ON THE ORBITS OF COMPUTABLE ENUMERABLE SETS

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ABSTRACT. The goal of this paper is to show there is a single orbit of the c.e. sets with inclusion, $\mathcal{E}$, such that the question of membership in this orbit is Σ_1^1-complete. This result and proof have a number of nice corollaries: the Scott rank of $\mathcal{E}$ is $\omega_1^{\mathrm{CK}}+1$; not all orbits are elementarily definable; there is no arithmetic description of all orbits of $\mathcal{E}$; for all finite $\alpha \geq 9$, there is a properly Δ_α^0 orbit (from the proof).

1. Introduction

In this paper we work completely within the c.e. sets with inclusion. This structure is called $\mathcal{E}$.

Definition 1.1. $A \approx \hat{A}$ iff there is a map, Φ, from the c.e. sets to the c.e. sets preserving inclusion, $\subseteq$, (so $\Phi \in \mathrm{Aut}(\mathcal{E})$) such that $\Phi(A) = \hat{A}$.

By Soare [19], $\mathcal{E}$ can be replaced with $\mathcal{E}^*$, $\mathcal{E}$ modulo the filter of finite sets, as long as A is not finite or cofinite. The following conjecture was made by Ted Slaman and Hugh Woodin in 1989.

Conjecture 1.2 (Slaman and Woodin [18])**.** *The set* $\{\langle i, j\rangle : W_i \approx W_j)\}$ *is* Σ_1^1*-complete.*

This conjecture was claimed to be true by the authors in the mid 1990s; but no proof appeared. One of the roles of this paper is to correct that omission. The proof we will present is far simpler than all previous (and hence unpublishable) proofs. The other important role is to prove a stronger result.

Theorem 1.3 (The Main Theorem)**.** *There is a c.e. set A such that the index set* $\{i : W_i \approx A\}$ *is* Σ_1^1*-complete.*

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As mentioned in the abstract this theorem does have a number of nice corollaries.

Corollary 1.4. *Not all orbits are elementarily definable; there is no arithmetic description of all orbits of $\mathcal{E}$.*

Corollary 1.5. *The Scott rank of $\mathcal{E}$ is $\omega_1^{\mathrm{CK}} + 1$.*

Proof. Our definition that a structure has Scott rank $\omega_1^{\mathrm{CK}} + 1$ is that there is an orbit such that membership in that orbit is Σ^1_1-complete. There are other equivalent definitions of a structure having Scott Rank $\omega_1^{\mathrm{CK}} + 1$ and we refer the readers to Ash and Knight [1]. □

Theorem 1.6. *For all finite $\alpha > 8$ there is a properly Δ^0_α orbit.*

Proof. Section 3 will focus on this proof. □

1.1. **Why Make Such a Conjecture?** Before we turn to the proof of Theorem 1.3, we will discuss the background to the Slaman-Woodin Conjecture. Certainly the set $\{\langle i, j\rangle : W_i \approx W_j)\}$ is Σ^1_1. Why would we believe it to be Σ^1_1-complete?

Theorem 1.7 (Folklore[1]). *There is a computable listing, $\mathcal{B}_i$, of computable Boolean algebras such that the set $\{\langle i, j\rangle : \mathcal{B}_i \cong \mathcal{B}_j\}$ is Σ^1_1-complete.*

Definition 1.8. We define $\mathcal{L}(A) = (\{W \cup A : W \text{ a c.e. set}\}, \subseteq)$ and $\mathcal{L}^*(A)$ to be the structure $\mathcal{L}(A)$ modulo the ideal of finite sets, $\mathcal{F}$.

That is, $\mathcal{L}(A)$ is the substructure of $\mathcal{E}$ consisting of all c.e. sets containing A. $\mathcal{L}(A)$ is definable in $\mathcal{E}$ with a parameter for A. A set X is finite iff all subsets of X are computable. So being finite is also definable in $\mathcal{E}$. Hence $\mathcal{L}^*(A)$ is a definable structure in $\mathcal{E}$ with a parameter for A. The following result says that the full complexity of the isomorphism problem for Boolean algebras of Theorem 1.7 is present in the supersets of a c.e. set.

Theorem 1.9 (Lachlan [14]). *Effectively in i there is a c.e. set H_i such that $\mathcal{L}^*(H_i) \cong \mathcal{B}_i$.*

Corollary 1.10. *The set $\{\langle i, j\rangle : \mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)\}$ is Σ^1_1-complete.*

Slaman and Woodin's idea was to replace "$\mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)$" with "$H_i \approx H_j$". This is a great idea which we now know cannot work, as we discuss below.

[1]See Section 5.1 for more information and a proof.

Definition 1.11 (The sets disjoint from A)**.**

$$\mathcal{D}(A) = (\{B : \exists W(B \subseteq A \cup W \text{ and } W \cap A =^* \emptyset)\}, \subseteq).$$

Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$.

Lemma 1.12. *If A is simple then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{L}^*(A)$.*

A is $\mathcal{D}$*-hhsimple* iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Boolean algebra. Except for the creative sets, until recently all known orbits were orbits of $\mathcal{D}$-hhsimple sets. We direct the reader to Cholak and Harrington [5] for a further discussion of this claim and for an orbit of $\mathcal{E}$ which does not contain any $\mathcal{D}$-hhsimple sets. The following are relevant theorems from Cholak and Harrington [5].

Theorem 1.13. *If A is $\mathcal{D}$-hhsimple and A and $\hat{A}$ are in the same orbit then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$.*

Theorem 1.14 (using Maass [15])**.** *If A is $\mathcal{D}$-hhsimple and simple (i.e., hhsimple) then $A \approx \hat{A}$ iff $\mathcal{L}^*(A) \cong_{\Delta^0_3} \mathcal{L}^*(\hat{A})$.*

Hence the Slaman-Woodin plan of attack fails. In fact even more is true.

Theorem 1.15. *If A and $\hat{A}$ are automorphic then $\mathcal{E}_{\mathcal{D}(A)}$ and $\mathcal{E}_{\mathcal{D}(\hat{A})}$ are Δ^0_6-isomorphic.*

Hence in order to prove Theorem 1.3 we must code everything into $\mathcal{D}(A)$. This is completely contrary to all approaches used to try to prove the Slaman-Woodin Conjecture over the years. We will point out two more theorems from Cholak and Harrington [5] to show how far the sets we use for the proof must be from simple sets, in order to prove Theorem 1.3.

Theorem 1.16. *If A is simple then $A \approx \hat{A}$ iff $A \approx_{\Delta^0_6} \hat{A}$.*

Theorem 1.17. *If A and $\hat{A}$ are both promptly simple then $A \approx \hat{A}$ iff $A \approx_{\Delta^0_3} \hat{A}$.*

1.2. **Past Work and Other Connections.** This current paper is a fourth paper in a series of loosely connected papers, Cholak and Harrington [4], Cholak and Harrington [3], and Cholak and Harrington [5]. We have seen above that results from Cholak and Harrington [5] determine the direction one must take to prove Theorem 1.3. The above results from Cholak and Harrington [5] depend heavily on the main result in Cholak and Harrington [3] whose proof depends on special $\mathcal{L}$-patterns and several theorems about them which can be found in

Cholak and Harrington [4]. It is not necessary to understand any of the above-mentioned theorems from any of these papers to understand the proof of Theorem 1.3.

But the proof of Theorem 1.3 does depend on Theorems 2.16, 2.17, and 5.10 of Cholak and Harrington [5]; see Section 2.6.1. The proof of Theorem 1.6 also needs Theorem 6.3 of Cholak and Harrington [5]. The first two theorems are straightforward but the third and fourth require work. The third is what we call an "extension theorem." The fourth is what we might call a "restriction theorem"; it restricts the possibilities for automorphisms. Fortunately, we are able to use these four theorems from Cholak and Harrington [5] as *black boxes*. These four theorems provide a clean interface between the two papers. If one wants to understand the *proofs* of these four theorems one must go to Cholak and Harrington [5]; otherwise, this paper is completely independent from its three predecessors.

1.3. **Future Work and Degrees of the Constructed Orbits.** While this work does answer many open questions about the orbits of c.e. sets, there are many questions left open. But perhaps these open questions are of a more degree-theoretic flavor. We will list three questions here.

Question 1.18 (Completeness)**.** *Which c.e. sets are automorphic to complete sets?*

Of course, by Harrington and Soare [12], we know that not every c.e. set is automorphic to a complete set, and partial classifications of precisely which sets can be found in Downey and Stob [8] and Harrington and Soare [13, 11].

Question 1.19 (Cone Avoidance)**.** *Given an incomplete c.e. degree* $\mathbf{d}$ *and an incomplete c.e. set* A*, is there an* $\hat{A}$ *automorphic to* A *such that* $\mathbf{d} \not\leq_T \hat{A}$*?*

In a technical sense, these may not have a "reasonable" answer. Thus the following seems a reasonable question.

Question 1.20. *Are these arithmetical questions?*

In this paper we do not have the space to discuss the import of these questions. Furthermore, it not clear how this current work impacts possible approaches to these questions. At this point we will just direct the reader to slides of a presentation of Cholak [2]; perhaps a paper reflecting on these issues will appear later.

One of the issues that will impact all of these questions are which degrees can be realized in the orbits that we construct in Theorem 1.3

and 1.6. A set is *hemimaximal* iff it is the nontrivial split of a maximal set. A degree is *hemimaximal* iff it contains a hemimaximal set. Downey and Stob [8] proved that the hemimaximal sets form an orbit.

We will show that we can construct these orbits to contain at least a fixed hemimaximal degree (possibly along others) or contain all hemimaximal degrees (again possibly along others). However, what is open is if every such orbit must contain a representative of every hemimaximal degree or only hemimaximal degrees. For the proofs of these claims, we direct the reader to Section 4.

1.4. **Toward the Proof of Theorem 1.3.** The proof of Theorem 1.3 is quite complex and involves several ingredients. The proof will be easiest to understand if we introduce each of the relevant ingredients in context.

The following theorem will prove be to useful.

Theorem 1.21 (Folklore[2]). *There is a computable listing T_i of computable infinite branching trees and a computable infinite branching tree $T_{\Sigma^1_1}$ such that the set $\{i : T_{\Sigma^1_1} \cong T_i\}$ is Σ^1_1-complete.*

The idea for the proof of Theorem 1.3 is to code each of the above T_is into the orbit of A_{T_i}. Informally let $\mathcal{T}(A_T)$ denote this encoding; $\mathcal{T}(A_T)$ is defined in Definition 2.47. The game plan is as follows:

(1) **Coding:** For each T build an A_T such that $T \cong \mathcal{T}(A_T)$ via an isomorphism $\Lambda \leq_T \mathbf{0^{(2)}}$. (See Remark 2.48 for more details.)
(2) **Coding is preserved under automorphic images:** If $\hat{A} \approx A_T$ via an automorphism Φ then $\mathcal{T}(\hat{A})$ exists and $\mathcal{T}(\hat{A}) \cong T$ via an isomorphism Λ_Φ, where $\Lambda_\Phi \leq_T \Phi \oplus \mathbf{0^{(2)}}$. (See Lemma 2.49.)
(3) **Sets coding isomorphic trees belong to the same orbit:** If $T \cong \hat{T}$ via isomorphism Λ then $A_T \approx A_{\hat{T}}$ via an automorphism Φ_Λ where $\Phi_\Lambda \leq_T \Lambda \oplus \mathbf{0^{(2)}}$.

So $A_{T_{\Sigma^1_1}}$ and A_{T_i} are in the same orbit iff $T_{\Sigma^1_1}$ and T_i are isomorphic. Since the latter question is Σ^1_1-complete so is the former question.

We should also point out that work from Cholak and Harrington [5] plays a large role in part 3 of our game plan; see Section 2.6.1.

1.5. **Notation.** Most of our notation is standard. However, we have two trees involved in this proof. We will let T be a computable infinite branching tree as described above in Theorem 1.21. For the time being it will be convenient to think of the construction as occurring for each

[2]See Section 5.1 for more information and a proof.

tree *independently*, but this will later change in Section 2.4. Trees T we will think of as growing upward. There will also be the tree of strategies which will denote Tr (which will grow downward). λ is always the empty node (in all trees). It is standard to use $\alpha, \beta, \delta, \gamma$ to range over nodes of Tr. We will add the restriction that $\alpha, \beta, \delta, \gamma$ range *only* over Tr. We will use ξ, ζ, χ to range exclusively over T.

2. The Proof of Theorem 1.3

2.1. **Coding, The First Approximation.**

The main difficulty in this proof is to build a list of pairwise disjoint computable sets with certain properties to be described later. We are going to assume that we have this list of computable sets and slowly understand how these undescribed properties arise. For each node $\chi \in \omega^{<\omega}$ and each i, we will build disjoint computable sets $R_{\chi,i}$. Inside each $R_{\chi,i}$ we will construct a c.e. set $M_{\chi,i}$.

We need to have an effective listing of these sets. Fix a computable one-to-one onto listing $l(e)$ from positive integers to the set of pairs (χ, k), where $\chi \in \omega^{<\omega}$ and $k \in \omega$ such that for all χ and n, if $\xi \preceq \chi$, $m \leq n$, and $l(i) = (\chi, n)$, then there is a $j \leq i$ such that $l(j) = (\xi, m)$. Assume that $l(e) = (\chi, k)$; then we will let $R_{2e} = R_{\chi,2k}$, $R_{2e+1} = R_{\chi,2k+1}$, $M_{2e} = M_{\chi,2k}$, and $M_{2e+1} = M_{\chi,2k+1}$. Which listing of the Rs we use will depend on the situation. We do this as there will be situations where one listing is evidently better than the other.

Definition 2.1. M is *maximal* in R iff $M \subset R$, R is a computable set, and $M \sqcup \overline{R}$ is maximal.

The construction will ensure that either $M_{\chi,i}$ will be maximal in $R_{\chi,i}$ or $M_{\chi,i} =^* R_{\chi,i}$. If i is odd we will let $M_{\chi,i} = R_{\chi,i}$. In this case we say $M_{\chi,i}$ is *known to be computable*. This is an artifact of the construction; the odd sets are errors resulting from the tree construction. More details will be provided later.

To build $M_{\chi,i}$ maximal we will use the construction in Theorem 3.3 of Soare [20]. The maximal set construction uses markers. The marker Γ_e is used to denote the eth element of the complement of the maximal set. At stage s, the marker Γ_e is placed on the eth element of the complement of the maximal set at stage s. In the standard way, we allow the marker Γ_e to "pull" elements of $\overline{M}_s$ at stage $s+1$ such that the element marked by Γ_e has the highest possible e-state and dump the remaining elements into M.

However, at times we will have to destroy this construction of $M_{\chi,i}$ with some priority p. If we decide that we must destroy $M_{\chi,i}$ with some priority p at stage s we will just enumerate the element Γ_p is marking into $M_{\chi,i}$ at stage s. If this occurs infinitely many times then $M_{\chi,i} =^* R_{\chi,i}$. With this twist, we will just appeal to the construction in Soare [20].

To code T, for all χ, such that $\chi \in T$, we will build pairwise disjoint computably enumerable sets D_χ. We will let $A = D_\lambda$. If $l(i) = (\chi, 0)$ then we will let $D_i = D_\chi$. If $l(i) \neq (\zeta, 0)$ then we will let $D_i = \emptyset$. These sets will be constructed as follows.

Remark 2.2 (Splitting M). Let $l(j) = (\chi, i)$. We will use the Friedberg Splitting Theorem; we will split $M_{\chi,2i}$ into $i + 3$ parts. Again we will just appeal to the standard proof of the Friedberg Splitting Theorem. We will put one of the parts into D_χ. For $0 \leq l \leq i$, if $\chi\hat{\ } l \in T$ and there is a $j' < j$ such that $l(j') = (\chi\hat{\ } l, 0)$, then we put one of the parts into $D_{\chi\hat{\ } l}$. The remaining part(s) remain(s) disjoint from the union of the Ds; we will name this remaining infinite part $H_{\chi,i}$. This construction works even if we later decide to destroy $M_{\chi,i}$ by making $M_{\chi,i} =^* R_{\chi,i}$.

If $M_{\chi,i}$ is known to be computable, we will split $R_{\chi,i}$ into $i + 3$ *computable* parts distributed as above. However in this case we cannot appeal to the Friedberg Splitting Theorem since many of the elements in the D under question will have entered the Ds prior to entering $M_{\chi,i} = R_{\chi,i}$. We will have to deal with this case in more detail later.

Lemma 2.3. *This construction implies that* $\bigsqcup_\chi D_\chi \subseteq \bigsqcup_{(\chi,i)} (R_{\chi,i} - H_{\chi,i})$.

At this point we should point out a possible problem. If the list of computable sets is effective then we have legally constructed c.e. sets. If not, we could be in trouble.

However, we want our list to satisfy the following requirement. This requirement will have a number of roles. Its main function is to control where the sets W_e live within our construction.

Requirement 2.4. *For all e, there is an i_e such that either*

$$W_e \cup \bigsqcup_{j \leq i_e} R_i \cup \bigsqcup_{j \leq i_e} D_i =^* \omega, \text{ or} \tag{2.4.1}$$

$$W_e \subseteq^* \bigsqcup_{j \leq i_e} R_i, \text{ or} \tag{2.4.2}$$

$$W_e \subseteq^* \bigsqcup_{j \leq i_e} R_i \sqcup \Big(\bigsqcup_{j < i_e} D_i - \bigsqcup_{j \leq i_e} R_i \Big). \tag{2.4.3}$$

Equation (2.4.3) implies Equation (2.4.2), but this separation will be useful later. If Equation 2.4.1 holds, then there is a computable R_{W_e}

such that

$$R_{W_e} \subseteq \bigsqcup_{j \leq i_e} R_i \cup \bigsqcup_{j \leq i_e} D_i \text{ and } W_e \cup R_{W_e} = \omega. \tag{2.4.4}$$

If we have an effective list of all the R_e then we have an effective list of H_e. Let h_i be the ith element of H_i. Then the collection of all h_i is a computable set, say W_e. But e contradicts Requirement 2.4. It follows that our list *cannot* be effective, but it will be effective enough to ensure the D are computably enumerable.

At this point we are going to have to bite the bullet and admit that there will be an underlying tree construction. We are going to have to decide how the sets we want to construct will be placed on the tree.

Assume that α is in our tree of strategies and $l(|\alpha|) = (\chi, n)$. At node α we will construct two computable sets R_α and E_α. E_α will be the error forced on us by the tree construction. If $\chi \in T$ and $n = 0$ then at α we will also construct D_α.

Assume α is on the true path and $l(|\alpha|) = (\chi, n)$. Then $R_{\chi,2n} = R_\alpha$ and E_α is $R_{\chi,2n+1} = M_{\chi,2n+1} = E_\alpha$. This is the explanation of why $M_{\chi,i}$ is computable for i odd; $R_{\chi,i}$ is the error. If $\chi \in T$ and $n = 0$ then $D_\chi = D_\alpha$. Hence the listing of computable sets we want is along the true path. Therefore, from now on, when we mention $R_{\chi,i}$, D_χ, R_e, or D_e, we assume we are working along the true path. When we mention R_α or D_α we are working somewhere within the tree of strategies but not necessarily on the true path.

2.2. **Meeting Requirement 2.4.** Our tree of strategies will be a Δ^0_3 branching tree. Hence at α we can receive a guess to a finite number of Δ^0_3 questions asked at α^-. Using the Recursion Theorem these questions might involve the sets R_β, E_β, and D_β for $\beta \prec \alpha$. The correct answers are given along the true path, f. There is a standard approximation to the true path, f_s. Constructions of this sort are found all over the c.e. set literature.

These constructions are equipped with a computable position function $\alpha(x, s)$, the node in Tr where x is at stage s. All balls x enter Tr at λ. If the approximation to the true path is the left of x's position, x will be moved upward to be on this approximation and never allowed to move right of this approximation. To move a ball x downward from α^- to α, α must be on the approximation to the true path and x must be α^- allowed. When we α^- allow x depends on Equations 2.4.1 and 2.4.3.

So, formally, $\alpha(x, x) = \lambda$. If $f_{s+1} <_L \alpha(x, s)$ then we will let $\alpha(x, s+1) = f_{s+1} \cap \alpha(x, s)$. If $\alpha(x, s) = \alpha^-$, x has been α^- *allowed*,

$\alpha \subseteq f_s$, and, for all stages t, if $x \leq t < s$ then $f_t \not<_L \alpha$; then we will let $\alpha(x, s+1) = \alpha$.

Exactly when a ball will be α-allowed is the key to this construction and will be addressed shortly. However, given these rules, it is clear if $f <_L \alpha$ then there are no balls x with $\lim_s \alpha(x, s) = \alpha$ and if $\alpha <_L f_s$ then there are at most finitely many balls x with $\lim_s \alpha(x, s) = \alpha$. Of course, the question remains what happens at $\alpha \subset f$?

The question we ask at α^- is if the set of x such that there is a stage s with

$$\begin{gathered} x \in W_{e,s}, \alpha^- \subseteq \alpha(x, s), x \text{ is } \alpha^-\text{-allowed at stage } s, \\ \text{and } x \notin (\bigsqcup_{\beta \preceq \alpha^-} R_{\beta,s} \cup \bigsqcup_{\beta \preceq \alpha^-} E_{\beta,s} \cup \bigsqcup_{\beta \preceq \alpha^-} D_{\beta,s}) \end{gathered} \tag{2.4.5}$$

is infinite, where $e = |\alpha^-|$, a Π^0_2 question.

2.2.1. *A Positive Answer.* Assume that α believes the answer is yes. Then for each time $\alpha \subset f_s$, α will be allowed to pull three such balls to α. That is, α will look for three balls x_1, x_2, x_3 and stages t_1, t_2, t_3 such that Equation 2.4.5 holds for x_i and t_i, $x_i > s$, $\alpha(x_i, t_i) \not<_L \alpha$, $x_i \notin E_{\alpha,t_i} \cup R_{\alpha,t_i}$, and x is not α-allowed at stage t_i.

When such a ball x_i and stage t_i are found, we will let $\alpha(x_i, t_i+1) = \alpha$. For the first such ball x_1 we will add x_1 to E_α at stage t_1. Throughout the whole stagewise construction we will enumerate x_1 into various disjoint D_β at stage t_1 to ensure that $H_\alpha = E_\alpha - \bigsqcup_{\beta \preceq \alpha} D_\beta$ and, for each $\beta \preceq \alpha$, $D_\beta \cap E_\alpha$ is an infinite set. For the second such ball x_2 we will add x_2 to R_α at stage t_2. For the third such ball x_3 we will α-allow x_3 and place all balls y such that $\alpha(y, t_3) = \alpha$, $y \notin R_{\alpha,t_3}$, and y is not α-allowed into E_{α,t_3} (without any extra enumeration into the D_β).

It is not hard to see that when balls are α-allowed at stage s they are not in

$$\bigsqcup_{\beta \preceq \alpha^-} R_{\beta,s} \cup \bigsqcup_{\beta \preceq \alpha^-} E_{\beta,s} \cup \bigsqcup_{\beta \preceq \alpha} D_{\beta,s};$$

once a ball is α-allowed it never enters R_α or E_α, and, for almost all x, if $\lim_s \alpha(x, s) = \alpha$ then $x \in E_\alpha \sqcup R_\alpha$ (finitely many of the α-allowed balls may live at α in the limit).

Assume $\alpha \subset f$. Then every search for a triple of such balls will be successful; both R_α and E_α are disjoint infinite computable sets; infinitely many balls are α-allowed and hence almost of the α-allowed balls move downward in Tr; $E_\alpha - \bigsqcup_{\beta \preceq \alpha} D_\beta$ is infinite and computable; for each $\beta \preceq \alpha$, $D_\beta \cap E_\alpha$ is infinite and computable; $R_\alpha \subset W_e$, and most importantly, for all $\beta \succ \alpha$, $R_\beta \sqcup E_\beta \subseteq W_e$ and hence Equation 2.4.1 holds.

2.2.2. *A Negative Answer.* Assume that α believes the answer is no. Assume $\alpha \subset f$ and that infinitely many balls are α^- allowed. This is certainly the case if α^- corresponds to the above positive answer. If W_e intersect the sets of balls which are α^--allowed is finite then

$$W_e \subseteq^* \bigsqcup_{\beta \preceq \alpha^-} R_\beta \cup \bigsqcup_{\beta \preceq \alpha^-} E_\beta$$

and hence Equation 2.4.2 holds. Assume this is not the case. Since Equation 2.4.5 does not hold for infinitely many balls x and stages s, for almost all x if

$$x \in W_{e,s}, \alpha^- \subseteq \alpha(x,s), x \text{ is } \alpha^-\text{-allowed at stage } s,$$

then $x \in \bigsqcup_{\beta \preceq \alpha^-} D_{\beta,s}$. Hence,

$$W_e \subseteq^* \bigsqcup_{\beta \preceq \alpha^-} R_\beta \cup \bigsqcup_{\beta \preceq \alpha^-} E_\beta \cup \bigsqcup_{\beta \preceq \alpha^-} D_\beta$$

and Equation 2.4.3 holds.

Either way there are infinitely many balls x and stage s such that

$$\begin{gathered} \alpha^- \subseteq \alpha(x,s), x \text{ is } \alpha^-\text{-allowed at stage } s, \\ \text{and } x \notin \Big(\bigsqcup_{\beta \preceq \alpha^-} R_{\beta,s} \cup \bigsqcup_{\beta \preceq \alpha^-} E_{\beta,s} \cup \bigsqcup_{\beta \preceq \alpha^-} D_{\beta,s} \Big) . \end{gathered} \tag{2.4.6}$$

In the same way as when α corresponds to the positive answer, we will pull three such balls to α. The action we take with these balls is exactly the same as in the positive answer. Hence, among other things, infinitely many balls are α-allowed, allowing us to inductively continue.

2.2.3. *The maximal sets and their splits.* To build M_α we will appeal to the standard maximal set construction as suggested above. But we will label the markers as Γ_e^α or $\Gamma_e^{\chi,i}$ rather than Γ_e just to keep track of things. As suggested in Remark 2.2, to build the D_β within R_α, for $\beta \preceq \alpha$, we will appeal to the Friedberg Splitting Theorem.

At this point, we will step away from the construction and see what we have manged to achieve and what more needs to be achieved. We will be careful to point out where we use the above requirement and where it is not enough for our goals.

2.3. **A definable view of our coding.** For each $\chi \in T$ we will construct pairwise disjoint c.e. sets D_χ. The reader might wonder how this helps. In particular, how do these sets code T? Moreover, if $\hat{A}$ is in the orbit of A how do we recover an isomorphic copy of T? To address these issues, we will need some sort of "definable structure." Unfortunately, the definition of the kind of structure we need is rather involved. To motivate the definition, we need to recall how nontrivial splits of maximal sets behave and then see what the above construction does with these splits in a definable fashion.

Definition 2.5. A split D of M is a *Friedberg split* iff, for all W, if $W - M$ is not a c.e. set then neither is $W - D$.

Lemma 2.6 (Downey and Stob [8])**.** *Assume M is maximal in R. Then D is a nontrivial split of M iff D is a Friedberg split of M.*

Proof. In each direction we prove the counterpositive. Let $\breve{D}$ be such that $D \sqcup \breve{D} = M$.

Assume that D is not Friedberg. Hence for some W, $W - D$ is c.e. but $W - M$ is not. If $W \subseteq^* (M \cup \overline{R})$ then $(W - M) \subseteq^* \overline{R}$ and hence $W - M =^* W \cap \overline{R}$, a c.e. set. Therefore $\overline{M \sqcup \overline{R}} = (R - M) \subseteq^* W$. Therefore $D \sqcup ((W - D) \cup \breve{D} \cup \overline{R}) = \omega$ and D is computable.

The set $R - M$ is not a c.e. set. Assume D is computable. Then $R - D = R \cap \overline{D}$. Hence $\overline{D}$ witnesses that D is not a Friedberg split. ☐

Lemma 2.7. *Assume that M_i are maximal in R and D is a nontrivial split of both M_i. Then $M_1 =^* M_2$.*

Proof. $M_1 \cup \overline{R}$ is maximal. $\overline{M_1 \cup \overline{R}} = R - M_1$. Since $M_2 \cup \overline{R}$ is maximal either $M_1 \subseteq^* M_2$ or $(R - M_1) \subseteq^* M_2$. In the former case, $M_2 \subseteq^* M_1 \cup \overline{R}$ so $M_1 =^* M_2$.

Assume the later case. Let $D \cup \breve{D} = M_2$. Since D is a split of M_1, $(R - M_1) \subseteq^* \breve{D}$. Now $\breve{D} - M_1 = R - M_1$ is not c.e. set but $\breve{D} - D = \breve{D}$ is a c.e. set. So D is not a Friedberg split of M_1. So by Lemma 2.6, D is not nontrivial split of M_1. Contradiction. ☐

It turns out that we will need a more complex version of the above lemmas.

Definition 2.8. $W \equiv_{\mathcal{R}} \hat{W}$ iff $W \triangle \hat{W} = (W - \hat{W}) \sqcup (\hat{W} - W)$ is computable.

Lemma 2.9. *Assume that M_i is maximal in R_i and $D \cap R_i$ is a nontrivial split of M_i. Either*

(1) *there are disjoint $\tilde{R}_i$ such that $(M_i \cap \tilde{R}_i)$ is maximal in $\tilde{R}_i$, $D \cap \tilde{R}_i$ is a nontrivial split of M_i, and either $\tilde{R}_1 = R_1 - R_2$ and $\tilde{R}_2 = R_2$ or $\tilde{R}_1 = R_1$ and $\tilde{R}_2 = R_2 - R_1$, or*
(2) *$\tilde{M} = M_1 \cap M_2$ is maximal in $\tilde{R} = R_1 \cap R_2$. So $\tilde{R} - M_i =^* \tilde{R} - \tilde{M}$ and hence $\tilde{M} \equiv_{\mathcal{R}} M_1 \equiv_{\mathcal{R}} M_2$. Furthermore, if $R_1 = R_2$ then $\tilde{M} =^* M_1 =^* M_2$.*

Proof. $M_i \cup \overline{R}_i$ is maximal. $\overline{M_i \cup \overline{R}_i} = R_i - M_i$. $R_i - M_i$ is not split into two infinite pieces by any c.e. set. Since $M_2 \cup \overline{R}$ is maximal either $(M_1 \cup \overline{R}_1) \subseteq^* (M_2 \cup \overline{R}_2)$ or $(R_1 - M_1) \subseteq^* (M_2 \cup \overline{R}_2)$. If $(R_1 - M_1) \subseteq^* (M_2 \cup \overline{R}_2)$ then $(R_1 - M_1) \subseteq^* M_2$ or $(R_1 - M_1) \subseteq^* \overline{R}_2$.

Assume $(R_1 - M_1) \subseteq^* M_2$. So $M_2 - (M_1 \cup \overline{R}_1) = R_1 - M_1$ is not a c.e. set. Let $(D \cap R_2) \cup \tilde{D} = M_2$. Therefore $(R_1 - M_1) \subseteq^* \tilde{D}$ or $(R_1 - M_1) \subseteq^* (D \cap R_2)$. In the former case $(D \cap R_2) - (M_1 \cup \overline{R}_1) = \emptyset$ is a c.e. set. In the latter case $\tilde{D} - (M_1 \cup \overline{R}_1) = \emptyset$ is a c.e. set. Either way, by Lemma 2.6, $(D \cap R_2)$ is not a nontrivial split of M_2. Contradiction.

Now assume $(R_1 - M_1) \subseteq^* \overline{R}_2$. Let $\tilde{R}_1 = R_1 - R_2$ and $\tilde{R}_2 = R_2$. Let $(D \cap R_1) \sqcup \tilde{D} = M_1$ be a nontrivial split. Let $\tilde{M} = M_1 - R_2$. Then $(D \cap \tilde{R}_1) \sqcup (\tilde{D} - R_2) = \tilde{M}$ is a nontrivial split of $\tilde{M}$. (Otherwise $(D \cap R_1) \sqcup \tilde{D} = M_1$ is a trivial split.)

We can argue dually switching the roles of M_1 and M_2. We are left with the case $(M_1 \cup \overline{R}_1) \subseteq^* (M_2 \cup \overline{R}_2)$ and $(M_2 \cup \overline{R}_2) \subseteq^* (M_1 \cup \overline{R}_1)$. Hence $(M_1 \cup \overline{R}_1) =^* (M_2 \cup \overline{R}_2)$ and $R_1 - M_1 =^* R_2 - M_2$. Therefore $\tilde{M} = M_1 \cap M_2$ is maximal in $\tilde{R} = R_1 \cap R_2$. □

Definition 2.10. *D lives inside R witnessed by M* iff M maximal in R and $D \cap R$ is a nontrivial split of M.

By Lemma 2.7, if D lives in R witnessed by M_i then $M_1 =^* M_2$. Hence at times we will drop the "witnessed by M." If D lives in R then we will say D lives in R witnessed by M^R. The point is that M^R is well defined modulo finite difference.

Lemma 2.11. *If D lives in R_1, $R_1 \cap R_2 = \emptyset$, and $D \cap R_2$ is computable, then D lives in $R_1 \sqcup R_2$.*

Lemma 2.12. *If R is computable and $D \cap R$ is computable, then D does not live in R.*

Lemma 2.13. *If $\chi \in T$, then D_χ lives in $R_{\chi,2i}$ or $M_{\chi,i} =^* R_{\chi,i}$.*

Proof. Follows from the construction. □

Lemma 2.14. *For all $R_{\chi,i}$, if $M_{\chi,i}$ is maximal in $R_{\chi,i}$, there is a subset $H_{\chi,i} \subset M_{\chi,i}$ such that $H_{\chi,i}$ lives in $R_{\chi,i}$ and $H_{\chi,i} \cap \bigsqcup_\xi D_\xi = \emptyset$.*

Proof. Follows from the construction. □

Lemma 2.15. *If $D_\xi \cap R_{\chi,i} \neq \emptyset$, then $\xi = \chi$ or $|\xi| = |\chi| + 1$. Furthermore, if D_ξ lives in $R_{\chi,i}$ then i is even.*

Proof. Again follows from the construction. □

Lemma 2.16. *If $\chi\hat{}l \in T$ then there is a least i' and j' such that $l(j') = (\chi, i')$, and, for all $i \geq 2i'$, $D_\chi \cap R_{\chi,i} \neq^* \emptyset$, $D_{\chi\hat{}l} \cap R_{\chi,i} \neq^* \emptyset$, and either both D_χ and $D_{\chi\hat{}l}$ live in $R_{\chi,i}$ or $M_{\chi,i} =^* R_{\chi,i}$. So, in particular, both D_χ and $D_{\chi\hat{}l}$ live in $R_{2j'}$ or $M_{2j'} =^* R_{2j'}$. Furthermore i' and j' can be found effectively.*

Proof. Assume $\chi\hat{}l \in T$. Let j be such that $l(j) = (\chi\hat{}l, 0)$. Let j' be the least such that $j < j'$ and $l(j') = (\chi, i')$. (See Section 2.2.3.) $\square$

Requirement 2.17. *For each $\chi \in T$ there are infinitely many i such that $M_{\chi,i} \neq^* R_{\chi,i}$.*

Currently we meet this requirement since if i is even then $M_{\chi,i} \neq^* R_{\chi,i}$. But for later requirements we will have to destroy some of these $M_{\chi,i}$, so some care will be needed to ensure that it is met.

The following definition is a complex inductive one. This definition is designed so that if A and $\hat{A}$ are in the same orbit witnessed by Φ we can recover a possible image for D_χ without knowing Φ. In reality, we want more: we want to be able to recover T. But the ability to recover T will take a lot more work. In any case, the definition below is only a piece of what is needed.

Definition 2.18.

(1) *An $\mathcal{R}^A$ list* (or, equivalently, an $\mathcal{R}^{D_\lambda}$ list) is an infinite list of disjoint computable sets R_i^A such that, for all i, A lives in R_i^A witnessed by M_i^A and, for all computable R, if A lives in R witnessed by M then there is exactly one i such that $R - M =^* R_i^A - M_i^A$.

(2) We say that *D is a 1-successor of $\tilde{D}$ over some $\mathcal{R}^{\tilde{D}}$ list* if D and $\tilde{D}$ are disjoint, and, for almost all i, D lives in $R_i^{\tilde{D}}$.

(3) Let D be a 1-successor of $\tilde{D}$ witnessed by an $\mathcal{R}^{\tilde{D}}$ list. *An $\mathcal{R}^D$ list over an $\mathcal{R}^{\tilde{D}}$ list* is an infinite list of disjoint computable sets R_i^D such that, for all i, D lives in R_i^D and, for all computable R, if D lives in R then there is exactly one i such that exactly one of $R - M =^* R_i^D - M_i^D$ or $R - M =^* R_i^{\tilde{D}} - M_i^{\tilde{D}}$ hold.

Lemma 2.19. *If $\chi \in T$, then let $R_e^{D_\chi} = R_{\chi,g(e)}$, where $g(e)$ is the eth set of all those $R_{\chi,i}$ where $M_{\chi,i} \neq^* R_{\chi,i}$. (By Requirement 2.17, such a g exists). This list is an $\mathcal{R}^{D_\chi}$ list over $\mathcal{R}^{D_{\chi^-}}$ (where $\mathcal{R}^{D_\lambda^-}$ is the empty list.)*

Proof. We argue inductively. We are going to take two lists $\mathcal{R}^{D_{\chi^-}}$ and $\mathcal{R}^{D_\chi}$ and merge them to get a new list. To each set of this new list we will add at most finitely different $R_{\xi,j}$, where for all i,

$R_{\xi,j} - M_{\xi,j} \neq^* R_i^{D_{\chi^-}} - M_i^{D_{\chi^-}}$ and $R_{\xi,j} - M_{\xi,j} \neq^* R_i^{D_\chi} - M_i^{D_\chi}$ such that all such $R_{\xi,j}$ are added to some set in our new list. Call the nth set of this resulting list $\tilde{R}_n$. By Lemmas 2.13 and 2.11 and Definition 2.18, D_χ lives in almost all $\tilde{R}_n$.

Fix R such that D_χ lives in R. For each n, apply Lemma 2.9 to R and $\tilde{R}_n$. If case (2) applies, then R behaves like $\tilde{R}_n$ and we are done. Otherwise we can assume R is disjoint from $\tilde{R}_n$.

If this happens for all n then R and $\bigsqcup_i \tilde{R}_i$ are disjoint. Split R into two infinite computable pieces R_1 and R_2. Since $\bigsqcup D \subseteq \bigsqcup \tilde{R}$, R_i cannot be a subset of $\bigsqcup D$. Therefore $R_i \not\subseteq^* \bigsqcup \tilde{R} \cup \bigsqcup D$. Furthermore, $R_i \cup \bigsqcup \tilde{R} \cup \bigsqcup D \neq^* \omega$. But assuming that we meet Requirement 2.4 this cannot occur. Contradiction. □

Corollary 2.20. *Assume $\chi\hat{\ }l \in T$. By Lemmas 2.19 and 2.16, $D_{\chi\hat{\ }l}$ is a 1-successor of D_χ over $\mathcal{R}^{D_\chi}$. Furthermore, if F is finite then $D_{\chi\hat{\ }l} - \bigsqcup_{i\in F} R_i$ is a 1-successor of D_χ over $\mathcal{R}^{D_\chi}$.*

Corollary 2.21. *If disjoint D_i are 1-successors of $\tilde{D}$ over $\mathcal{R}^{\tilde{D}}$ then so is $D_1 \sqcup D_2$. In particular, for all $\chi, \zeta \in T$, if $\chi \neq \zeta$ and $|\chi| = |\xi|$, then $D = D_\chi \sqcup D_\xi$ is a 1-successor of D_{χ^-} over $\mathcal{R}^{D_{\chi^-}}$ and the elementwise union of the lists $\mathcal{R}^{D_\chi}$ and $\mathcal{R}^{D_\xi}$ is an $\mathcal{R}^D$ list over $\mathcal{R}^{D_{\chi^-}}$.*

Lemma 2.22. *If χ does not have a successor in T then there are no 1-successors of D_χ over $\mathcal{R}^{D_\chi}$.*

Proof. Assume that D is a 1-successor of D_χ over $\mathcal{R}^{D_\chi}$. By Requirement 2.4, there is finite F such that $D \subseteq^* \bigsqcup_{j\in F} R_j \cup \bigsqcup_{j\in F} D_j$. Since D is a 1-successor of D_χ, so is $D - \bigsqcup_{j\in F} R_j$. Since D and D_χ are disjoint we can assume that if $l(j) = (\chi, 0)$ then $j \notin F$. Now if $j \in F$ then $D_j \cap R_{\chi,i} = \emptyset$. Contradiction. □

Definition 2.23.

(1) D is a 0-*successor* witnessed by $\mathcal{R}^D$ iff $D = A$ and the lists, $\mathcal{R}^A$ and $\mathcal{R}^D$, are identical.
(2) D is a 1-successor of A over $\mathcal{R}^A$ was defined in Definition 2.18 (2).
(3) Let $\tilde{D}$ be a n-successor of A witnessed by $\mathcal{R}^W$. If an $\mathcal{R}^{\tilde{D}}$ list over $\mathcal{R}^W$ exists and D is a 1-successor of $\tilde{D}$ over $\mathcal{R}^{\tilde{D}}$, then D *is an $n+1$-successor of A witnessed by $\mathcal{R}^{\tilde{D}}$.*
(4) D is a *successor* of A iff, for some $n \geq 0$, D is an n-successor.

Corollary 2.24. *Let $\chi \in T$. Then D_χ is a $|\chi|$-successor of A over $\mathcal{R}^{D_{\chi^-}}$. Furthermore, if F is finite then $D_\chi - \bigsqcup_{i\in F} R_i$ is a $|\chi|$-successor of A over $\mathcal{R}^{D_{\chi^-}}$.*

Corollary 2.25. *For all $\chi, \zeta \in T$, if $\chi \neq \zeta$ and $|\chi| = |\xi|$, then $D_\chi \sqcup D_\xi$ is a $|\chi|$-successor of A witnesses by $\mathcal{R}^{D_{\chi^-}}$.*

We want to transfer these results to the hatted side. We want to find n-successors of $\hat{A}$, *without using the* Φ, witnessing A and $\hat{A}$ are in the same orbit. Just from knowing A and $\hat{A}$ are in the same orbit we want to be able to recover all successors of $\hat{A}$. But first we need the following lemmas.

Lemma 2.26 (Schwarz, see Theorem XII.4.13(ii) of Soare [20])**.** *The index set of maximal sets is Π^0_4-complete and hence computable in $\mathbf{0^{(4)}}$.*

Lemma 2.27. *The index set of computable sets is Σ^0_3-complete and hence computable in $\mathbf{0^{(3)}}$.*

Corollary 2.28. *The set $\{\langle e_1, e_2\rangle : W_{e_1}$ lives in $W_{e_2}\}$ is Σ^0_5 and hence computable in $0^{(5)}$.*

Lemma 2.29. *An $\mathcal{R}^{\hat{A}}$ list exists and can be found in an oracle for $0^{(5)}$.*

Proof. First we know $\mathcal{R}^{D_\lambda}$ is an $\mathcal{R}^A$ list. So $R_i^{\hat{D}_\lambda} = \Phi(R_i^{D_\lambda})$ is an $\mathcal{R}^{\hat{D}_\lambda}$ list. Hence an $\mathcal{R}^{\hat{A}}$ list exists. However, using Φ in this fashion does not necessarily bound the complexity of $\mathcal{R}^{\hat{A}}$.

Inductively, using an oracle for $0^{(5)}$, we will create an $\mathcal{R}^{\hat{A}}$ list. Assume that $\hat{R}_i^{\hat{A}}$ are known for $i < j$, and that for $e < j$, if $\hat{A}$ lives in W_e then there is an $i < j$ such that $W_e - \hat{M}^{W_e} =^* \hat{R}_i^{\hat{A}} - \hat{M}^{\hat{R}_i^{\hat{A}}}$. Look for the least $e \geq j$ such that $\hat{A}$ lives in W_e and for all $i < j$ such that $W_e - \hat{M}^{W_e} \neq^* \hat{R}_i^{\hat{A}} - \hat{M}^{\hat{R}_i^{\hat{A}}}$. Such an e must exist since an $\mathcal{R}^{\hat{A}}$ lists exists. Let $\hat{R}_j^{\hat{A}} = W_e$. Apply the hatted version of Lemma 2.9 to get the $\hat{R}_j^{\hat{A}}$ disjoint from $\hat{R}_i^{\hat{A}}$. □

Definition 2.30. Let g be such that $W_{g(i)} = R_i^{\hat{\hat{D}}}$. Then we will say that g is a *presentation* of $\mathcal{R}^{\hat{\hat{D}}}$.

Lemma 2.31. *Let $\hat{\hat{D}}$ and an $\mathcal{R}^{\hat{\hat{D}}}$ list be given. Assume that g is a presentation of $\mathcal{R}^{\hat{\hat{D}}}$. Then all the 1-successors of $\hat{\hat{D}}$ over $\mathcal{R}^{\hat{\hat{D}}}$ can be found using an oracle for $(g \oplus 0^{(5)})^{(2)}$.*

Proof. Asking "whether an e such that $W_e = W_{g(i)}$ and $\hat{D}$ lives in W_e" is computable in $g \oplus 0^{(5)}$. $\hat{D}$ is a 1-successor of $\hat{\hat{D}}$ over $\mathcal{R}^{\hat{\hat{D}}}$ iff there is a k, for all $i \geq k$, [there is an e such that $W_e = W_{g(i)}$ and $\hat{D}$ lives in W_e]. □

Corollary 2.32. *The* 1*-successors of* $\hat{A}$ *can be found with an oracle for* $0^{(7)}$.

A word of caution: For all $\chi \in T$ of length one, $\Phi(D_\chi)$ is a 1-successor of $\hat{A}$ and, for $\Phi(D_\chi)$, an infinite $\mathcal{R}^{\Phi(D_\chi)}$ list over $\mathcal{R}^{\hat{A}}$ exists. But, by Corollary 2.25, not every 1-successor $\hat{D}$ of $\hat{A}$ is the image of some such D_χ even modulo finite many $R_{\xi,i}$. Furthermore, there is no reason to believe that if $\hat{D}$ is a 1-successor of $\hat{A}$ that an $\mathcal{R}^{\hat{D}}$ list over $\mathcal{R}^{\hat{A}}$ exists. Unfortunately, we must fix this situation before continuing.

Definition 2.33. Let D_1 and D_2 be 1-successors of $\tilde{D}$ over some $\mathcal{R}^{\tilde{D}}$ list. Let an $\mathcal{R}^{D_i}$ list be given. D_1 and D_2 are *T-equivalent* iff for almost all m, there is an n such that $R^{D_1}_m - M^{R^{D_1}_m} =^* R^{D_2}_n - M^{R^{D_2}_n}$ and for almost all m, there is an n such that $R^{D_2}_m - M^{R^{D_2}_m} =^* R^{D_1}_n - M^{R^{D_1}_n}$.

Lemma 2.34. *If $\chi \in T$ and F is finite then D_χ and $D_\chi - \bigsqcup_{i\in F} R_i$ are T-equivalent* 1*-successors of D_{χ^-} over $\mathcal{R}^{D_{\chi^-}}$.*

Lemma 2.35. *For all $\chi, \zeta \in T$, if $\chi \neq \xi$ and $|\chi| = |\xi|$, then D_χ, D_ξ and $D_\chi \sqcup D_\xi$ are pairwise T-nonequivalent* 1*-successors of D_{χ^-} over $\mathcal{R}^{D_{\chi^-}}$.*

Lemma 2.36. *D_1 and D_2 are T-equivalent iff their automorphic images are T-equivalent.*

Lemma 2.37. *Whether "$\hat{D}_1$ and $\hat{D}_2$ are T-equivalent" can be determined with an oracle for $(g_1 \oplus g_2 \oplus \tilde{g} \oplus 0^{(5)})^{(2)}$, where g_i and $\tilde{g}$ are representatives of needed lists.*

So D_χ and $D_\chi - R_i$ are T-equivalent. Therefore, we need to look at the T-equivalence class of D_χ rather than just D_χ; D_χ is just a nice representative of the T-equivalence class of D_χ. T-equivalence allows us to separate D_χ for χ of the same length; they are not T-equivalent. However, we cannot eliminate the image of the disjoint union of two different D_χ as a possible successor of the image of $\hat{D}_{\chi^-}$. For that we need another notion.

Definition 2.38. Let D be a 1-successor of $\tilde{D}$ over some $\mathcal{R}^{\tilde{D}}$ list. Let an $\mathcal{R}^D$ list be given. We say that D is *atomic* iff for all nontrivial splits $D_1 \sqcup D_2 = D$, if D_i is a 1-successor of $\tilde{D}$ then, for almost all m, D_i lives in R^D_m.

Lemma 2.39. *Assume D is an atomic* 1*-successor of $\tilde{D}$ over some $\mathcal{R}^{\tilde{D}}$, an $\mathcal{R}^D$ list exists, and $D_1 \sqcup D_2$ is a nontrivial split of D. If D_i is a* 1*-successor of $\tilde{D}$ then an $\mathcal{R}^{D_i}$ list exists and D and D_i are T-equivalent.*

Definition 2.40. A T-equivalent class $\mathcal{C}$ is called an *atomic T-equivalent class* if every member of $\mathcal{C}$ is atomic.

The following lemma says that the notion of being atomic indeed eliminates the disjoint union possibility.

Lemma 2.41. *If $\chi \neq \xi$ and $|\chi| = |\xi|$ then $D_\chi \sqcup D_\xi$ is not atomic.*

Lemma 2.42. *Let D be a 1-successor of $\tilde{D}$ over some $\mathcal{R}^{\tilde{D}}$ list. Let an $\mathcal{R}^{D}$ list be given. Then D is atomic iff its automorphic image is atomic.*

Lemma 2.43. *Let $\hat{D}$ be a 1-successor of $\hat{\hat{D}}$ over some $\mathcal{R}^{\hat{\hat{D}}}$ list. Let an $\mathcal{R}^{\hat{D}}$ list be given. Determining "whether $\hat{D}$ is atomic" can be done using an oracle for $(g \oplus \tilde{g} \oplus 0^{(5)})^{(3)}$, where g and $\tilde{g}$ are representatives of needed lists.*

Unfortunately, with the construction as given so far, there is no reason to believe that each D_χ is atomic. We are going to have to modify the construction so that each D_χ is atomic. Thus, we are going to have to add this as another requirement.

Requirement 2.44. *Fix χ such that $\chi \in T$. Then D_χ is an atomic 1-successor of D_{χ^-}.*

We will have to modify the construction so that we can meet the above requirement. This will be done in Section 2.5. Until that section, we will work under the assumption we have met the above requirement.

These next two lemmas must be proved simultaneously by induction. They are crucial in that they reduce the apparent complexity down to something arithmetical.

Lemma 2.45. *Fix an automorphism Φ of $\mathcal{E}$ taking A to $\hat{A}$. Let $\mathcal{C}_{n+1}$ be the class formed by taking all sets of the form $\Phi(D_\chi)$, where $\chi \in T$ and has length $n+1$, and closing under T-equivalence. The collection of all atomic $n+1$-successors of $\hat{A}$ and $\mathcal{C}_{n+1}$ are the same class.*

Proof. For the base case, by Lemma 2.29, an $\mathcal{R}^{\hat{A}}$ list exists. Now apply Lemma 2.42. For the inductive case, use the following lemma, and then Lemma 2.42. □

Lemma 2.46. *Let $\hat{\hat{D}}$ be an atomic n-successor of $\hat{A}$ witnessed by $\mathcal{R}^{\hat{W}}$. Assume an $\mathcal{R}^{\hat{\hat{D}}}$ list over $\mathcal{R}^{\hat{W}}$ exists and $\hat{D}$ is an atomic 1-successor of $\hat{\hat{D}}$ over $\mathcal{R}^{\hat{\hat{D}}}$. (Then $\hat{D}$ is an atomic $n+1$-successor of $\hat{A}$ witnessed by $\mathcal{R}^{\hat{\hat{D}}}$.) Then an $\mathcal{R}^{\hat{D}}$ list over $\mathcal{R}^{\hat{\hat{D}}}$ can be constructed with an oracle for $g \oplus 0^{(5)}$, where g is representative for $\mathcal{R}^{\hat{\hat{D}}}$.*

Proof. First we will show an $\mathcal{R}^{\hat{D}}$ list must exist. By the above lemma, $\hat{D}$ is T-equivalent to $\Phi(D_\chi)$, where χ has length $n+1$. An $\mathcal{R}^{D_\chi}$ list exists; hence, so does an $\mathcal{R}^{\hat{D}}$ list.

Because of the given properties of $\hat{\hat{D}}$, the $\mathcal{R}^{\hat{W}}$ list, and $\mathcal{R}^{\hat{\hat{D}}}$, if $\hat{R}$ is a set in the $\mathcal{R}^{\hat{D}}$ list, then $\hat{\hat{D}}$ does not live in $\hat{R}$. (This is true for the pre-images of these sets and hence for these sets.)

Inductively using an oracle for $g \oplus 0^{(5)}$ we will create an $\mathcal{R}^{\hat{D}}$ list. Assume that $\hat{R}_i^{\hat{D}}$ are known for $i < j$ and that for $e < j$ if $\hat{D}$ lives in W_e then there is an $i < j$ such that $W_e - \hat{M}^{W_e} =^* \hat{R}_i^{\hat{D}} - \hat{M}^{\hat{R}_i^{\hat{D}}}$. Look for the least $e \geq j$ such that $\hat{D}$ lives in W_e, $\hat{\hat{D}}$ does not live in W_e, and for all $i < j$ such that $W_e - \hat{M}^{W_e} \neq^* \hat{R}_i^{\hat{D}} - \hat{M}^{\hat{R}_i^{\hat{D}}}$. Such an e must exist. Let $\hat{R}_j^{\hat{D}} = W_e$. Apply the hatted version of Lemma 2.9 to get the $\hat{R}_j^{\hat{A}}$ disjoint from $\hat{R}_i^{\hat{A}}$. □

Definition 2.47. Let $\mathcal{T}(A)$ denote the class of atomic T-equivalence classes of successors (of A) with the binary relation of 1-successor restricted to successors of A.

Remark 2.48. So the map $\Lambda(\chi) = D_\chi$ is a map from T to $\mathcal{T}(A)$ taking a node to a representative of an atomic T-equivalent class of successors. Furthermore, ζ is an immediate successor of χ iff D_ζ is a 1-successor of D_χ. Hence Λ is an isomorphism. Recall $D_\chi = D_\alpha$ if $l(\alpha) = (\chi, 0)$. Hence Λ is computable along the true path which is computable in $\mathbf{0^{(2)}}$.

Lemma 2.49. *If A and $\hat{A}$ are in the same orbit witnessed by Φ then $\mathcal{T}(\hat{A})$ must exist and must be isomorphic to $\mathcal{T}(A)$ via an isomorphism induced by Φ and computable in $\Phi \oplus 0''$. The composition of this induced isomorphism and the above Λ is an isomorphism between $\mathcal{T}(\hat{A})$ and T. (This addresses part two of our game plan.)*

Our coding is not elementary; it is not even in $\mathcal{L}_{\omega_1,\omega}$. The coding depends on the infinite lists $\mathcal{R}^{D_\chi}$. One cannot say such a list exists in $\mathcal{L}_{\omega_1,\omega}$. It is open if there another coding of T which is elementary or in $\mathcal{L}_{\omega_1,\omega}$. This is another excellent open question.

Lemma 2.50. *$\mathcal{T}(\hat{A})$ has a presentation computable in $\mathbf{0^{(8)}}$.*

2.4. **More Requirements; The Homogeneity Requirements.** Let $\chi, \xi \in T$ be such that $\chi^- = \xi^-$. Then in terms of the above coding, the atomic T-equivalence classes of D_χ and D_ξ cannot be differentiated. For almost all i, D_χ and D_ξ live in $R_{\chi^-,2i}$ (if $M_{\chi^-,i}$ is maximal in $R_{\chi^-,i}$) and

$$\text{for all } i(D_\chi \text{ lives in } R_{\chi,i} \text{ iff } D_\xi \text{ lives in } R_{\xi,i}). \tag{2.50.1}$$

In this sense, these sets are homogeneous. What we are about to do has the potential to destroy this homogeneity. We must be careful not to destroy this homogeneity.

In fact, we must do far more than just restore this homogeneity. For each T_i we will construct an A_{T_i}. For all $\chi^{T_k} \in T_k$, we will construct $D_{\chi^{T_k}}$, $R_{\chi^{T_k},i}$, and $M_{\chi^{T_k},i}$. In order to complete part 3 of our game plan (that is, sets coded by isomorphic trees belong to the same orbit) we must ensure that the following homogeneity requirement holds.

Requirement 2.51. *For all $k, \hat{k}$, if $\chi^{T_k} \in T_k$, $\chi^{T_{\hat{k}}} \in T_{\hat{k}}$, and $|\chi^{T_k}| = |\chi^{T_{\hat{k}}}|$ then, for all i,*

$$M_{\chi^{T_k},i} \textit{ is maximal in } R_{\chi^{T_k},i} \textit{ iff } M_{\chi^{T_{\hat{k}}},i} \textit{ is maximal in } R_{\chi^{T_{\hat{k}}},i}, \textit{ and}$$

$$M_{\chi^{T_k},i} =^* R_{\chi^{T_k},i} \textit{ iff } M_{\chi^{T_{\hat{k}}},i} =^* R_{\chi^{T_{\hat{k}}},i}.$$

Remark 2.52. We cannot overstate the importance of this requirement. It is key to the construction of *all* of the needed automorphisms; see Section 2.6.3. Note that we use Section 2.6.3 twice; once in this proof and once in the proof of Theorem 1.6.

One consequence of this requirement is that we must construct all the sets, $D_{\chi^{T_k}}$, $R_{\chi^{T_k},i}$, and $M_{\chi^{T_k},i}$, simultaneously using the same tree of strategies. Up to this point we have been working with a single T. To dovetail all the trees into our construction at the node $\alpha \in Tr$ where $|\alpha| = k$ we will start coding tree T_k. Since at each node we only needed answers to a finite number of Δ^0_3 questions, this dovetailing is legal in terms of the tree argument. Note that each tree T gets its own copy of ω to work with.

So at each $\alpha \in Tr$, we will construct, for $k < |\alpha|$, $R^k_\alpha, E^k_\alpha, M^k_\alpha$, and D^k_α as above. The eth marker for M^k_α be will denoted $\Gamma^{\alpha,k}_e$. Assume that $\alpha \subset f$, $k < |\alpha|$, and $l(|\alpha| - k) = (\chi, i)$; then $R^k_\alpha = R_{\chi^{T_k},2i}$, $M^k_\alpha = M_{\chi^{T_k},2i}$, $\Gamma^{\alpha,k}_e = \Gamma^{\chi^{T_k},2i}_e$, $E_\alpha = R_{\chi^{T_k},2i+1} = M_{\chi^{T_k},2i+1}$, and if $i = 0$ then $D_\alpha = D_{\chi^{T_k}}$. In the following, when the meaning is clear, we drop the subscript k and assume we are working with a tree T.

2.5. **Meeting the Remaining Requirements.** The goal in this section is to understand what it takes to show D_χ is atomic when $\chi \in T \cup \{\lambda\}$. We have to do this and meet Requirements 2.17 and 2.51. Since we will meet Requirement 2.17, an $\mathcal{R}^{D_\chi}$ list exists. D_χ is potentially not atomic witnessed by an c.e. set $W \subseteq D_\chi$ if the set of i such that W lives in an $R^{D_\chi}_i$ is an infinite coinfinite set. We must make sure that W behaves cohesively on the sets $R^{D_\chi}_i$.

We will meet Requirement 2.44 by a e-state argument on the $R_{\chi,2i}$s; this is similar to a maximal set construction. With the maximal set construction, for each e, s and x, there are 2 states, either state 0 iff $x \notin W_{e,s}$ or 1 iff $x \in W_{e,s}$. Here the situation is more complex.

R has state 1 w.r.t. a single e iff $W_e \cap R =^* \emptyset$. R has state 2 w.r.t. a single e iff $W_e \cap R \neq^* \emptyset$. R has state 3 w.r.t. a single e iff there is an $\breve{e}$ such that $W_e \sqcup W_{\breve{e}} = M^R$. R has state 4 w.r.t. a single e iff there is an $\breve{e}$ such that $W_e \sqcup W_{\breve{e}} = R$. If the highest state of R is 3 w.r.t. a single e then W_e is a nontrivial split of M^R. Determining the state of R w.r.t. e is Σ^0_3. (The state 0 will be used later.)

Let $s_{e'}$ be the state of R w.r.t. e'. The e-state of R is the string $s_0 s_1 s_2 \dots s_e$. An e-state σ_1 is *greater* than σ_2 iff $\sigma_1 <_L \sigma_2$. We will do an e-state construction along the true path for the tree T_k.

Assume $\alpha \in Tr$, $e = |\alpha| - k$, and $l(e) = (\chi, i)$. Since at α we can get answers to a finite number of Δ^0_3 questions, at α we will have encoded answers to which, if any, $\beta \prec \alpha$, if $l(|\beta| - k) = (\chi, i')$ is $W_e \cap R^k_\beta$ is infinite; for which of the above βs and for which $j < e$, does W_l, for $l < e$, witness that W_j is a split of M^k_β; and for which of the above βs, and for which $j < e$, does W_l, for $l < e$, witness that W_j is a split of R^k_β?

Using this information, α will determine $\beta^{\alpha,k}_0, \beta^{\alpha,k}_1, \dots$ such that $R^k_{\beta^{\alpha,k}_e}$ has the greatest possible e-state according to the information encoded at α. This listing does not change w.r.t. stage. For all other $\beta \prec \alpha$ such that $l(|\beta| - k) = (\chi, i')$, when $\alpha \subseteq f_s$, α will dump $\Gamma^{\beta,k}_{|\alpha|}$ into M^k_β. If $\alpha \subset f$ then, for the above β, $R^k_\beta =^* M^k_\beta$.

One can show, for each e, there is an $\alpha_e \subset f$ such that, for all γ with $\alpha_e \preceq \gamma \subset f$, $\beta^{\alpha_e,k}_e = \beta^{\gamma,k}_e$. Hence $M^k_{\alpha_e}$ is maximal in $R^k_{\alpha_e}$ and Requirement 2.17 is met. In addition, one can show that, for almost all i, such that $M_{\chi^{T_k},i} \neq^* R_{\chi^{T_k},i}$, $R_{\chi^{T_k},i}$ have the same e-state and hence D_χ is atomic.

However, Equation 2.50.1 no longer holds and hence Requirement 2.51 is not met. The problem is that we can dump $M_{\chi^{T_k},i}$ into $R_{\chi^{T_k},i}$ without dumping $M_{\chi^{T_{\hat{k}}},i}$ into $R_{\chi^{T_{\hat{k}}},i}$.

The solution is that when we dump $M_{\chi^{T_k},i}$ we also must dump $M_{\chi^{T_{\hat{k}}},i}$, for all possible $\chi^{T_{\hat{k}}}$. This means that we have to do the above e-state construction for T_k simultaneously for all T_k. So, for each n, we have *one* e-state construction, for all $D_{\chi^{T_k}}$ and $D_{\xi^{T_k}}$, for all k and for all $\chi^{T_k}, \xi^{T_k} \in T_k$ with $|\chi^{T_k}| = |\xi^{T_k}| = n$.

To do this we need the following notation: Let $\{\xi_i : i \in \omega\}$ be a computable listing of all nodes of length n in $\omega^{<\omega}$. Fix some nice one-to-one onto computable listing, $\langle -, -, - \rangle$, of all triples (e, k, l) and, furthermore, assume if (e, k, l) is the mth triple listed then $\langle e, k, l \rangle = m$.

Assume $l(|\beta|) = (\xi, i)$ and $|\xi| = n$. If there is a $\beta' \preceq \beta$ and a $k \leq |\beta|$ such that $l(|\beta'| - k) = (\xi', i)$ (the same i as above) and $|\xi'| = n$ and, furthermore, β' is the lth such β' then the state of β w.r.t. $\langle e, k, l \rangle$ is the state of $R^k_{\beta'}$ w.r.t. e. Otherwise the state of β w.r.t. $\langle e, k, l \rangle$ is 0. Let $s_{\langle e', k', l' \rangle}$ be the state of β w.r.t. $\langle e', k', l' \rangle$. The $\langle e, k, l \rangle$-state of β is the string $s_{\langle e_0, k_0, l_0 \rangle} s_{\langle e_1, k_1, l_1 \rangle} s_{\langle e_2, k_2, l_2 \rangle} \ldots s_{\langle e, k, l \rangle}$.

Using the additional information we encoded into α for the single e-state construction, α has enough information to determine the $\langle e, k, l \rangle$-state of $\beta \preceq \alpha$. Using this information, α will determine $\beta^{\alpha}_{\langle e_0, k_0, l_0 \rangle}, \beta^{\alpha}_{\langle e_1, k_1, l_1 \rangle}, \ldots$ such that $\beta^{\alpha}_{\langle e, k, l \rangle}$ has the greatest possible $\langle e, k, l \rangle$-state according the information encoded at α. Again this listing does not change w.r.t. stage.

For all other $\beta \prec \alpha$ such that $l(|\beta|) = (\xi_{j'}, i')$ and $|\xi_{j'}| = n$, when $\alpha \subseteq f_s$, for all k, for all $\chi \in T^k$ of length n, α will dump $\Gamma_{|\alpha|}^{\chi^{T_k}, 2i'}$ into $M_{\chi^{T_k}, 2i'}$. If $\alpha \subset f$ then, for the above i', for all k, for all $\chi \in T^k$ of length n, $M_{\chi^{T_k}, 2i'} =^* R_{\chi^{T_k}, 2i'}$.

One can show, for each $\langle e, k, l \rangle$, there is an $\alpha_{\langle e, k, l \rangle} \subset f$ such that for all γ with $\alpha_{\langle e, k, l \rangle} \preceq \gamma \subset f$, $\beta^{\alpha_{\langle e,k,l \rangle}}_{\langle e, k, l \rangle} = \beta^{\gamma}_{\langle e, k, l \rangle}$. Assume $l(|\beta^{\alpha_{\langle e,k,l \rangle}}_{\langle e, k, l \rangle}|) = (\chi, i)$. Then, for all k, for all $\chi^{T_k} \in T^k$ of length n, $M_{\chi^{T_k}, 2i}$ is maximal in $R_{\chi^{T_k}, 2i}$. Hence Requirements 2.17 and 2.51 are met.

In addition, one can show that, for all e, for all k, for all $\chi^{T_k} \in T^k$ of length n, for almost all i, if $M_{\chi^{T_k}, 2i'}$ is maximal in $R_{\chi^{T_k}, 2i'}$ then $R_{\chi^{T_k}, i}$ has the same state w.r.t. e and, hence, $D_{\chi^{T_k}}$ is atomic. Thus Requirement 2.44 is met.

2.6. **Same Orbit.** Let T and $\hat{T}$ be isomorphic trees via an isomorphism Λ. We must build an automorphism Φ_Λ of $\mathcal{E}$ taking A to $\hat{A}$. We want to do this piecewise. That is, we want to build isomorphisms between the $\mathcal{E}^*(D_\chi)$ and $\mathcal{E}^*(\hat{D}_{\Lambda(\chi)})$ and piece them together in some fashion to get an automorphism. Examples of automorphisms constructed in such a manner can be found in Section 5 of Cholak et al. [6] and Section 7 of Cholak and Harrington [5].

In reality $T = T_k$ and $\hat{T} = T_{\hat{k}}$. The sets in question for T_k are $D_{\chi^{T_k}}$, $R_{\chi^{T_k}, i}$, and $M_{\chi^{T_k}, i}$. Here we will just drop the T_k superscript from χ. The sets in question for $T_{\hat{k}}$ are $D_{\chi^{T_{\hat{k}}}}$, $R_{\chi^{T_{\hat{k}}}, i}$, and $M_{\chi^{T_{\hat{k}}}, i}$. Here we will "hat" the sets involved and drop the $T_{\hat{k}}$ superscript from χ.

However, before we shift to our standard notation changes we would like to point out the following. Since Λ is an isomorphism between T_k and $T_{\hat{k}}$, $|\chi_k^T| = |\Lambda(\chi^{T_k})|$. Therefore, by Requirement 2.51, for all i,

$$M_{\chi^{T_k},i} \text{ is maximal in } R_{\chi^{T_k},i} \text{ iff } M_{\Lambda(\chi^{T_k}),i} \text{ is maximal in } R_{\Lambda(\chi^{T_k}),i},$$

$$\text{and } M_{\chi^{T_k},i} =^* R_{\chi^{T_k},i} \text{ iff } M_{\Lambda(\chi^{T_k}),i} =^* R_{\Lambda(\chi^{T_k}),i}.$$

2.6.1. *Extendible algebras of computable sets.* The workhorse for constructing Φ_Λ is the following theorem and two lemmas.

Theorem 2.53 (Theorem 5.10 of Cholak and Harrington [5])**.** *Let $\mathcal{B}$ be an extendible algebra of computable sets and similarly for $\hat{\mathcal{B}}$. Assume the two are extendibly isomorphic via Π. Then there is a Φ such that Φ is a Δ^0_3 isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$, Φ maps computable subsets to computable subsets, and, for all $R \in \mathcal{B}$, $(\Pi(R)-\hat{A}) \sqcup \Phi(R \cap A)$ is computable (and dually).*

Lemma 2.54. *Let $\chi \in T$. The collection of all $R_{\chi,i}$ forms an extendible algebra, $\mathcal{B}_\chi$, of computable sets.*

Proof. Apply Theorem 2.17 of Cholak and Harrington [5] to $A = \omega$ to get an extendible algebra of $\mathcal{S}_{\mathcal{R}}(\omega)$ of all computable sets with representation B. Let $j \in B_\chi$ iff there is an $i \leq j$ such that $S_j = R_{\chi,i}$. Now take the subalgebra generated by B_χ to get $\mathcal{B}_\chi$. ☐

Lemma 2.55. *Let $\chi \in T$; then the join of $\mathcal{B}_{\chi^-}$ and $\mathcal{B}_\chi$ is an extendible algebra of computable sets, $\mathcal{B}_{\chi^- \oplus \chi}$.*

Proof. See Lemma 2.16 of Cholak and Harrington [5]. ☐

Lemma 2.56. *For all i, if $R_{\xi,j} \neq_{\mathcal{R}} R_{\chi,i}$ and $R_{\xi,j} \neq_{\mathcal{R}} R_{\chi^-,i}$ then $D_\chi \cap R_{\xi,j} = \emptyset$.*

Proof. See Lemma 2.15. ☐

Lemma 2.57. *If $\chi, \xi \in T$ and $|\chi| = |\xi|$ then $\mathcal{B}_{\chi^- \oplus \chi}$ and $\hat{\mathcal{B}}_{\xi^- \oplus \xi}$ are extendibly isomorphic via $\Phi_{\chi,\xi}$ where $\Phi_{\chi,\xi}(R_{\chi^-,i}) = \hat{R}_{\xi^-,i}$ and $\Phi_{\chi,\xi}(R_{\chi,i}) = \hat{R}_{\xi,i}$. Furthermore, $\Phi_{\chi,\xi}$ is Δ^0_3.*

2.6.2. *Building Φ_Λ on the Ds and Ms.* The idea is to use Theorem 2.53 to map $\mathcal{E}^*(D_\chi)$ to $\mathcal{E}^*(\hat{D}_{\Lambda(\chi)})$. By the above lemmas, there is little question that the extendible algebras we need are some nice subalgebras of $\mathcal{B}_{\chi^- \oplus \chi}$ and $\hat{\mathcal{B}}_{\Lambda(\chi^-) \oplus \Lambda(\chi)}$ and the isomorphism between these nice subalgebras is induced by the isomorphism $\Phi_{\chi,\Lambda(\chi)}$.

We will use the following stepwise procedure to define part of Φ_Λ. This is not a computable procedure but computable in $\Lambda \oplus 0''$. χ is added to $\mathcal{N}$ at step s iff we determined the image of D_χ (modulo finitely many $R_{\chi^-,j}$). The parameter $i_{\chi,s}$ will be used to keep track of the $M_{\chi,i}$ which we have handled and will be increasing stepwise. This procedure does not completely define Φ_Λ; we will have to deal with those W which are not subsets of $\bigsqcup M \cup \bigsqcup D$.

Step 0: Let $\mathcal{N}_0 = \{\lambda\}$. By the above lemmas $\mathcal{B}_\lambda$ is isomorphic to $\hat{\mathcal{B}}_\lambda$ via $\Phi_{\lambda,\lambda}$. Let $i_{\lambda,0} = 0$. Now apply Theorem 2.53 to define Φ_Λ for $W \subseteq A = D_\lambda$ and dually.

Step $s+1$: *Part* $\chi\hat{\ }s$: For each $\chi \in \mathcal{N}_s$ such that $\chi\hat{\ }s \in T$ do the following: Add $\chi\hat{\ }s$ to $\mathcal{N}_{s+1}$. Let $i_{\chi\hat{\ }s,s+1} = 0$. Apply Lemma 2.16 to $\chi\hat{\ }s$ to get i'. Apply the hatted version of Lemma 2.16 to $\Lambda(\chi\hat{\ }s)$ to get $\hat{i}'$. Let $i_{\chi,s+1}$ be the max of i', $\hat{i}'$ and $i_{\chi,s}+1$. Let $\mathcal{B}^*_{\chi,\chi\hat{\ }s}$ be the extendible algebra generated by $R_{\chi,i}$, for $i \geq i_{\chi,s+1}$, and, for all j, $R_{\chi\hat{\ }s,j}$. Define $\mathcal{B}^*_{\Lambda(\chi),\Lambda(\chi\hat{\ }s)}$ in a dual fashion. Now $\Phi_{\chi\hat{\ }s,\Lambda(\chi\hat{\ }s)}$ induces an isomorphism between these two extendible algebras. Now apply Theorem 2.53 to define Φ_Λ for $W \subseteq \left(D_{\chi\hat{\ }s} - \bigsqcup_{i<i_{\chi,s+1}} R_{\chi,i}\right)$ and Φ_Λ^{-1} for $\hat{W} \subseteq \left(\hat{D}_{\Lambda(\chi\hat{\ }s)} - \bigsqcup_{i<i_{\chi,s+1}} \hat{R}_{\Lambda(\chi),i}\right)$.

Step $s+1$: *Part* $i_{\chi,s+1}$: For all $\chi \in \mathcal{N}_s$ and for all i such that $i_{\chi,s} \leq i < i_{\chi,s+1}$, do the following: Let $S_{\chi,i} = \left(M_{\chi,i} - \bigsqcup_{\xi\in\mathcal{N}_s} D_\xi\right)$ and $\hat{S}_{\Lambda(\chi),i} = \left(\hat{M}_{\chi,i} - \bigsqcup_{\xi\in\mathcal{N}_s} \hat{D}_{\Lambda(\xi)}\right)$. So $H_{\chi,i} \subseteq S_{\chi,i}$ and $\hat{H}_{\chi,i} \subseteq \hat{S}_{\Lambda(\chi),i}$. $S_{\chi,i}$ and $\hat{S}_{\Lambda(\chi),i}$ are both infinite and furthermore, by Equation 2.50.1, the one is computable iff the other is computable.

Subpart H: If both $S_{\chi,i}$ and $\hat{S}_{\Lambda(\chi),i}$ are noncomputable then apply Theorem 2.53 (using the empty extendible algebras) to define Φ_Λ for $W \subseteq S_{\chi,i}$ and Φ_Λ^{-1} for $\hat{W} \subseteq \hat{S}_{\Lambda(\chi),i}$. If both $S_{\chi,i}$ and $\hat{S}_{\Lambda(\chi),i}$ are computable then such Φ_Λ can be found by far easier means.

One can show that $T = \lim_s \mathcal{N}_s$ and that, for all i, $\chi \in T$, there is step s such that, for all $t \geq s$, $i_{\chi,t} \geq i$. For all $\chi \in T$, let s_χ be the step that χ enters $\mathcal{N}$ and $s_{\chi,i}$ be the first stage such that $i_{\chi,s_{\chi,i}} > i$.

2.6.3. *Defining* Φ_Λ *on* $R_{\chi,i}$. Let $s = s_{\chi,i}$. By Section 2.6.2, Φ_Λ is defined on

$$M_{\chi,i} = S_{\chi,i} \sqcup \bigsqcup_{\xi\in\mathcal{N}_s} (R_{\chi,i} \cap D_\xi);$$

$$\Phi_\Lambda(M_{\chi,i}) = \hat{S}_{\Lambda(\chi),i} \sqcup \bigsqcup_{\xi\in\mathcal{N}_s} \Phi_\Lambda(R_{\chi,i} \cap D_\xi).$$

Hence Φ_Λ is defined on subsets W of $M_{\chi,i}$. Furthermore, if such a W is computable so is $\Phi_\Lambda(W)$.

Let $\xi \in \mathcal{N}_s$. Then

$$R_{\chi,i} \cap \left(D_\xi - \bigsqcup_{j < i_{\xi^-, s_\xi}} R_{\xi^-,j} \right) = R_{\chi,i} \cap D_\xi,$$

$$\hat{R}_{\Lambda(\chi),i} - \left(\hat{D}_{\Lambda(\xi)} - \bigsqcup_{j < i_{\xi^-, s_\xi}} \hat{R}_{\Lambda(\xi^-),j} \right) = \hat{R}_{\Lambda(\chi),i} - \hat{D}_{\Lambda(\xi)},$$

and $\Phi_{\chi,\Lambda(\chi)}(R_{\chi,i}) = \hat{R}_{\Lambda(\chi),i}$. Therefore, by Theorem 2.53,

$$\hat{R}_{\Lambda(\chi),i} - \hat{D}_{\Lambda(\xi))} \sqcup \Phi_\Lambda(R_{\chi,i} \cap D_\xi) = \hat{X}_\xi$$

is computable. Since $\Phi_\Lambda(R_{\chi,i} \cap D_\xi) \subset \hat{D}_{\Lambda(\xi)}$, $\hat{R}_{\Lambda(\chi),i} \triangle \hat{X}_\xi \subseteq \hat{D}_{\Lambda(\xi)}$ (recall $\triangle$ is the symmetric difference between two sets). Fix computable sets $\tilde{R}^{in}_\xi$ and $\tilde{R}^{out}_\xi$ such that $\hat{X}_\xi = (\hat{R}_{\Lambda(\chi),i} \sqcup \tilde{R}^{in}_\xi) - \tilde{R}^{out}_\xi$.

Consider the computable set

$$\tilde{R} = (\hat{R}_{\Lambda(\chi),i} \sqcup \bigsqcup_{\xi \in \mathcal{N}_s} \tilde{R}^{in}_\xi) - \bigsqcup_{\xi \in \mathcal{N}_s} \tilde{R}^{out}_\xi.$$

Then

$$\tilde{R} - \bigsqcup_{\xi \in \mathcal{N}_s} \Phi_\Lambda(R_{\chi,i} \cap D_\xi) = \hat{S}_{\Lambda(\chi),i} \sqcup \left(\hat{R}_{\Lambda(\chi),i} - M_{\Lambda(\chi),i} \right).$$

Therefore

$$\tilde{R} - \Phi_\Lambda(M_{\chi,i}) = \hat{R}_{\Lambda(\chi),i} - M_{\Lambda(\chi),i}.$$

Since $M_{\chi,i}$ is maximal in $R_{\chi,i}$ or $M_{\chi,i} =^* R_{\chi,i}$, if $W \subseteq R_{\chi,i}$ either $W \subseteq^* M_{\chi,i}$ or there is computable R such that $R \subseteq M_{\chi,i}$ and $R_W \cup W = R_{\chi,i}$. In the former case, $\Phi_\Lambda(W)$ is defined. In the latter case, let

$$\Phi_\Lambda(W) = (\tilde{R} - \Phi_\Lambda(R_W)) \sqcup \Phi_\Lambda(W \cap R_W).$$

Hence $\Phi_\Lambda(R_{\chi,i}) = \tilde{R}$.

Since Λ is an isomorphism between T and $\hat{T}$, $|\chi| = |\Lambda(\chi)|$. Therefore, as we noted above, by Requirement 2.51, either $M_{\chi,i}$ is maximal in $R_{\chi,i}$ and $\hat{M}_{\chi,i}$ is maximal in $\hat{R}_{\Lambda(\chi),i}$ or $M_{\chi,i} =^* R_{\chi,i}$ and $\hat{M}_{\chi,i} =^* \hat{R}_{\Lambda(\chi),i}$. In either case, Φ_Λ induces an isomorphism between $\mathcal{E}^*(R_{\chi,i})$ and $\mathcal{E}^*(\tilde{R})$. Φ_Λ^{-1} on $\mathcal{E}^*(\hat{R}_{\Lambda(\chi),i})$ is handled in the dual fashion.

2.6.4. *Putting Φ_Λ together.* By Requirement 2.4 and our construction, for all e, there are finite sets F_D and F_R such that either

$$W_e \subseteq^* \left(\bigsqcup_{\chi \in F_D} D_\chi \cup \bigsqcup_{(\chi,i)\in F_R} R_{\chi,i} \right) \tag{2.57.1}$$

or there is an R_{W_e} such that

$$R_{W_e} \subseteq \left(\bigsqcup_{\chi \in F_D} D_\chi \cup \bigsqcup_{(\chi,i)\in F_R} R_{\chi,i} \right) \text{ and } W_e \cup R_{W_e} = \omega. \tag{2.57.2}$$

It is possible to rewrite the set

$$\bigsqcup_{\chi \in F_D} D_\chi \cup \bigsqcup_{(\chi,i)\in F_R} R_{\chi,i}$$

as

$$\bigsqcup_{\chi \in F_D} \left(D_\chi - \bigsqcup_{(\xi,j)\in F_\chi} R_{\xi,j} \right) \sqcup \bigsqcup_{(\chi,i)\in F_R^*} R_{\chi,i}, \tag{2.57.3}$$

where $F_R^* \subseteq F_R \cup \bigcup_{\chi \in F_D} F_\chi$ and F_χ is finite and includes the set $\{(\chi^-, l) : l < i_{\chi^-, s_\chi}\}$. Φ_Λ as defined in the Section 2.6.2 is well behaved on the first union in Equation 2.57.3 and, furthermore, on these unions computable sets are sent to computable sets. Similarly, by Section 2.6.3, Φ_Λ is well behaved on the second union in Equation 2.57.3 and, furthermore, on these unions computable sets are sent to computable sets.

If Equation 2.57.1 for e hold, then $\Phi(W_e)$ is determined. Otherwise Equation 2.57.2 holds and map $W_e = \overline{R_{W_e}} \sqcup (W \cap R_{W_e})$ to $\overline{\Phi(R_{W_e})} \sqcup \Phi(W \cap R_{W_e})$. Φ_Λ^{-1} is handled in the dual fashion. So Φ_Λ is an automorphism.

3. Invariants and Properly Δ^0_α orbits

It might appear that $\mathcal{T}(A)$ is an invariant which determines the orbit of A. But there is no reason to believe for an arbitrary A that $\mathcal{T}(A)$ is well defined. The following theorem shows that $\mathcal{T}(\hat{A})$ is an invariant as far as the orbits of the A_Ts are concerned. In Section 3.2, we prove a more technical version of the following theorem.

Theorem 3.1. *If $\hat{A}$ and A_T are automorphic via Ψ and $T \cong \mathcal{T}(\hat{A})$ via Λ then $A_T \approx \hat{A}$ via Φ_Λ where $\Phi_\Lambda \leq_T \Lambda \oplus \mathbf{0}^{(8)}$.*

Proof. See Section 3.1. ☐

Theorem 3.2 (Folklore[3])**.** *For all finite α there is a computable tree T_{i_α} from the list in Theorem 1.21 such that, for all computable trees T, T and T_{i_α} are isomorphic iff T and T_{i_α} are isomorphic via an isomorphism computable in deg$(T) \oplus 0^{(\alpha)}$. But, for all $\beta < \alpha$ there is an i^*_β such that $T_{i^*_\beta}$ and T_{i_α} are isomorphic but are not isomorphic via an isomorphism computable in $0^{(\beta)}$.*

It is open if the above theorem holds for all α such that $\omega \geq \alpha < \omega_1^{\mathrm{CK}}$. But if it does then so does the theorem below.

Theorem 3.3. *For all finite $\alpha > 8$ there is a properly Δ^0_α orbit.*

Proof. Assume that $A_{T_{i_\alpha}}$ and $\hat{A}$ are automorphic via an automorphism Φ. Hence, by part 2 of the game plan, $\mathcal{T}(\hat{A})$ and T_{i_α} are isomorphic. Since $\mathcal{T}(\hat{A})$ is computable in $0^{(8)}$, $\alpha > 8$, and by Theorem 3.2, $\mathcal{T}(\hat{A})$ and T_{i_α} via a $\Lambda \leq_T 0^{(\alpha)}$. By Theorem 3.1, $\hat{A}$ and $A_{T_{i_\alpha}}$ are automorphic via an automorphism computable in $0^{(\alpha)}$.

Fix β such that $8 \geq \beta < \alpha$. By part 3 of the game plan and the above paragraph, $A_{T_{i_\alpha}}$ and $A_{T_{i^*_\beta}}$ are automorphic via an automorphism computable in $0^{(\alpha)}$. Now assume $A_{T_{i^*_\beta}} \approx A_{T_{i_\alpha}}$ via Φ. By Lemma 2.49, $\mathcal{T}(A_{T_{i^*_\beta}}) \cong T_{i_\alpha}$ via Λ_Φ, where $\Lambda_\Phi \leq_T \Phi \oplus \mathbf{0^{(2)}}$. Since $\mathcal{T}(A_{T_{i^*_\beta}})$ is computable in $0^{(8)}$ and $\mathcal{T}(A_{T_{i^*_\beta}})$ is isomorphic to $T_{i^*_\beta}$ via an isomorphism computable in $0^{(\beta)}$ (part 1 of the game plan), by Theorem 3.2, $\Lambda_\Phi >_T 0^{(\beta)}$. Hence $\Phi >_T 0^{(\beta)}$. $\square$

3.1. **Proof of Theorem 3.1.** For A_T the above construction gives us a $\mathbf{0}''$ listing of the sets D_χ, $R_{\chi,i}$, and $M_{\chi,i}$. So they are available for us to use here. Our goal here is to redo the work in Section 2.6 without having a $\mathbf{0}''$ listing of the sets $\hat{D}_\chi$, $\hat{R}_{\chi,i}$, and $\hat{M}_{\chi,i}$. Our goal is to find a suitable listing of these sets and the isomorphisms $\Phi_{\chi,\Lambda(\chi)}$. And then start working from Section 2.6.2 onward to construct the desired automorphism using the replacement parts we have constructed. We work with an oracle for Λ and $0^{(8)}$.

Λ is an isomorphism between T and $\mathcal{T}(\hat{A})$. By Lemma 2.50, using $\mathbf{0}^{(8)}$ as an oracle, we can find a representative of each atomic T-equivalence class of n-successors of $\hat{A}$. Furthermore, we can assume that when choosing a representative we always choose a maximal representative of terms of T-equivalence. Hence we can consider Λ as a map that takes D_χ to a representative of the equivalent class which codes χ. Let $\hat{D}_{\Lambda(\chi)} = \Lambda(D_\chi)$.

[3] See Section 5.2 for more information and a proof.

We recall that each $R_{\chi,i}$ is broken into a number of pieces. First there is a subset $M_{\chi,i}$ which is either maximal in $R_{\chi,i}$ or almost equal to $R_{\chi,i}$. $M_{\chi,i}$ is split into several parts; $H_{\chi,i}$ and if $\xi = \chi\hat{\ }l \in T$ and $l^{-1}(\xi,0) \leq l^{-1}(\chi,i)$ or $\xi = \chi$ then $D_\xi \cap M_{\chi,i} = D_\xi \cap R_{\chi,i}$ is a infinite split of $M_{\chi,i}$; $M_{\chi,i}$ is computable iff all of these pieces are computable. Effectively in each χ and i we can give a finite set $F_{\chi,i}$ such that

$$R_{\chi,i} = (R_{\chi,i} - M_{\chi,i}) \sqcup H_{\chi,i} \sqcup \bigsqcup_{\xi \in F_{\chi,i}} (D_\xi \cap R_{\chi,i})$$

and either, for all $\xi \in F_{\chi,i}$, $M_{\chi,i}$ is maximal in $R_{\chi,i}$ and $D_\xi \cap R_{\chi,i}$ is a nontrivial split of $M_{\chi,i}$ or, for all $\xi \in F_{\chi,i}$, $M_{\chi,i} = R_{\chi,i}$ and $D_\xi \cap R_{\chi,i}$ is computable. Now we must find $\hat{R}_{\Lambda(\chi),i}$ such that it has the same properties.

We need the following two lemmas. The first follows from the definition of an extendible subalgebra. The second lemma follows from the construction of A_T and the fact that, for almost all i, D_ξ lives in $R_{\xi^-,i}$ iff D_{ξ^-} lives in $R_{\xi^-,i}$. The second part of the second lemma follows in particular from the homogeneity requirements.

Lemma 3.4. *The collection of the sets*

$$\begin{gathered}\{(R_{\xi^-,i} \cap D_\xi) : i \geq j\}, \{(\overline{R}_{\xi^-,i} \cap D_\xi) : i \geq j\}, \\ \{(R_{\xi,i} \cap D_\xi) : i \geq 0\}, \text{ and } \{(\overline{R}_{\xi,i} \cap D_\xi) : i \geq 0\}\end{gathered} \tag{3.4.1}$$

form an extendible subalgebra, $\mathbb{B}_{\xi,j}$, of the splits of D_ξ.

Lemma 3.5. *If $|\xi| = |\zeta|$ then there is a $j_{\xi,\zeta}$ such that $\mathbb{B}_{\xi,j_{\xi,\zeta}}$ is extendibly Δ^0_3-isomorphic to $\mathbb{B}_{\zeta,j_{\xi,\zeta}}$ via the identity map. (The identity map sends $R_{\xi,i} \cap D_\xi$ to $R_{\zeta,i} \cap D_\zeta$, etc.) Furthermore, for all i, D_ξ lives in $R_{\chi,i}$ iff D_ζ lives in $R_{\chi,i}$ and, for all $i \geq j_{\xi,\zeta}$, D_ξ lives in $R_{\chi^-,i}$ iff D_ζ lives in $R_{\chi^-,i}$.*

Now we must use another theorem from Cholak and Harrington [5].

Theorem 3.6 (Theorem 6.3 of Cholak and Harrington [5])**.** *Assume D and $\hat{D}$ are automorphic via Ψ. Then D and $\hat{D}$ are automorphic via Θ where $\Theta \restriction \mathcal{E}(D)$ is Δ^0_3.*

Lemma 3.7. *For some j_ξ, there is an extendible subalgebra, $\hat{\mathbb{B}}_{\Lambda(\xi),j_\xi}$, of the splits of $D_{\Lambda(\xi)}$ which is extendibly Δ^0_3 isomorphic via Θ_ξ to $\mathbb{B}_{\xi,j_\xi}$. Furthermore, for all $i \geq j_\xi$, $D_\xi \cap R_{\xi^-,i}$ is the split of a maximal set iff $\Theta_\xi(D_\xi \cap R_{\xi^-,i})$ is the split of a maximal set, and $D_\xi \cap R_{\xi^-,i}$ is computable iff $\Theta_\xi(D_\xi \cap R_{\xi^-,i})$ is computable. And, for all i, $D_\xi \cap R_{\xi,i}$ is the split of a maximal set iff $\Theta_\xi(D_\xi \cap R_{\xi,i})$ is the split of a maximal set, and*

$D_\xi \cap R_{\xi,i}$ is computable iff $\Theta_\xi(D_\xi \cap R_{\xi,i})$ is computable. Moreover, we can find j_ξ, $\hat{\mathbb{B}}_{\Lambda(\xi),j_\xi}$, and Θ_ξ with an oracle for $\mathbf{0}^{(8)}$.

Proof. Recall A and $\hat{A}$ are automorphic via Ψ and the image of a D_ξ must also code a node of length $|\xi|$. By Lemma 2.45, $\hat{D}_{\Lambda(\xi)}$ is the preimage under Ψ of some $D_{\Psi^{-1}(\Lambda(\xi))} =^* D_\eta - \bigsqcup_{j<j'} R_{\eta^-,j}$, where $|\eta| = |\xi|$. Now apply Theorem 3.6 to get Θ_ξ. Find the least j_ξ such that, for all $i \geq j_\xi$, $D_{\Lambda(\xi)}$ lives in $R_{\Lambda(\xi)^-,i}$ iff $D_{\Lambda(\xi)^-}$ lives in $R_{\Lambda(\xi)^-,i}$ and similarly for $D_{\Psi^{-1}(\Lambda(\xi))}$ and $D_{\Psi^{-1}(\Lambda(\xi))^-}$, and D_ξ and D_{ξ^-}. The image of $\mathbb{B}_{\Psi^{-1}(\Lambda(\xi)),j_\xi}$ under Θ_ξ is an extendible subalgebra $\hat{\mathbb{B}}_{\Lambda(\xi),j_\xi}$ and, furthermore, these subalgebras are extendibly Δ^0_3-isomorphic. By Lemma 3.5, $\mathbb{B}_{\xi,j_\xi}$ is extendibly Δ^0_3-isomorphic to $\mathbb{B}_{\Psi^{-1}(\Lambda(\xi)),j_\xi}$. Since Θ_ξ is an automorphism the needed homogeneous properties are preserved.

Now that we know these items exist we know that we can successfully search for them. Look for an j_ξ and Θ_ξ such that $\Theta_\xi(\mathbb{B}_{\xi,j_\xi}) = \hat{\mathbb{B}}_{\Lambda(\xi),j_\xi}$ is extendibly Δ^0_3-isomorphic to $\mathbb{B}_{\xi,j_\xi}$ via Θ_ξ; these items also satisfiy the second sentence of the above lemma and the additional property that, for all $\hat{R}$, if $\hat{R}$ is an infinite subset of $D_{\Lambda(\xi)}$ then there are finitely many $\tilde{R}_i$ such that $\hat{R} \subseteq^* \bigcup \Theta_\xi(\tilde{R}_i)$. Since, by Requirement 2.4, this last property is true of D_ξ, and Θ_ξ is generated by an automorphism, it also must be true of $D_{\Lambda(\xi)}$. This extra property ensures that Θ_ξ is onto. By carefully counting quantifiers we see that $\mathbf{0}^{(8)}$ is more than enough to find these items. $\square$

Let $\tilde{F}_{\chi,i}$ be such that $\xi \in \tilde{F}_{\chi,i}$ iff $\xi \in F_{\chi,i}$ and $i \geq j_\xi$. For all χ and i, let

$$\breve{\hat{H}}_{\Lambda(\chi),i} = \bigsqcup_{\xi \in \tilde{F}_{\chi,i}} \Theta_\xi(D_\xi \cap R_{\chi,i}).$$

Either $\breve{\hat{H}}_{\Lambda(\chi),i}$ is computable or the split of a maximal set. This follows from the projection through the above lemmas of the homogeneity requirements. In the latter case, $\breve{\hat{H}}_{\Lambda(\chi),i}$ lives inside $\hat{\omega}$.

We repeatly apply the dual of Lemma 2.9 to all those $\breve{\hat{H}}_{\Lambda(\chi),i}$ who live inside $\hat{\omega}$ to get $\tilde{R}_{\Lambda(\chi),i}$ which are all pairwise disjoint. This determines the $\tilde{\hat{M}}_{\Lambda(\chi),i}$ which witness that $\breve{\hat{H}}_{\Lambda(\chi),i}$ lives in $\tilde{R}_{\Lambda(\chi),i}$. Let $\breve{\hat{R}}_{\Lambda(\chi),i}$ be a computable infinite subset of $\tilde{\hat{M}}_{\Lambda(\chi),i} - \breve{\hat{H}}_{\Lambda(\chi),i}$ (we call this *set subtraction*). Let $\hat{R}_{\Lambda(\chi),i} = \tilde{R}_{\Lambda(\chi),i} - \breve{\hat{R}}_{\Lambda(\chi),i}$. $\breve{\hat{H}}_{\Lambda(\chi),i}$ lives inside $\hat{R}_{\Lambda(\chi),i}$. In this case, again, by the dual of Lemma 2.9, we have determined $\hat{M}_{\Lambda(\chi),i}$ and hence we have determined $\hat{H}_{\Lambda(\chi),i}$.

So it remains to find $\hat{R}_{\Lambda(\chi),i}$ and $\hat{M}_{\Lambda(\chi),i}$, where $\breve{\hat{H}}_{\Lambda(\chi),i}$ is computable. For such i once we find $\hat{R}_{\Lambda(\chi),i}$ we will let $\hat{R}_{\Lambda(\chi),i} = \hat{M}_{\Lambda(\chi),i}$.

By Requirement 2.4 and our construction, for all e, there are finite sets F_D and F_R such that either

$$W_e \subseteq^* \left(\bigsqcup_{\chi \in F_D} D_\chi \cup \bigsqcup_{(\chi,i)\in F_R} R_{\chi,i} \right),$$

or there is an R_{W_e} such that

$$R_{W_e} \subseteq \left(\bigsqcup_{\chi \in F_D} D_\chi \cup \bigsqcup_{(\chi,i)\in F_R} R_{\chi,i} \right) \text{ and } W_e \cup R_{W_e} = \omega.$$

By Lemma 2.45, as a collection the $\hat{D}_{\Lambda(\chi)}$s are the isomorphic images of the collection of the D_χ and similarly with the collection of all $R_{\chi,i}$s. Hence we should be able to define $\hat{R}_{\Lambda(\chi),i}$, where $\breve{\hat{H}}_{\Lambda(\chi),i}$ is computable such that, for all e, there are finite sets $\hat{F}_D$ and $\hat{F}_R$ with either

$$\hat{W}_e \subseteq^* \left(\bigsqcup_{\chi \in \hat{F}_D} \hat{D}_{\Lambda(\chi)} \cup \bigsqcup_{(\chi,i)\in \hat{F}_R} \hat{R}_{\Lambda(\chi),i} \right), \tag{3.7.1}$$

or there is an $R_{\hat{W}_e}$ such that

$$R_{\hat{W}_e} \subseteq \left(\bigsqcup_{\chi \in \hat{F}_D} \hat{D}_{\Lambda(\chi)} \cup \bigsqcup_{(\chi,i)\in \hat{F}_R} \hat{R}_{\Lambda(\chi),i} \right) \text{ and } \hat{W}_e \cup R_{\hat{W}_e} = \hat{\omega}. \tag{3.7.2}$$

Fix some nice listing of the (χ, i) such that $\hat{R}_{\Lambda(\chi),i}$ has yet to be defined (as above). Assume that (χ, i) is the eth member in our list and the first $e-1$ of $\hat{R}_{\Lambda(\chi),i}$ have been defined such that, for all $e' < e$, one of the two equations above hold. For all e, either there are finitely many (ξ, j) where $\hat{R}_{\Lambda(\xi),j}$ is defined such that $\hat{R}_{\Lambda(\xi),j} \cap \hat{W}_e \neq^* \emptyset$ or, for almost all (ξ, j), where $\hat{R}_{\Lambda(\xi),j}$ is defined, $\hat{R}_{\Lambda(\xi),i} \subseteq^* \hat{W}_e$ (this is true for any possible pre-image of $\hat{W}_e$).

In the first case find a computable $\hat{R}$, a finite $\hat{F}_R$, and a finite $\hat{F}_D$ such that if $(\xi, j) \in \hat{F}_R$ then $\hat{R}_{\Lambda(\xi),j}$ is defined; if $\hat{R}_{\Lambda(\xi),j}$ is defined then $\hat{R} \cap \hat{R}_{\Lambda(\xi),j} = \emptyset$; $\breve{\hat{H}}_{\Lambda(\chi),i} \subseteq \hat{R}$; $(\hat{R} - \breve{\hat{H}}_{\Lambda(\chi),i}) \cap \bigsqcup_\xi \hat{D}_{\Lambda(\xi)} = \emptyset$ (these last three clauses are possible because of the above set subtraction); and

$$\hat{W}_e \subseteq^* \left(\hat{R} \cup \bigsqcup_{\xi \in \hat{F}_D} \hat{D}_{\Lambda(\xi)} \cup \bigsqcup_{(\xi,i)\in \hat{F}_R} \hat{R}_{\Lambda(\xi),i} \right).$$

In the second case find a computable $\hat{R}$, a finite $\hat{F}_R$, and a finite $\hat{F}_D$ such that all of the above but the last clause above hold and

$$\overline{\hat{W}_e} \subseteq^* \left(\hat{R} \cup \bigsqcup_{\xi \in \hat{F}_D} \hat{D}_{\Lambda(\xi)} \cup \bigsqcup_{(\xi,i) \in \hat{F}_R} \hat{R}_{\Lambda(\xi),i} \right).$$

Either way let $R_{\Lambda(\chi),i} = \hat{R}$. Since the sets we have defined so far cannot be all the images of the $R_{\xi,l}$, there must be enough of $\hat{\omega}$ for us to continue the induction.

Now we have to find a replacement for the isomorphisms given to us by Lemma 2.57; we cannot. But as we work through Section 2.6.2 we see that we want to apply Theorem 5.10 of Cholak and Harrington [5] to $D_\xi - \bigsqcup_{j<j_\xi} R_{\xi^-,j}$ and $D_{\Lambda(\xi)} - \bigsqcup_{j<j_\xi} \Theta_\xi(R_{\xi^-,j})$, we need these isomorphisms to meet the hypothesis, and, furthermore, this is the only place these isomorphisms are used. However, the first step of the proof of Theorem 5.10 of Cholak and Harrington [5] is to use the given isomorphisms (given by Lemma 2.57) to create an extendible isomorphism between extendible subalgebra generated by $R_{\chi,i} \cap D_\xi$ and the one generated by $\hat{R}_{\chi,i} \cap \hat{D}_{\Lambda(\xi)}$ and, furthermore, this is the only place these given isomorphisms are used in the proof. These subalgebras are $\mathbb{B}_{\xi,j_\xi}$ and $\hat{\mathbb{B}}_{\Lambda(\xi),j_\xi}$ which are isomorphic via Θ_ξ. Hence we can assume that we can apply Theorem 5.10 of Cholak and Harrington [5].

At this point we have all the needed sets and isomorphisms with the desired homogeneity between these sets (in terms of Requirement 2.51). Now we have enough to apply part 3 of our game plan to construct the desired automorphism. That is, start working from Section 2.6.2 onward to construct the desired automorphism.

3.2. **A Technical Invariant for the orbit of A_T.** The goal of this section is to prove a theorem like Theorem 3.1 but without the hypothesis that A and $\hat{A}$ are in the same orbit. Reflecting back through the past section we see that the fact that A and $\hat{A}$ are in the same orbit was used twice: in the proof of Lemma 3.7 and in showing that Equations 3.7.1 and 3.7.2 hold. Hence we assume these two items would allow us to weaken the hypothesis as desired. Since the notation from the above section is independent of the fact that A and $\hat{A}$ are in the same orbit we borrow it wholesale for the following.

Theorem 3.8. *Assume*
(1) $T \cong \mathcal{T}(\hat{A})$ *via* Λ,
(2) *the conclusion of Lemma 3.7 (the whole statement of the lemma is the conclusion), and*
(3) *Equations 3.7.1 and 3.7.2 hold.*

Then $A_T \approx \hat{A}$ *via* Φ_Λ *where* $\Phi_\Lambda \leq_T \Lambda \oplus \mathbf{0}^{(\mathbf{8})}$.

Corollary 3.9. $A_T \approx \hat{A}$ *iff*

(1) $T \cong \mathcal{T}(\hat{A})$ *via* Λ,
(2) *the conclusion of Lemma 3.7 (the whole statement of the lemma is the conclusion), and*
(3) *Equations 3.7.1 and 3.7.2 hold.*

4. Our Orbits and Hemimaximal Degrees

A set is *hemimaximal* iff it is the nontrivial split of a maximal set. A degree is *hemimaximal* iff it contains a hemimaximal set.

Let T be given. Construction A_T as above. For all i, either A_T lives in R_i or $A_T \cap R_i$ is computable. If A_T lives in R_i then $A_T \cap R_i$ is a split of maximal set $M \sqcup \overline{R}_i$ and hence $A_T = (A_T \cap R_i)$ is a hemimaximal set. $A_T = \bigsqcup_{i\in\omega}(A_T \cap R_i)$ where $A_T \cap R_i$ is either hemimaximal or computable. So the degree of A_T is the infinite join of hemimaximal degrees. It is not known if the (infinite) join of hemimaximal degrees is hemimaximal. Moreover, this is not an effective infinite join. But if we control the degrees of $A_T \cap R_i$ we can control the degree of A_T.

Theorem 4.1. *Let* H *be hemimaximal. We can construct* A_T *such that* $A_T \equiv_T H$. *Call this* A_T, A^H_T, *to be careful.*

Proof. Consider those α and k such that $l(|\alpha| - k) = (\lambda, n)$, for some n. Only at such α do we construct pieces of $D^k_\lambda = A_{T_k}$. Uniformly we can find partial computable mapping, p^k_α, from ω to R^k_α such that if R^k_α is an infinite computable set then p^k_α is one-to-one, onto, and computable. Since H is hemimaximal there is a maximal set M and a split $\breve{H}$ witnessing that H is hemimaximal. Then $p^k_\alpha(M) \sqcup \overline{R}^k_\alpha$ is maximal and $p^k_\alpha(H)$ is nontrivial split of $p^k_\alpha(M) \sqcup \overline{R}^k_\alpha$ with the same degree as H.

The idea is that at α we would like to let $M^k_\alpha = p^k_\alpha(M)$ but because of the dumping this does not work. Dumping allows us to control whether $R^k_\alpha =^* M^k_\alpha$ or not. Let $\tilde{M}^k_\alpha = p^k_\alpha(M)$. If

$$\overline{p^k_\alpha(M_s)} \cap R^k_\alpha = \{m^{\alpha,k}_0, m^{\alpha,k}_1, m^{\alpha,k}_2, \ldots\}$$

then place the marker $\Gamma^{\alpha,k}_e$ on $m^{\alpha,k}_e$ at stage s. Now when dumping the element marked by marker $\Gamma^{\alpha,k}_e$ we will just dump that *single* element (this not the case in the standard dumping arguments). Now assume that the dumping is done effectively (this is the case in the construction of A_T). Let $M^k_{\alpha,s+1} = \tilde{M}^k_{\alpha,s+1} \cup M^k_{\alpha,s}$ plus those m^α_e which are dumped via $\Gamma^{\alpha,k}_e$ at stage $s+1$. M^k_α is c.e. and $\tilde{M}^k_\alpha \subseteq M^k_\alpha$. Since $\tilde{M}^k_\alpha \sqcup \overline{R}^k_\alpha$

is maximal, either $M^k_\alpha =^* \tilde{M}^k_\alpha$ or $M^k_\alpha =^* R^k_\alpha$. In the first case $p^k_\alpha(H)$ and $p^k_\alpha(\breve{H}) \sqcup \overline{R}^k_\alpha$ are nontrivial splits of M^k_α. The second case occurs iff there is least $\Gamma^{\alpha,k}_e$ which is dumped into M^k_α infinitely often. The above construction of M^k_α is uniformly in α.

In Section 2.2.3, when we construct M^k_α and its splits, rather than using the maximal set construction and the Friedberg splitting construction, we use the above construction of M^k_α; we will put the split $p^k_\alpha(H)$ into $D^k_\lambda = A_T$ and use the Friedberg splitting construction to split $p^k_\alpha(\breve{H})$ into enough pieces as determined by the construction. □

There is no reason to believe that if $\hat{A}$ is in the same orbit as A^H_T that $\hat{A} \equiv_T H$. Nor is there a reason to believe $\hat{A}$ must have hemimaximal degree. Notice that for each H we have a separate construction. Hence the homogeneity requirement need not hold between these different constructions. Therefore, we cannot prove that the sets A^H_T are in the same orbit. It might be that for $H \neq \tilde{H}$, that A^H_T and $A^{\tilde{H}}_T$ are in different orbits. We conjecture, using Corollary 3.9, it is possible to construct two different versions of A_T which are not in the same orbit. But we can do the following.

Theorem 4.2. *There is an A_T whose orbits contain a representative of every hemimaximal degree.*

Proof. The idea is for all hemimaximal H to do the above construction simultaneously. This way the homogeneous requirement will be met between the different A^H_Ts.

Notice the above construction is uniformly in the triple $e = \langle m, h, \breve{h}\rangle$ where $W_m = M, W_h = H$, and $W_{\breve{h}} = \breve{H}$.

We want to reorder the trees from Theorem 1.21. Let $\tilde{T}_{\langle e,i\rangle} = T_i$. Now do the construction in Section 2 with two expectations: use the trees $\tilde{T}_{\langle e,i\rangle}$ and, for those α and k such that $l(|\alpha| - k) = (\lambda, n)$, for some n, we use the construction of M^k_α outlined in the proof of Theorem 4.1.

For all i and e coding a hemimaximal set we construct a set $A_{\tilde{T}_{\langle e,i\rangle}}$. If e' codes another hemimaximal set then $A_{\tilde{T}_{\langle e,i\rangle}}$ and $A_{\tilde{T}_{\langle e',i\rangle}}$ are in the same orbit.

If e' does not code sets such that $W_m = W_h \sqcup W_{\breve{h}}$ then construction of $A_{\tilde{T}_{\langle e',i\rangle}}$ is impaired but this does not impact the simultaneous construction of the other $A_{\tilde{T}_{\langle e,i\rangle}}$. □

5. On the Isomorphism Problem for Boolean Algebras and Trees

5.1. Σ^1_1**-completeness.** We think it is well known that the isomorphism problem for Boolean Algebras and Trees are Σ^1_1-complete, at least in the form stated in Theorems 1.7 and 1.21. We have searched for a reference to a proof for these theorems without success. It seems very likely that these theorems were known to Kleene. There are a number of places where something very close to what we want appears; for example, see the example at the end of Section 5 of Goncharov et al. [10] and surely there are earlier examples (for example, White [21]). All of these work by coding the Harrison ordering, as will the construction below. To be complete we include a proof in this section. We are thankful to Noam Greenberg for providing the included proof.

Remark 5.1 (Notation). For cardinals κ, λ, etc. (we use 2 and ω), a *tree on* $\kappa \times \lambda$ is a downward-closed subset of

$$\bigcup_{n<\omega} \kappa^n \times \lambda^n,$$

so that the set of paths of the tree is a closed subset of $\kappa^\omega \times \lambda^\omega$. We may use more or fewer coordinates. For a tree R, $[R]$ is the set of paths through R. For a subset A of a product space $\kappa^\omega \times \lambda^\omega$ (for example), pA is the projection of A onto the first coordinate.

Lemma 5.2. *There is an effective operation I such that, given a computable infinite-branching tree T, $I(T)$ is a computable linear ordering such that*

(1) *if T is well-founded then $I(T)$ is a well-ordering;*
(2) *if T is not well-founded then $I(T) \cong \omega_1^{\mathrm{CK}}(1+\mathbb{Q})$.*

Proof. Suppose that a computable tree $T_0 \subseteq \omega^{<\omega}$ is given. Unpair to get a tree T_1 on $2 \times \omega$ such that $[T_0] = \{X \oplus f : (X, f) \in [T_1]\}$.

Now let $T_2 = T_1 \times 2^{<\omega}$, the latter inserted as a second coordinate (so $T_2 = \{(\sigma, \tau, \rho) : (\sigma, \rho) \in T_1 \ \&\ \tau \in 2^{<\omega} \ \&\ |\tau| = |\sigma| = |\rho|\}$.) Let T_3 be the tree on $2 \times \omega$ which is obtained by pairing the first two coordinates of T_2.

The class HYP of hyperarithmetic reals is Π^1_1, and so $p[T_3] - \mathrm{HYP}$ is Σ^1_1; let T_4 be a computable tree such that $p[T_4] = p[T_3] - \mathrm{HYP}$.

Let L_5 be the Kleene-Brouwer linear ordering obtained from T_4; finally, let $I(T) = L_5\omega = L_5 + L_5 + \cdots$.

The point is this: $p[T_2] = p[T_1] \times 2^\omega$. Thus if T is not well-founded, then $p[T_1]$ is nonempty and so $p[T_2]$ is uncountable and so $p[T_4]$, and hence $[T_4]$, is nonempty. If T is well-founded then $p[T_4]$ is empty; that is, T_4 is well-founded. Also, $p[T_4]$ contains no hyperarithmetic sets, and so T_4 has no hyperarithmetic paths.

It follows that if T is well-founded then L_5, and so $I(T)$, is a well-ordering. If T is not well-founded then L_5 is a computable linear ordering which is not a well-ordering but has no hyperarithmetic infinite descending chains, that is, a Harrison linear ordering. This has order-type $\omega_1^{\mathrm{CK}}(1+\mathbb{Q})+\gamma$ for some computable ordinal γ. For any computable γ we have $\gamma+\omega_1^{\mathrm{CK}}=\omega_1^{\mathrm{CK}}$ (as ω_1^{CK} is closed under addition) and so $I(T)$ has ordertype $\omega_1^{\mathrm{CK}}(1+\mathbb{Q}+1+\mathbb{Q}+1+\mathbb{Q}+\cdots)\cong\omega_1^{\mathrm{CK}}(1+\mathbb{Q})$. □

Corollary 5.3. *For any Σ_1^1 set A, there is a computable sequence $\langle L_n\rangle$ of (computable) linear orderings such that, for all n,*

(1) *if $n\in A$ then $L_n\cong\omega_1^{\mathrm{CK}}(1+\mathbb{Q})$;*
(2) *if $n\notin A$ then L_n is a well-ordering.*

Proof. Let A be a Σ_1^1 set. There is a computable sequence $\langle T_n\rangle$ of trees on ω such that, for all n, $n\notin A$ iff T_n is well-founded. Now apply I to each T_n. □

Corollary 5.4 (Theorem 1.21)**.** *There is a computable tree T on ω such that the collection of computable trees S which are isomorphic to T is Σ_1^1-complete.*

Proof. Use the operation that converts a linear ordering L to the tree T_L of finite descending sequences in L. The point is that if L is an ordinal then T_L is well-founded and so cannot be isomorphic to $T_{\omega_1^{\mathrm{CK}}(1+\mathbb{Q})}$. □

Corollary 5.5 (Theorem 1.7)**.** *There is a computable Boolean algebra B such that the collection of Boolean algebras C that are isomorphic to B is Σ_1^1-complete.*

Proof. Similar; use the interval algebra B_L. If L is an ordinal then B_L is superatomic. □

5.2. **Π_n^0-completeness.** Again we believe it is known that there are trees T_{Π_n} such that the isomorphism problem for T_{Π_n} is Π_n^0-complete, at least in the form stated in Theorem 3.2. The closest we could find was work in White [21], which does not quite work. To be complete we include a proof in this section. The details are similar in style but different from what is found in [21]. The trees in [21] do not provide precise bounds; they are hard for the appropriate class but not known to be complete (see Remark 5.10). We wonder if Theorem 3.2 is true for all computable ordinals, the case $\alpha=\omega$ being a good test case. The following construction is joint work with Noam Greenberg. The following lemma is well known, but we include a proof for completeness; it is a partial version of uniformalization.

Lemma 5.6. *Let $A(n,x)$ be a Π^0_1 relation. Then there is a Π^0_1 partial function f such that* $\operatorname{dom} A = \operatorname{dom} f$.

Proof. We give an effective construction of a computable predicate R such that $f(n) = x \iff \forall y\, R(n,x,y)$. If $n \geq s$ or $x \geq s$ then $R(n,x,s)$ always holds; so to make R computable, at stage s of the construction we define $R(n,x,s)$ for all $x, n < s$. In fact, for all $n < s$, at stage s we define $R(n,x,s)$ to hold for at most one $x < s$. This will imply that f is indeed a function.

Let S be a computable predicate such that $A(n,x) \iff \forall y\, S(n,x,y)$.

For every n and x we have a moving marker $c(n,x)$. We start with $c(n,x) = x$. At stage s, for every $n < s$, find the least $x < s$ such that for all $y < s$ we have $S(n,x,y)$ (if one exists). For $x' \neq x$, initialize $c(n,x')$ by redefining it to be large. Now define R by letting $R(n,c(n,x),s)$ hold but $R(n,z,s)$ not hold for all $z < s$ different from $c(n,x)$.

Let $n < \omega$. Suppose that $n \in \operatorname{dom} f$. For all $s > \max\{n, f(n)\}$, $R(n,f(n),s)$ holds, which means that at stage s, $f(n) = c(n,x)$ for some x. Different markers get different values and so there is just one such x, independent of s. By the instructions, for all $s > \max\{n, f(n)\}$, for all $y < s$, $S(n,x,y)$ holds; this shows that $n \in \operatorname{dom} A$.

Suppose that $n \in \operatorname{dom} A$. Let x be the least such that for all y, $S(n,x,y)$ holds. There is some stage after which $c(n,x)$ does not get initialized (wait for some stage s that bounds, for all $z < x$, some y such that $S(n,z,y)$ does not hold). Let s be the last stage at which $c(n,x)$ gets initialized. At stage s, a final, large value $a = c(n,x)$ is chosen. For all $t > a$, $R(n,a,t)$ holds because $t > s$. Thus a witnesses that $n \in \operatorname{dom} f$. $\square$

By relativizing the above to $\mathbf{0}^{(n-2)}$, we see that for every $n \geq 2$, for every Σ^0_n set A, there is a Π^0_{n-1} function f such that $A = \operatorname{dom} f$.

A *tree* is a downward closed subset of $\omega^{<\omega}$. The collection $Tree$ of all computable trees (i.e., indices for total, computable characteristic functions of trees) is Π^0_2. For any tree T, let Isom_T be the collection of $S \in Tree$ which are isomorphic to T.

Lemma 5.7. *Let T_{Π_2} be the infinite tree of height 1.* $\mathrm{Isom}_{T_{\Pi_2}}$ *is Π^0_2-complete.*

Proof. A tree is isomorphic to T_{Π_2} iff it has height 1 and it is infinite. Certainly this is a Π^0_2 property.

Let A be a Π^0_2 set; say that $A(n) \iff \forall x \exists y R(n,x,y)$ where R is computable. For n and s, let $l(n,s)$ be the greatest l such that for all

$x \leq l$ there is some $y < s$ such that $R(n, x, y)$ holds. Say that s is expansionary for n if $l(n, s) > l(n, s - 1)$.

For each n define a tree $T_{2,A}(n)$: this is a tree of height 1, and a string $\langle s \rangle$ is on the tree iff s is expansionary for n. Then $n \mapsto T_{2,A}(n)$ reduces A to $\mathrm{Isom}_{T_{\Pi_2}}$. □

For the next level we use trees of height 2. We use two trees: the tree T_{Π_3} is the tree of height 2 such that for each n there are infinitely many level 1 nodes which have exactly n children, and no level 1 node has infinitely many children. The tree T_{Σ_3} is like T_{Π_3}, except that we add one level 1 node which has infinitely many children.

Lemma 5.8. $\mathrm{Isom}_{T_{\Pi_3}}$ *is* Π^0_3 *and* $\mathrm{Isom}_{T_{\Sigma_3}}$ *is* $\Pi^0_3 \wedge \Sigma^0_3$.

Proof. If T is a computable tree, then the predicate "$\langle x \rangle$ has exactly n children in T" is Σ^0_2, uniformly in a computable index for T. So is the predicate "$\langle x \rangle$ has finitely many children in T". The predicate "there are infinitely many level 1 nodes on T which have n children" is Π^0_3.

Also, to say that the height of a tree T is at most 2 is Π^0_1 (once we know that $T \in Tree$).

A tree T is isomorphic to T_{Π_3} if it has height at most 2 and for every n, there are infinitely many level 1 nodes on T which have n children, and every level 1 node on T has finitely many successors.

The predicate "$\langle x \rangle$ has infinitely many children in T" is Π^0_2; and so the predicate "at most one level 1 node on T has infinitely many children" is Π^0_3.

A tree T is isomorphic to T_{Σ_3} if it has height at most 2 and for every n, there are infinitely many level 1 nodes on T which have n children, at most one level 1 node on T has infinitely many children, and some level 1 node has infinitely many children. The last condition is Σ^0_3 and all previous ones are Π^0_3. □

Lemma 5.9. $(\Sigma^0_3, \Pi^0_3) \leq_1 (\mathrm{Isom}_{T_{\Sigma_3}}, \mathrm{Isom}_{T_{\Pi_3}})$.

Proof. Let A be a Σ^0_3 set. By Lemma 5.6, there is some Π^0_2-definable function f such that $A = \mathrm{dom}\, f$.

For any n, we define a tree $T_{3,A}(n)$ of height 2. First, it contains a copy of T_{Π_3}. Then, for every x, there is a level 1 node $\langle m_x \rangle$ such that $T_{3,A}(n)[m_x] = T_{2,f}(n, x)$ (that is, for all y, $\langle m_x, y \rangle \in T_{3,A}(n)$ iff $\langle y \rangle \in T_{2,f}(n, x)$.)

Then $n \mapsto T_{3,A}(n)$ reduces $(A, \neg A)$ to $(\mathrm{Isom}_{T_{\Sigma_3}}, \mathrm{Isom}_{T_{\Pi_3}})$ because for all but perhaps one x we have $T_{2,f}(n, x)$ finite. □

Remark 5.10 (Walker's T_{Σ_3}). Walker defined his T_{Σ_3} such that it has infinitely many T_{Π_2} children. Walker's $\mathrm{Isom}_{T_{\Sigma_3}}$ is be Π^0_4. The above lemma still holds (via a slightly different reduction) but we only get hardness not completeness. It is not known if Walker's T_{Σ_3} is Π_4-complete. To avoid using infinitely many T_{Π_2} children we have to be more careful. Here we get around this problem by using Lemma 5.6.

We can now lift it up.

Lemma 5.11. *For all $n \geq 3$ there are trees T_{Σ_n} and T_{Π_n} such that*

(1) $\mathrm{Isom}_{T_{\Pi_n}}$ *is* Π^0_n*;*
(2) $\mathrm{Isom}_{T_{\Sigma_n}}$ *is* $\Pi^0_n \wedge \Sigma^0_n$*;*
(3) $(\Sigma^0_n, \Pi^0_n) \leq_1 (\mathrm{Isom}_{T_{\Sigma_n}}, \mathrm{Isom}_{T_{\Pi_n}})$.

Thus $\mathrm{Isom}_{T_{\Pi_n}}$ is Π^0_n-complete.

Proof. By induction; we know this for $n = 3$.

The tree $T_{\Pi_{n+1}}$ is a tree of height n which has infinitely many level 1 nodes, the tree above each of which is T_{Σ_n}. The tree $T_{\Sigma_{n+1}}$ is the tree $T_{\Pi_{n+1}}$, together with one other level 1 node above which we have T_{Π_n}.

A tree T is isomorphic to $T_{\Pi_{n+1}}$ iff it has infinitely many level 1 nodes (this is Π^0_2!), and for every level 1 node $\langle x\rangle$, the tree $T[x]$ above $\langle x\rangle$ is isomorphic to T_{Σ_n}.

A tree T is isomorphic to $T_{\Sigma_{n+1}}$ iff it has infinitely many level 1 nodes; for every level 1 node $\langle x\rangle$, the tree $T[x]$ is isomorphic to either T_{Σ_n} or to T_{Π_n}; there is at most one $\langle x\rangle$ such that $T[x]$ is isomorphic to T_{Π_n}; and there is some $\langle x\rangle \in T$ such that $T[x] \cong T_{\Pi_n}$.

Note again that if we had infinitely many T_{Π_n}s (which is what White's trees had) then we'd have had to pay another quantifier.

The reduction is similar to that of the case $n = 3$: given a Σ^0_{n+1} set A, we get a Π^0_n function f such that $A = \mathrm{dom}\, f$; we construct $T_{n+1,A}(m)$ to be a tree such that for all x, $\langle x\rangle \in T_{n+1,A}(m)$ and the tree $T_{n+1,A}(m)[x] = T_{n,f}(m, x)$. $\square$

For the case $\alpha \geq \omega$, the situation is murkier. Using the trees from White [21], for example, gives a reduction of, say, $\Sigma^0_{\omega+1}$ to a tree T such that Isom_T is computable from something like $\mathbf{0}^{(\omega+3)}$. With more work it seems that this can be reduced to $\mathbf{0}^{(\omega+2)}$, but it seems difficult to reduce this to $\mathbf{0}^{(\omega)}$. We remark that "things catch up with themselves" at limit levels which is why we get $+2$ for $\alpha \geq \omega$.

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