ON HYPER-TORRE ISOLS

ROD DOWNEY

§1. Introduction. As Dekker [3] suggested, certain fragments of the isols can exhibit an arithmetic rather more resembling that of the natural numbers than the general isols do. One such natural fragment is Barback’s “tame models” (cf. [2], [6] and [7]), whose roots go back to Nerode [8]. In this paper we study another variety of such fragments: the hyper-torre isols introduced by Ellentuck [4]. Let $Y$ denote an infinite isol with $D(Y)$ the collection of all isols $A \leq f_A(Y)$ for some recursive and combinational unary function $f$. (Here, as usual, $f_A$ is the Myhill-Nerode extension of $f$ to the isols).

(1.1) DEFINITION [4]. An isol $Y$ is called hyper-torre if $Y$ is infinite, regressive and for all $m \geq 1$, recursive sets $\alpha \subseteq (\omega)^m$ and $A \in (D(Y))^m$ we have $A \in \alpha_A \cup \bar{\alpha}_A$.

The pretty fact discovered by Ellentuck concerning such $Y$’s is

THEOREM (ELLENTUCK [4]). If $Y$ is hyper-torre then the universal theory of $(D(Y), +, \cdot)$ is the same as $(\omega, +, \cdot)$.

Although Ellentuck directly constructed such isols, it was subsequently discovered that in 1976 Harrington (cf. [5, Chapter 20]) had constructed a hyper-torre isol. In a remarkable theorem Barback [1] (see [5, Chapter 20]) showed that $Y$ is hyper-torre iff $Y$ is infinite, regressive, and hereditarily odd-even (i.e. for all $A \leq Y$ either $A$ is even or $A$ is odd). Harrington constructed a hereditarily odd-even isol. Harrington’s construction is a quite elaborate minimal-degree type tree construction, and Ellentuck’s is a modification of this. Both constructions seem very different from the types of construction that lead to $\Sigma^0_1$ or $\Sigma^1_1$ sets. It has since become a well-known open question in isol theory whether or not hyper-torre isols can exist in the co-simple isols. This appears as Question 9 in [5]. In this paper we solve McLaughlin’s question affirmatively by proving

(1.2) THEOREM. There exists a co-simple hyper-torre isol.

In §2 we give the proof. In §3 we discuss some variations. Our construction uses a $0''$-priority argument, and we refer to reader to Soare [9] for any unexplained notation or terminology, and for motivation for this technique. The heart of the

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paper is the discussion preceding the formal construction. As usual all computations etc. are bounded by s at stage s.

We let \( \{ \phi_e \}_{e \in \omega} \) be a standard enumeration of all one-to-one partial recursive functions, and we let \( \lambda \) denote the empty string.

§2. The proof. We shall build an r.e. set \( A = \bigcup_s A_s \) in stages. At each stage s, we let \( \{ a_{i,s} : i \in \omega \} \) enumerate \( A_s \). The desired set will be \( A = \{ a_i : i \in \omega \} \), where \( a_i = \lim_{s \to \infty} a_{i,s} \). We must build \( A \) co-traceable satisfying the requirements below:

\( N_e : \) \( \lim_{s \to \infty} a_{e,s} = a_e \) exists.

\( R_e : \) If \( \text{dom} \, \phi_e \supset A \) then \( \phi_e(\bar{A}) \cap 2\omega \) is either even or odd.

\( P_e : \) \( \text{card}(W_e) = \infty \to W_e \cap A = \emptyset \).

To ensure that \( A \) is co-traceable we use a dump construction for \( A \). Namely if \( a_t = \mu x (x \in A_{t+1} - A_t) \) then \( i < s \) and \( A_{s+1} = A_s \cup \{ a_j : i \leq j \leq s \} \). It is then easy to see that \( \bar{A} \) is traceable. The relevant retracting function \( g \) can be defined as follows. Let \( z \) be given. To define \( g(z) \), go to stage \( z \). If \( z \notin \bar{A} \) define \( g(z) \) arbitrarily. If \( z \in \bar{A} \) then \( z = a_{r,s} \) for some \( r \). If \( i = 0 \) define \( g(z) = z \), and otherwise define \( g(z) = a_{r-1,s} \).

To meet the \( P_e \) we wait for an unrestrained element \( x \) to occur in \( W_e,s \) (whilst \( W_e,s \cap A_x = \emptyset \)) and enumerate \( x \) into \( A_{s+1} \), causing \( W_e,s+1 \cap A = \emptyset \). As we shall see, this requirement interacts quite strongly with the \( N_e \) and \( R_e \) of higher priority. As a consequence there will be several "guessed" versions of the restraint \( \text{"r}(e,s)\) attempting to protect \( a_{r,s} \) for \( j \leq e \). The supremum of the overall restraint \( \text{inf}[\text{r}(e,s)] \) is finite, so we can eventually meet \( P_e \).

Now we turn to the key requirements, the \( R_e \). First we give the basic module.

The fundamental idea we use to meet \( R_e \) is what we call binding. Define

\[ l(e,s) = \max \{ x : (\forall y < x)(\phi_{a,x}(a_{y,s}) \downarrow) \} \]

The process is quite simple for a single \( R_e \). For the sake of \( R_e \) we shall wait till there occurs a stage \( s \) and least unbound numbers \( a_{i,s}, a_{j,s} \) with \( i < j < l(e,s) \) and such that \( \phi_{a,e}(a_{i,s}) \neq \phi_{a,e}(a_{j,s}) \in 2\omega \). We call such a stage e-expansory since we know we have two new numbers to deal with. At this stage \( s \) we declare \( a_{i,s} \) and \( a_{i,s} \) as bound (together); we then promise that \( a_{i,s} \in A \) iff \( a_{j,s} \in A \) (or, in effect, we extend a partial recursive function \( h \) we are defining for the sake of \( R_e \) by setting \( h(a_{i,s}) = a_{j,s} \)). The reader should note that for a single \( R_e \) the effect is that if card \( (\phi_e(\bar{A}) \cap 2\omega) = \infty \) then \( \phi_e(\bar{A}) \cap 2\omega \) is even. This follows since if \( a_i \in A \) and \( \phi_e(a_i) \in 2\omega \) then \( a_i \) is bound to some unique \( a_j \).

How does this strategy cohere with the other requirements? For the \( P_e \) and \( N_k \) cooperating with \( R_e \) the problem is this. Suppose \( P_e \) has lower priority than \( N_k \), so that \( P_e \) does not wish to move \( a_{k,s} \). Suppose \( P_e \) has been assigned restraint \( \text{r}(j,s) \) and \( a_{m,s} > \text{r}(j,s) \), so that \( a_{m,s} \) is currently free to be used to \( P_e \). Now perhaps \( R_e \) acts and binds \( a_{n,s} \) to \( a_{m,s} \), but the problem is that \( n \leq k \). Now if we enumerate \( z \leq a_{m,s} \) into \( A_{t+1} = A_t \) we must fulfill our \( R_e \) commitments (perhaps \( e < k \)). Thus we must also enumerate \( a_{n,s} \) into \( A \) too allowing \( P_e \) to injure \( N_k \) due to its interaction with \( R_e \).

The solution to this dilemma is to simply reset the restraint and so ensure that if \( z \) enters \( A_{t+1} - A_t \) for \( t > s \) then \( z = a_{p,s} \) for some \( p > m \) such that, for all \( q \geq p \), \( a_{q,s} \) is not bound to some \( a_{r,s} \) for \( j \leq k \).
For the sake of the "\(\alpha\)-module" below we shall introduce a little more terminology. In the "\(\alpha\)-module", it is no longer possible to use pairs bound together. Rather we will use even finite collections of elements bound in a block. To facilitate this we refer to the least element \(x\) of a collection that is \(e\)-bound (in the pair situation above this refers to \(a_{i,s}\)) as the lower boundary of its block. Let \(x = a_{i,s}\). If \(a_{k,s}\) is the least element with \(k > i\) such that \(a_{k,s}\) is not in \(x\)'s \(e\)-block then we refer to \(a_{k-1,s}\) as the upper boundary of \(x\)'s \(e\)-block. Note that we will try to ensure that there are an even number of \(y\) between these boundaries, that is, in \(x\)'s \(e\)-block with \(\varphi_\varepsilon(y) \in 2\alpha\). Also the upper boundary of a block is determined by the next block. We shall promise that if any element of an \(e\)-block enters, all of the \(e\)-block enters together.

(2.1) The \(\alpha\)-module and coherence of the \(R_e\). Now we discuss the \(\alpha\)-module, that is, the modifications to the basic module to allow it to cohere with all the other \(R_f\) along the "true path" of the construction. Consider two versions \(R_e, R_f\) with \(e < f\). The problem with the basic module is this.

Consider the situation which might occur, for example, for all \(j \leq s\) with \(j \equiv 0, 1\) (mod 4), \(\varphi_{a,4}(a_{j,s}) \in 2\alpha\), and with \(j \equiv 2, 3\) we have \(\varphi_{a,2}(a_{j,s}) \in 2\alpha\). There may occur some \(n\) such that, if we bound \(e\) and \(f\) as in the basic module, we would have \(a_{n,s}\) \(e\)-bound to \(a_{n+4,s}, a_{n+5,s}\), to \(a_{n+4,s}, a_{n+5,s}\), to \(a_{n+12,s}\), etc.; and perhaps \(a_{n+3,s}\) \(f\)-bound to \(a_{n+6,s}, a_{n+7,s}\) to \(a_{n+10,s}, a_{n+11,s}\), to \(a_{n+12,s}\), etc. Then it is easy to see that these bindings have become interlocked, and if, for example, we enumerated \(a_{n+14,s}\) into \(A_{s+1} - A_s\) then this would cause \(a_{n+11,s}\) and so (by the dump) \(a_{n+12,s}\) and eventually \(a_{n,s}\) to enter \(A_{s+1} - A_s\).

Situations like this mean that \(R_f\) must be more careful with its bindings. The fundamental idea for the \(\alpha\)-module is that, if \(R_f\) is guessing infinitary \(R_e\) behaviour, then \(R_f\) must attempt to bind within \(e\)-boundaries. \(R_f\) views this as follows. If \(x < y\) are \(e\)-bound, then if \(x\) is \(f\)-bound to some \(q\) with \(z < y\) then \(R_f\) has really bound all of \(x, z, y, q\) even if \(x < z\).

Now if \(x\) lies within a previously defined \(f\)-block we have, in turn, really extended a previously set \(f\)-block. Suppose that the \(e\)-blocks (at stage \(s\)) appear as \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), where \(x_i, y_i\) denotes the lower (upper) \(e\)-boundary. Thus we note that \(y_1 = a_{j-1,s}\) if \(x_s = a_{j,s}\). Our task is to try to define our \(f\)-blocks so as not to interlock all of \(A\).

Now the reader should note that if for each \(i\) there are only an even number of \(z\) with \(z = a_{k,s}\) some \(k, x_i \leq z \leq y_i\) and \(\varphi_{f,i}(z) \in 2\alpha\), there is no real problem. We can simply use the same boundaries as \(e\) to be the boundaries of the \(f\)-blocks. Then we keep happy whilst getting no interlocks since none of \(e\)'s boundaries are crossed by \(f\)-bindings. The only remaining problem is if for some \(i, (x_i, y_i)\) contains only an odd number of \(z = a_{k,s}\) with \(\varphi_{f,i}(z) \in 2\alpha\). The key rule is that we only ever bind even numbers of such \(z\). Without loss, we may suppose that \((x_1, y_1)\) is the least such \(e\)-block with an odd number. Our solution to this problem is this. We begin anew defining a new blocking predicated on the assumption that \((x_1, y_1)\) is the only such \(f\)-odd \(e\)-block. Thus provided \((x_3, y_3), \ldots, \) all turn out to be \(f\)-even we do not \(f\)-block \((x_1, y_1)\) at all but simply work on \((x_j, y_j)\) for \(j \geq 2\). Obviously if this outcome is the correct one then again there are no problems; \(\varphi_\varepsilon(\overline{A})\) will be even plus a finite number. Finally, should we see some least \((x_j, y_j)\) for \(j \geq 2\) with \((x_j, y_j)\) also \(f\)-odd, we shall cancel our previous \(f\)-blockings and define our new \(f\)-block as \((x_1, y_1)\). Note
at although our new $f$-block consists of $j$ $e$-blocks the trade-off is that we have no
ending commitment to $(x_1, y_1)$ any more. (It is this outcome which necessitates
changing our strategy from the basic module.) There is obviously no problem with $n$
$2$ strategies since they will be inductively defined as above. Since each of the above
determined by infinite recursive collections of $\Sigma_1$-events, the strategies can be, as
usual, handled by a standard $\Pi_2$ guessing tree. We expect that readers familiar with
such arguments will provide the details for themselves. However, for completeness,
e give some formal details below.

Let $A = \{oe, o, e, w\}$ ordered in the manner given (i.e. $oe <_A o <_A e <_A w$). The
terpretation here is that $w$ means “wait”, $e$ means “even”, $o$ means “odd, the rest
then” and $oe$ means “infinitely many changes from odd to even”. The priority tree is

$A^<_w$.

We refer to $\sigma, \tau \in T$ as *guesses* and let $\sigma \leq \tau$ mean that $\sigma$ is an initial segment of $\tau$.
the priority ordering $\leq_L$ is the standard lexicographic ordering: thus $\sigma \leq_L \tau$ if
or $\exists y(x^i \leq \sigma \& \gamma^i \leq \tau \& i <_L j)$. If $\sigma \in T$ and $lh(\sigma) = e$ we denote $\sigma$ to
satisfy $R_\sigma$, and $\sigma^i$ for $i \in A$ are the outcomes of $\sigma$. We replace the idea of $e$-bound
\& $i$-bound for such $\sigma$. Of course $\sigma$ encodes the guess as to the behaviour of
the other priority requirements (which act infinitely often). For $j \in \{oe, o, e\}$ we let
$\sigma^i, i, s$ and $y(\sigma^i, i, s)$ denote, respectively, the lower and upper boundaries of
the irreducible block. For $\sigma^o$ we will also define a critical block $Q(\sigma, s)$ which will be the
unique pending odd block for which we are waiting for another odd block. We let

$B(\sigma^i, i, s) = \{z: z = a_{k,s} \text{ for some } k \& x(\sigma^i, i, s) \leq z \leq y(\sigma^i, i, s)\}$.

In the construction to follow we use the phrase “initialize $\sigma$”. This means cancel all
strains $r(\sigma, s)$ and declare as undefined all $x(\sigma, i, s), y(\sigma, i, s), \text{ etc.}$ Also define $c(\sigma, s)$
the current state of the control of the $\sigma$-module) to be *w*. Note that in the
struction to follow any parameter not specifically reset is simply extended to the
xt stage without change.

If (for example) $x = x(\sigma, c, s)$ is defined and we enumerate $x$ into $A_{s+1}$ it then (of
ourse) becomes undefined. Moreover if $\sigma = \tau^o$ in the above and $x \in Q(\sigma, s)$, then
 reset $c(\tau, s+1) = oe$ (from $o$).

We say $P_e$ requires attention at stage $t$ of stage $s$ if $W_{\epsilon,s} \cap A_s = \emptyset$ and

$3x(\sigma \in W_{\epsilon,s} \& x > r(\sigma, s), \text{ where } \sigma = \sigma(t, s) \text{ and } e + 1 = lh(\sigma))$.

$(t, s)$ is defined in the construction.)

**Construction. Stage 0.** Define $\sigma_0 = \lambda$ and initialize all $\sigma \in T$.

*Stage s + 1.* At stage $s + 1$ we proceed in stages $t \leq s + 1$.

*Substage 0. Define $\sigma(0, s + 1) = \lambda$. *

*Substage t + 1, part 1.* Let $\sigma = \sigma(t, s + 1)$ and $lh(\sigma) = f$. Adopt the first case
low to pertain to $\sigma$, defining $m(\sigma, s + 1) = \max\{r(y, u), a_{\gamma,s}, y(\gamma, h, u): \gamma \leq_L \sigma \& \leq
lh(\sigma) \& x(\gamma, h, s + 1) \leq \min\{r(\gamma, u), a_{\gamma,s}\} \& u \leq s + 1\}$.

*Case 1. $\forall t \exists \sigma(\tau^w \leq \sigma)$. *

*Subcase 1. $c(\sigma, s) = w$. See if there exist least $k > j \geq m(\sigma, s + 1)$ with
$s(a_{k,s}) \leq 0$ and $\Phi(\sigma, a_{k,s}) \in 2\omega$ for all $q \in \{i, j, k\}$. If so define $y(\sigma^e, 0, s + 1) = a_{k-i,s}
and $x(\sigma^e, 0, s + 1) = \min\{a_{k,s}, m(\sigma, s + 1) < a_{k,s}\}$. Note that $p \leq i$ and we have$\Phi(\sigma^e, 0, s + 1)$. \
Now set $\sigma(t+1, s+1) = \sigma^e$ and if $t = s$ define $\sigma_{t+1} = \sigma^e$. Set $r(\sigma^e, s + 1) = y(\sigma^e, 0, s + 1) = \delta(\sigma^e, s + 1)$ and $c(\sigma, s+1) = e$. Go to part 2.

If no $k, j, i$ exist as above then define $\sigma(t + 1, s + 1) = \sigma^w$ and keep $c(\sigma, s+1) = w$. Now set $r(\sigma, s+1) = m(\sigma^w, s + 1)$ and if $t = s$ define $\sigma_{t+1} = \sigma(t + 1, s + 1)$. Now go to part 2.

Subcase 2. $c(\sigma, s) \neq w$. We claim $c(\sigma, s) = e$ in this case. (The reader should verify this as an easy induction on the construction.) There will be a greatest block $B(\sigma^e, g, s)$ defined at stage $s$. Let $y(\sigma^e, g, s) = a_{p,s}$. See if there exist least $k > j > i > p$ such that $\varphi_{\epsilon, \eta}(a_{p,s}) \downarrow$ and $\varphi_{\epsilon, \eta}(a_{p,s}) \in 2\omega$ for $q \in \{i, j, k\}$. If so, define

$$x(\sigma^e, g + 1, s + 1) = a_{p+1,s} \quad \text{and} \quad y(\sigma^e, g + 1, s + 1) = a_{k-1,s}.$$

Now set $\sigma(t + 1, s + 1) = \sigma^e$. (This is impossible here. To have $c(\sigma, s) = e$ necessitates a previous visit to $\sigma$.) Now go to part 2.

If no such $k, j, i$ exist as above, then define $\sigma(t + 1, s + 1) = \sigma^w$ and go to part 2. (Note that here $c(\sigma, s+1) = e$ and $r(\sigma^w, s + 1)$ remain the same for all $q$.)

Case 2. $\exists \tau \subseteq \sigma \land (\tau^w \notin \sigma)$. Let $\tau$ denote the longest such $\tau$ and let $\tau^q \subseteq \sigma$. Note that $q \notin \{\varnothing, o, e\}$. Adopt the first case below to pertain.

Subcase 1. $c(\sigma, s+1) = w$. See if there exists a least block $B(\tau^q, i, s + 1)$ with $l(f, s + 1) > y(\tau^q, i, s + 1)$ and $x(\tau^q, i, s + 1) > m(\sigma, s + 1)$. If none exists define $\sigma(t + 1, s + 1) = \sigma^w$ and $r(\sigma^w, s + 1) = m(\sigma^w, s + 1)$. If $t = s$ define $\sigma_{t+1} = \sigma^w$. Go to part 2.

If one exists, let $d = \operatorname{card}\{z : \varphi_{\epsilon}(z) \in 2\omega \land z \in \tilde{A}_x \land x \leq z \leq y\}$. If $d$ is even, define $x(\sigma^o, 0, s + 1) = x$ and $y(\sigma^o, 0, s + 1) = y$ if set $\sigma(t + 1, s + 1) = \sigma^e$ and $c(\sigma, s + 1) = y$ and $r(\sigma^e, s + 1) = m(\sigma^e, s + 1)$ and $C(\sigma, s + 1) = 0$. If $t = s$ define $\sigma_{s+1} = \sigma^o$. Go to part 2.

Subcase 2. $c(\sigma, s + 1) = e$. Let $B(\sigma^e, i, s)$ be the largest currently defined $\sigma^e$-block. Then $y(\tau^q, i, s) = y(\tau^q, j, s + 1)$ for some $j$. If

$$l(f, s + 1) > y = y(\tau^q, j + 1, s + 1)$$

(this and is defined), let

$$d = \operatorname{card}\{z : \varphi_{\epsilon}(z) \in 2\omega \land z \in \tilde{A}_x \land \tau^q \subseteq \omega \land x \leq z \leq y\}.$$

If $d$ is even, define $x(\sigma^o, i + 1, s + 1) = x$ and $y(\sigma^o, i + 1, s + 1) = y$ and set $\sigma(t + 1, s + 1) = \sigma^e$. If $d$ is odd, define

$$x(\sigma^o, k, s + 1) = x(\sigma^o, k, s + 1), \quad y(\sigma^o, k, s + 1) = y(\sigma^o, k, s + 1)$$

for all $k \leq i$, and $C(\sigma, s + 1) = 0$. Also define $x(\sigma^o, i + 1, s + 1) = x$ and $y(\sigma^o, i + 1, s + 1) = y$ and set $C(\sigma, s + 1) = 0$. Go to part 2.

If $l(f, s + 1) > y$ define $\sigma(t + 1, s + 1) = \sigma^w$.

Subcase 3. $c(\sigma, s + 1) = o$. Let $B(\sigma^o, i, s)$ denote the largest currently defined $\sigma^o$-block. Then $y(\sigma^o, i, s) = y(\tau^q, j, s + 1)$ for some $j$. See if $l(f, s + 1) > y = y(\tau^q, j + 1, s + 1)$. If not, define $\sigma(t + 1, s + 1) = \sigma^w$ and go to part 2.

If so, let $d = \operatorname{card}\{z : \varphi_{\epsilon}(z) \in 2\omega \land x(\tau^q, j, s + 1) \leq z \leq y \land z \in \tilde{A}_x\}$.
Subcase 1. \(d\) is even. Define \(\gamma(x^{\omega}, o, i + 1, s + 1) = \gamma\) and \(x(x^{\omega}, o, i + 1, s + 1) = x\). Set \(\sigma(t + 1, s + 1) = \sigma^{\omega}\) and go to part 2.

Subcase 2. \(d\) is odd. Now \(Q(\sigma, s + 1)\) is defined and

\[
Q(\sigma, s + 1) = B(\sigma^{\omega}, o, k, s + 1)
\]

for some \(k \leq i\). Define

\[
x(\sigma^{\omega}e, i + 1, s + 1) = x(\sigma^{\omega}e, i + 1, s + 1) = x(\sigma^{\omega}o, k, s + 1),
\]

and \(y(\sigma^{\omega}e, i + 1, s + 1) = y(\sigma^{\omega}e, i + 1, s + 1) = y\). For all \(m \leq i\) define

\[
x(\sigma^{\omega}o, m, s + 1) = x(\sigma^{\omega}o, m, s + 1) = x(\sigma^{\omega}o, m, s + 1)
\]

and similarly \(y\). Notice that under this identification we give \(x(\sigma^{\omega}e, n, s + 1)\) the same priority as \(x(\sigma^{\omega}o, n, s + 1)\), and so we do not cancel \(x(\sigma^{\omega}e, n, s + 1)\) henceforth unless we also cancel \(x(\sigma^{\omega}e, n, s + 1)\). Set \(r(\sigma^{\omega}o, s + 1) = m(\sigma^{\omega}o, s + 1)\). Now define \(\sigma^{\omega}o = \sigma(t + 1, s + 1)\) and \(c(\sigma, s + 1) = \epsilon\).

Part 2. Having computed \(\sigma = \sigma(t + 1, s + 1)\), see if \(P_{\text{ht}(\sigma)} - 1\) requires attention. Let \(f = \text{lh}(\sigma) - 1\). If not, go to substage \(t + 1\) unless \(t = s\). If \(t = s\) go to part 3. If \(P_f\) requires attention via \(z\), say, let \(z = a_{i,z}\). For any \(\tau^\gamma q \leq \tau^\gamma\) and \(q \in \{\gamma, o, e, o\}\), if \(a_{i,z} \leq y(\tau^\gamma q, n, s + 1)\) declare \(a_{i,z}\) as bound to \(x(\tau^\gamma q, n, s + 1)\). Find the least such \(x\). Then (by induction) \(x = a_{p,s}\) for some \(g\). We set \(A_{x+1} = A_x \cup \{a_{p,s} : p \leq s\}\). Now set \(\sigma_{s+1} = \sigma(t + 1, s + 1)\). Go to part 3.

Part 3. Initialize all \(\gamma \not\leq \tau^\gamma\).

End of construction.

Verification (sketch). Let \(\beta\) denote the true path of the construction. Thus \(\beta \in [T]\) is defined inductively via \(\lambda \leq \beta\). If \(\tau \leq \beta\) then \(\tau^\gamma \leq \beta\) if there are infinitely many \(\tau^\gamma\)-stages (that is, when \(\tau^\gamma = \sigma(t + 1, s + 1)\)) and only finitely many \(\tau^\gamma\)-stages for \(\tau^\gamma < \tau^\gamma\). To see that \(\beta\) exists, first note that \(P_1\) can receive attention at most once. Thus \(\beta\) exists since once \(\text{lh}(\sigma_{s+1}) > z\), we can have \(\text{lh}(\sigma_{s+1}) \leq z\) only if \(P_2\) receives attention for some \(i \leq z - 1\). To see that \(P_2\) is met it suffices to argue that \(f = \text{lh}(\tau)\) if \(\sigma = \tau^\gamma n \leq \beta\) then \(\text{lim}_n r(\gamma, s)\) exists for \(\gamma \leq \tau^\gamma n\).

For an induction, find a stage \(s_0\) such that for all \(s > s_0\)

\[
a) \gamma \leq \tau^\gamma \sigma \& \gamma \not\in \beta \rightarrow s \text{ is not a } \gamma\text{-stage},
b) \forall j \leq \text{lh}(\sigma)(P_j \text{ does not receive attention at stage } s), \text{ and}
c) (\forall j)(r(\gamma, s) = r(\gamma, s_0)).
\]

Note that a), b) and c) ensure that for all \(\gamma \leq \tau^\gamma \sigma\) if \(\gamma \not\in \beta\) then \(r(\gamma, s) = r(\gamma, s_0)\). Restraints are only reset at \(\gamma\)-stages for those \(\gamma \leq \tau^\gamma \sigma\) with \(\gamma \not\in \beta\) by \(\sigma = \tau^\gamma w\) then \(r(\sigma, s_0) = r(\sigma)\). Let \(R = \max\{a_{\text{ht}(\sigma)}, r(\gamma, s_0) : \gamma \leq \tau^\gamma \sigma\}\). If \(\sigma \not\in \tau^\gamma o\) compute a \(\sigma\)-stage \(s > s_0\) where \(x(\sigma, n, s) > R\). Then \(r(\sigma, s) = r(\sigma)\), since this will be specifically protected in the construction.

It is implicit here that in the construction, until we see such an \(x(\sigma, n, s)\), the \(P_j\) of lower priority than \(\sigma\) are restrained from enumerating \(x(\sigma, i, s)\). It follows that \(\text{lim}_n r(\sigma, s) = r(\sigma)\) exists and similarly all the \(N_{\epsilon}\) are met.

Finally, the \(R_e\) are met for exactly the reasons discussed before the formal construction. If \(\tau^\gamma e\) or \(\tau^\gamma oe\) are on the true path then almost all elements are in even blocks. If \(\tau^\gamma o \leq \beta\), then one block \(\text{lim}_n Q(\sigma, s) = Q(\sigma)\) is odd and all those exceeding
it are even. The formal verification of this provides absolutely no further insight, and we leave it to the reader.

§3. Variations and comments. One can add permitting to the above construction as follows. For the sake of $P_e$ one defines followers. For example, use $x = x(\sigma, i, s)$ to indicate $x$ is the current ith follower at guess $\sigma$, where $e = \text{lh}(\sigma)$. A follower is realized when $\bar{x} \in W_{\sigma, e}$ for some $x(\sigma, i - 1, s) < \bar{x} < x(\sigma, i, s)$. Whilst $A_\sigma \cap W_{\sigma, e} = \emptyset$, when $x(\sigma, i, s)$ becomes realized we initialize all $y \leq \sigma$ and define a new follower $x(\sigma, i + 1, s)$ in such a way that it does not violate $x(\sigma, i, s)$'s block and ensure that $y \leq \sigma$ cannot violate $x(\sigma, i + 1, s)$ by restraint. To meet $P_e$ we enumerate $y \in W_{\sigma, e}$ as before; if $W_{\sigma, e} \cap A_\sigma = \emptyset$, $y$ is unrestrained (as before), and additionally ask that $y > \bar{x}$ for some realized $x$ with $\bar{x}$ permitted we will enumerate as before (defining $\sigma_s$ to be $\sigma$). With this modification we get

(3.1) THEOREM. Let $C$ be any r.e. nonrecursive set. Then there exists a co-simple hyper-torre isol $A \leq_T C$.

Here, of course, we use the fact that retraceable isols have well-defined degrees (see [5, Chapter 5]).

I do not know if all nonzero r.e. degrees contain $\Pi_1$ hyper-torre isols. Indeed it is not even clear if $0'$ contains such an isol. It seems to me that a new construction would be needed to get this. I conjecture that not all r.e. nonzero degrees contain such isols.

REFERENCES


MATHEMATICS DEPARTMENT
VICTORIA UNIVERSITY
WELLINGTON, NEW ZEALAND