in A_{s+1} - A_s \);

(ii) \( x > \alpha(e, s) \) and \( l(e, s) = x + 1 \) where

\[
l(e, s) = \max \{ y : \exists \sigma \forall z < y (\hat{\Delta}_e(B \oplus \sigma; z) = A_s(z)) \};
\]

(iii) (ii) does not apply and \( x = \alpha(e, s) \).

We shall then define

\[
r(e, s) = 1 + \max \{ \gamma_{e,s}(z) : z \leq \alpha(e, s) \}
\]

and

\[
R(e, s) = \max \{ r(j, s) : j \leq e \}.
\]

There are two crucial observations regarding the relationship of \( \alpha, r \) and \( A \).

\( (2.7) \) If \( l(e, s) > x, t_1, t_2 > s \) and \( \alpha(e, t_1) = \alpha(e, t_2) = x \) then \( r(e, t_1) = r(e, t_2) \). That is, once we see \( l(e, s) > x \) we always know what \( r(e, t) "\) will be", should \( x \) be the least number to occur in \( A_{t+1} - A_t \) for \( t > s \). We denote this by \( m(e, x) \), that is, we define \( m(e, x) = r(e, t) = 1 + \max \{ \gamma_e(z) : t \leq x \} \)
Note that \( y = \alpha(e, s + 1) \leq \alpha(e, s) \) if \( y = \mu z(z \in A_{s+1} - A_s) < \alpha(e, s) \). In particular, we ignore the \( B \)-side when it comes to dropping \( \alpha \).

**Construction, stage \( s + 1 \).** If \( W_{e,s+1} \cap B_s = \emptyset \) then put \( x \in B_{s+1} - B_s \) if \( x > 2e, x > R(e, s + 1), A_{s+1}[x] \neq A_s[x] \) and \( x \in W_{e,z} \) and \( x \) is least with these properties.

**Verification.** We only sketch some points due to their similarities with [13]. The reader should note that (2.7) allows us to show that all the \( P_e \) are met, by a permitting argument: For suppose \( P_e \) fails to be met. Let \( z \in \omega \) be given. Let \( s_1 \) be a stage such that

\[
\forall t \geq s_1(\alpha(j, t) = \alpha(j, s_1)) \quad \text{for } j \leq e \text{ with } l(j, s) \to \infty.
\]

To decide if \( z \in A \) or not find a stage \( s = s(z) > s_1 \) such that \( l(j, s) > z \) for all \( j \) with \( j \leq e \) and \( l(j, s) \to \infty \) (so that \( m(j, z) \) of (2.7) is defined) and such that \( y \in W_{e,z} \) with \( y > \max(2e, s_1, m(j, z); j \leq e) \). By the observation (2.7) should ever \( z \in A_s \), the restraints all drop so that we will be free to add \( y \) to \( A \) meeting \( P_e \). Hence \( A_s[z] = A[z] \) and so \( A \) is recursive, a contradiction.

Finally we verify \( N_e \). Suppose \( C \) is any set with \( \hat{1}_e(B \oplus C) = A \). Notice that appropriate \( \sigma \) exist to satisfy (ii) of the definition of \( \alpha(e, s) \) and so \( l(e, z) \to \infty \). Let \( z \) be given. Let \( \sigma(z) \) denote \( C[\gamma_e(z)] \). To \( C \)-recursively compute \( A(z) \) find the least stage \( s = s(z) \) such that

(i) all the \( P_j \) for \( j < e \) cease activity, and

(ii) \( \alpha(e, s) > z \) and \( \forall y \leq z(\hat{1}_e(x, B \oplus \sigma(z); y) = A_s(y)) \).

We claim that \( A_s[z] = A[z] \): For suppose otherwise. Let \( \hat{z} \leq z \) be the least number with \( \hat{z} \in A - A_s \). By (2.8) we see that for all \( i \geq s \), \( \alpha(e, t) \geq \hat{z} \), and furthermore by (2.7), \( r(e, t) \geq m(e, \hat{z}) \). In particular, \( B_i[\gamma_e(\hat{z})] = B[\gamma_e(\hat{z})] \).

But now we see that \( \hat{z} \)'s entry into \( A \) causes the (preserved) disagreement

\[
\hat{1}_e(B \oplus C; \hat{z}) = 0 \neq 1 = A(\hat{z}).
\]

We get the following slightly strengthened form of Theorem A:

**Theorem A'.** There exists an r.e. degree \( a \neq 0 \) such that for all r.e. degrees \( b \neq 0 \) with \( b < a \) there exists \( c \leq b \) with \( c \) a strong anticupping witness for \( a \). That is, strong anticupping witnesses are downward dense below \( a \).

**Proof.** Let \( A \) be r.e. and of strongly contiguous degree. Let \( B \) be r.e. with \( \emptyset \nless T B \nless T A \). By contiguity, \( B \leq_w A \). Now by (2.3)' let \( C \) be an r.e. set \( \leq_w B \) such that for all sets \( D \) if \( C \oplus D \geq_w B \), \( D \geq_w B \). Now suppose for some set \( E \) we have \( E \oplus C =_T A \). By strong contiguity, \( E \oplus D =_w A \). Hence by choice of \( C \), \( E \geq_w B \) and so \( E =_w A \).
There are various other applications of the above approach. One must decide whether or not the permitting-type reductions built in the appropriate r.e. \( W \)-degree constructions may be replaced by \( \Delta_2^0 \) permitting. Obviously, not all results on \( W \) may be changed in this way. For example Lachlan [La2] has shown that not every degree in \( W \) bounds a minimal pair (in \( W \)) (strictly speaking this is a \( T \)-degree result that also must work in \( W' \)), yet well known cone-avoidance full approximation arguments show that

\[
\forall a, b \in W (0 < a < b \rightarrow \exists c \in D W (c \cap a = 0 \text{ and } 0 < c < b)).
\]

In fact we may choose \( c \) of minimal \( T \)-degree. We refer the reader to [12] and [9]. One nice corollary of (2.3)' is:

(2.9) Theorem. Suppose \( a \) and \( b \) are \( W \)-degrees with \( a \geq b > 0 \) and \( b \) r.e. Suppose that \( c \) is a \( T \)-degree with \( c \geq 0' \). Then the elementary theories of the upper semilattices \( [0, a] \) and \( [0, c] \) are different. In the language \( L(\leq, \lor, 0, 1) \) the difference occurs by the two quantifier level.

Proof. By Posner and Robinson [15, Theorem 3] the following sentence \( \gamma \) is not satisfiable in \([0, c]::

\[
\gamma = \exists x (x \neq 0 \text{ and } \forall y (y \lor x \geq 1 \rightarrow y \geq 1)).
\]

However, by (2.3)', \( \gamma \) is satisfiable in \([0, a]::

To close this paper, we shall briefly point out a couple of further applications of \( \Delta_2^0 \) transfer techniques. One example—transferring "backwards"—concerns the structure of \( W \)-degrees in a given degree. An r.e. degree \( a \) is strongly \( W \)-bottomed if there is an r.e. set \( A \) of degree \( a \) such that for all sets \( B \) of degree \( a \), \( A \leq_W B \). It is unknown whether there is a nice characterization of such degrees. It is conjectured that they all must be \( \text{low}_2 \), since all contiguous degrees are \( \text{low}_2 \) (Cohen [4]). We prove a weaker result.

(2.10) Theorem. No high degree is strongly \( W \)-bottomed.

Proof. Let \( A \) of degree \( a \) be the r.e. strong \( W \)-bottom. Let \( B \leq_W A \) be a global antichipping witness for \( A \) given by (2.3)' . Notice \( B \triangleleft_T A \). Now by Epstein's theorem [9] there is an \( \Delta_2^0 \) set \( C \) such that \( C \triangleleft_T A \) and \( C \ominus B \equiv_T A \). Now \( A \leq_W C \ominus B \) by choice of \( A \). But then \( A \leq_W C \) by choice of \( B \), a contradiction.

As our last example we again modify a construction from [13]

(2.11) Theorem. Every strongly contiguous r.e. degree \( a \) strongly splits over all lesser \( \Delta_2^0 \) degrees in the sense that if \( b \) is a \( \Delta_2^0 \) degree \( < a \) then there exist r.e. degrees \( a_1, a_2 \) with \( a_1 \cup a_2 = a \) and \( b \cup a_1, b \cup a_2 < a \).
This result follows from:

(2.12) **Lemma.** All r.e. \( W \)-degrees strongly split as above over all lesser \( \Delta^0_2 \) \( W \)-degrees.

**Proof.** We briefly indicate how to modify the proof from [13] using a marking function \( \alpha(e, i, s) \) as in (2.3)'. Let \( A = \bigcup_i A_i \) and \( B = \lim_i B_i \) be given recursive enumerations with \( B \prec_w A \). We need to construct a r.e. splitting \( A = A_0 \cup A_1 \) satisfying

\[
R_{e, i}; \hat{\Phi}_e(B \oplus A_i) \neq A_{i-1}.
\]

Now we define \( \alpha(e, i, s) \). Let \( \alpha(e, i, s) = 0 \) and let \( \alpha(e, i, s + 1) \) be the least \( y \) such that one of the following holds:

(i) \( y < \alpha(e, i, s) \) and \( y \in A_{i-1, s+1} - A_{i-1, s} \),

(ii) \( y \geq \alpha(e, i, s) \) and \( l(e, i, s) = y \) where

\[
l(e, i, s) = \max\{ z : \forall z < y(\hat{\Phi}_{e, s+1}(B_{s+1} \oplus A_{i, s+1}; y) = A_{i-1, s+1}(y))\}.
\]

(iii) (ii) does not pertain and \( y = \alpha(e, i, s) \).

Let \( r(e, i, s) \) be \( 1 + \sum_{z \leq \alpha(e, i, s)} \gamma_{e, i}(z) \).

Now one performs the usual Sacks splitting construction, but with \( r(e, i, s) \) in place of the usual Sacks restraints. Then a permitting argument ensures that all the \( R_{e, i} \) above are eventually met by a finite restraint (or else \( A \leq_w B \)). We refer the reader to [13] for further details.

The famous nonsplitting result of Lachlan [11] shows that (2.11) fails for arbitrary r.e. a (even for b r.e.). We do not know if theorem (2.11) is valid if we replace "a strongly contiguous" by "a low_2". The relevant result here is Bickford and Mills' [3] and Harrington's (unpublished) result that all r.e. low_2 degrees split over all lesser ones.

**References**


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