Splitting Theorems and the Jump Operator*

R. G. Downey†
Department of Mathematics
Victoria University of Wellington
P. O. Box 600
Wellington
New Zealand

Richard A. Shore¶
Department of Mathematics
White Hall
Cornell University
Ithaca NY 14853
USA

Abstract

We investigate the relationship of (jumps of) the degrees of splittings of a computably enumerable set and the degree of the set. We prove that there is a high computably enumerable set whose only proper splittings are low_2.

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1 Introduction

All sets and degrees will be computably enumerable unless otherwise stated. We say that A_1 and A_2 *split* A, written $A = A_1 \sqcup A_2$, if $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$. Such a splitting is called *proper* if both A_1 and A_2 are noncomputable. Ever since Friedberg [6] proved that any noncomputable set has a proper splitting, splitting theorems have been intimately related to the development of classical computability theory. We refer the reader to Downey and Stob [5] for a survey.

The present paper is concerned with a question of Remmel who asked if a high set could always be properly split into two sets one of which is high. This question is related to earlier work of Ladner [8] on mitotic sets (see also Downey-Slaman [4]), and later work of Lerman and Remmel [9] and Ambos-Spies and Fejer [2] on the universal splitting property. Recall that a set A is called mitotic if it has a proper splitting $A_1 \sqcup A_2 = A$ with $A_1 \equiv_T A_2 \equiv_T A$, and A has the universal splitting property if for all $C \leq_T A$, there is a splitting $A_1 \sqcup A_2 = A$ with $A_1 \equiv_T C$. Ladner proved that not all computably enumerable sets are mitotic, and, indeed, $\mathbf{0}'$ contains a nonmitotic set.

Ambos-Spies [1] proved that mitoticity could fail quite dramatically by constructing a complete A such that for any splitting $A_1 \sqcup A_2 = A$, one of A_1 or A_2 is low. On the other hand, Ambos-Spies's construction could not be used to solve Remmel's question since his set A, being complete, is of promptly simple degree while Downey and Stob [5] proved that if A has promptly simple degree then there is a proper splitting $A_1 \sqcup A_2 = A$ such that $A_1 \equiv_T A$. Ingrassia and Lempp [7] provided a counterexample to a stronger version of Remmel's question by constructing a computably enumerable set A such that for all nontrivial proper splittings $A_1 \sqcup A_2 = A$, $A_1', A_2' <_T A'$.

The goal of the present paper is to prove the following theorem.

Theorem 1.1 *There is a high computably enumerable set A such that if $A_1 \sqcup A_2 = A$ is a proper splitting of A, then both A_1 and A_2 are low$_2$.*

Corollary 1.2 *There is a high computably enumerable set A such that for* *all* $n \geq 1$*, if* $A_1 \sqcup A_2$ *is a proper splitting of* A *then* $A_i^{(n)} <_T A^{(n)}$ *for* $i = 1, 2$*.*

We remark that Cooper, Lachlan and Slaman have claimed (personal communication) that for all nonlow computably enumerable sets A, there is a proper splitting $A_1 \sqcup A_2 = A$ with A_1 nonlow. Given this result, ours is the strongest possible negative answer to Remmel's question. Moreover, our result together with that of Cooper et. al. completely answers all possible versions of Remmel's question in terms of the jump classes.

Our notation is standard and follows Soare [11]. As usual all computations etc. are bounded by the stage number and uses are monotone in both the argument and stage number. A number in brackets at the end of an expression such as $\Phi_i^{W_e}(y)\,[s]$ indicates that all computations and approximations to sets are to be understood as defined at stage s.

2 The Requirements and Construction

2.1 The Requirements and Intuition

We build a set A and a reduction Γ in stages to satisfy the following requirements:

$$R_e : \lim_s \Gamma^A(e, s) = Tot(e).$$

$$N_e : W_e \sqcup V_e = A \to (W_e \text{ low}_2. \vee .V_e \equiv_T \emptyset).$$

Here of course, Tot denotes the Π_2^0-complete index set $\{e : \varphi_e \text{ total}\}$. We decompose the negative requirements N_e into further subrequirements of the form

$$N_{e,i} : W_e \sqcup V_e = A \to (V_e \equiv_T \emptyset. \vee .[\limsup_{s\to\infty} \ell(\tau, s) \to \infty \to \Phi_i^{W_e} \text{ is total}]).$$

Here $\ell(\tau, s)$ denotes the length of convergence

$$\max\{x : \forall y \leq x(W_{e,s} \sqcup V_{e,s} = A_s \upharpoonright \varphi_{i,s}(y) \wedge \Phi_i^{W_e}(y) \downarrow [s])\}$$

measured at the node τ on the true path devoted to $N_{e,i}$. Note that this will make W_e low$_2$ if V_e is noncomputable since, as usual for an infinite injury argument, the true path, TP, is recursive in $0''$ and hence we can answer the question "Is $\Phi_i^{W_e}$ total?" recursively in $0''$.

The priority tree will have 3 types of nodes :

- β nodes for the sake of R_e with outcomes $\infty <_L f$.
- τ nodes for the sake of $N_{e,i}$ with outcomes $\infty <_L f$.
- α nodes living below $\tau\hat{}\infty$ also devoted to $N_{e,i}$ via subrequirements $N_{e,i,j}$. Such an α will be trying to preserve a computation of the form $\Phi_i^{W_e}(j)$ or trying to to demonstrate that $V_e \equiv_T \emptyset$. These nodes have outcomes $s <_L g$. The unique τ node associated with α will be denoted by $\tau(\alpha)$. For the i and e associated with τ, on the true path, an outcome s will demonstrate that $\Phi_i^{W_e}(j)\downarrow$. The outcome g demonstrates that V_e is computable.

The action of a β node is as usual. We must build a Δ_2^A approximation to Tot via Γ. We may as well assume that $\varphi_e(x)\uparrow [0]$ for all x. At stage 0, we will define $\gamma^A(e,x)[0] = \langle e, x, 0\rangle$. As with the standard thickness lemma, the basic idea is that when we see $\varphi_e(y)\downarrow$ for all $y \leq x$, we will enumerate some $g \leq \gamma(e,x)$ into $A[s]$ allowing us to redefine $\Gamma^A(e,x) = 1$. This will be the only reason we will change the value of $\Gamma^A(e,x)[s]$. (But not the only reason we might change $\gamma(e,x)[s]$.) If we succeed for almost all x then

$$\varphi_e \text{ is total iff } \lim_x \Gamma^A(e,x) = 1.$$

Thus A will be high as $\emptyset''$ will be Δ_2^A. In the construction to follow, $\gamma(e,x)[s]$ can also be changed for the sake of the $N_{f,j,k}$ of *lower* priority (which are defined precisely below). However, this action will be controlled by $\tau(\alpha)$), and we will certainly ensure that $\lim_s \gamma(e,x)[s]$ exists.

Below the infinite outcome of a τ node, that is where the length of convergence looks infinite infinitely often, there will be a tree of α nodes each devoted to some k, that is some subrequirement $N_{e,i,k}$ of $N_{e,i}$. These nodes will be devoted to requirements of the form

$$V_e \equiv_T \emptyset \vee \Phi_i^{W_e}(k) \downarrow .$$

The infinite outcome for an α node is the g outcome which corresponds to a global win for $\tau(\alpha)$ in the sense that it will witness the fact that $V_{e(\alpha)}$ is computable. Naturally, below the infinite outcome of an α node we will have no nodes devoted to N_e.

Associated with α will be a marker $m(\alpha, s)$ which represents an attempt to compute an initial segment of V_e based on the assumption that (α is on TP and) we fail to force convergence of $\Phi_i^{W_e}(k)$. That is, $m(\alpha, s)$ will represent the size of the domain of α's current recursive description of V_e. If we have α on the TP and $m(\alpha, s) \to \infty$ then V_e will be computable. The computation of $m(\alpha, s)$ and the actions of α described in more detail below.

There are two types of actions associated with α corresponding to the two types of positive requirements it must deal with.

A typical situation is the following. We have a node τ devoted to $N_{e,i}$. Naturally, it is able to guess at the behavior of higher priority R_f nodes, and will only use correspondingly τ-correct computations. However, if $\limsup_{s\to\infty} \ell(\tau, s) \to \infty$ we need to ensure either that W_e is low$_2$ or that V_e is computable. Thus we will need to deal with various β-nodes between $\tau\hat{\ }\infty$ and α as well as β nodes below α. (The point is that such nodes may be trying to put infinitely many elements into A whereas α is trying to preserve computations.) So suppose that we have

$$\tau\hat{\ }\infty \subset \beta_1\hat{\ }\infty \subset \beta_2\hat{\ }f \subset \beta_3\hat{\ }\infty \subset \alpha.$$

The way that α deals with these β-nodes between it and τ is the following. We reach τ (i. e. s is a τ stage) and it is expansionary with $\ell(\tau, s) > k$. We also assume that $\ell(\tau, s) > m(\alpha, s) + 1$ via τ-correct computations. What α would now like to do is to preserve its computation from W_e,

$$\Phi_i^{W_e}(k) \downarrow [s].$$

But it cannot really stop the β_i from putting their numbers (which may well be below the use $\varphi_i(k)[s]$) into A. What α tries to do is to lift the relevant

β_i-uses above $\varphi_i(k)[s]$. If α succeeds, then the computation $\Phi_i^{W_e}(k) \downarrow [s]$ becomes not only τ-correct, but α-correct, and hence α can preserve the computation forever. On the other hand, if α fails, we must arrange things so that we can increase $m(\alpha, s)$ thereby computing a longer initial segment of V_e. The trick is to control the number of elements to enter A.

In more detail, initially, $m(\alpha, 0) = k$. Now as above we reach τ and each of the β_i with $\tau\hat{}\infty \subseteq \beta_i\hat{}\infty \subseteq \alpha$ (in our case $i = 1, 3$) *indicate that they desire to change* $\Gamma^A(e(\beta_i), x(\beta_i))[s]$ *for some least* $x(\beta_i)$. (This is the main idea.) What α does is

- lift $m(\alpha, s+1) = m(\alpha, s) + 1$,
- request that a *single* number $z \leq \gamma(e(\beta_i), x(\beta_i))[s]$ enter $A[s]$, and
- lift $\gamma(e(\beta_i), x')$ above s for all x' greater than $x(\beta_i)$. (Here we assume that all the $\Gamma^A(e(\beta_i), x'')$ for $x'' < x(\beta_i)$ have been dealt with and are either permanently restrained or set to 1.)

Now, since the entry of z allows us to correct $\Gamma^A(e(\beta_i), x(\beta_i)) = 1[s]$, β_1 and β_3 are happy. Notice that since we lifted all the $\gamma(e(\beta_i), x')$ above s for all x' greater than $x(\beta_i)$, after stage s, the only numbers β_1 and β_3 will wish to put in will be numbers bigger than $s > m(\alpha, s)$. The relevance of this comes at the next $\tau\hat{}\infty$ stage s'. The single number z has entered A below $\varphi_i(m(\alpha, s))[s]$ between stages s and s'. Since W_e and V_e are disjoint, z has either entered W_e or V_e but not both. If z entered V_e then α can now successfully restrain the $\Phi_i^{W_e}(k) \downarrow [s']$ computation since it is identical to the $\Phi_i^{W_e}(k) \downarrow [s]$ computation. Now α can put finite restraint on the β' of lower priority, and the β of higher priority now only want to put numbers above s into A. (The β above τ were taken care of by the fact that we were dealing with τ-correct computations and $\ell(\tau, s) > k$; β between $\tau\hat{}\infty$ and α have had their $\gamma(e(\beta_i), x')$ lifted above s.) Then the next time we hit α we simply play outcome s defining a restraint $r(\alpha, s) = \varphi_i(k)[s]$.

On the other hand, it is possible that z went into W_e. In this case, when we reach α we play outcome g. Now we note that the only numbers that can be put into A by nodes β above α are bigger than $m(\alpha, s+1)$. The only numbers below $m(\alpha, s+1)$ which enter A after stage s must therefore come from nodes ν below α and can, like z above, only enter at stages at which we lift $m(\alpha, t)$ and hence as the single small number that enters between successive α stages. Assuming that g is the correct outcome, this single small number must enter W_e each time. If we assume that $m(\alpha, s) \to \infty$ then we can compute $V_e \upharpoonright p$ by simply waiting for a $\tau\widehat{\ }\infty$ stage with $\ell(\tau, s) > m(\alpha, s)$ and $m(\alpha, s) > p$.

The final point we need to notice is that it does not really matter what number $z \leq \gamma(e(\beta_i), x(\beta_i))[s]$ is used. We could equally well use some number requested by some β' below $\alpha\widehat{\ }g$. In this way we also get to meet the R_f of priority lower than α in case $\alpha\widehat{\ }g \subset TP$.

We now turn to the formal construction.

2.2 The Priority Tree

Define the priority tree as follows. If ν is on the priority tree and $|\nu| = 3e$, ν is devoted to R_e. Put $\nu\widehat{\ }\infty$ and $\nu\widehat{\ }f$ on the priority tree. ν is a β-node and $e(\nu) = e$.

Otherwise, we use two lists L_1 and L_2 to assign requirements to nodes. As usual the lists $L_1(\lambda) = L_2(\lambda) = \omega$. We use the convention that we do not change lists as we pass to the outcomes of a node unless specifically so instructed.

If $|\nu| \equiv 1 \bmod 3$ assign $N_{e,i}$ to ν where $\langle e, i\rangle$ is the least member of $L_1(\nu)$. Put $\nu\widehat{\ }\infty$ and $\nu\widehat{\ }f$ on the priority tree. Let $L_1(\nu\widehat{\ }\infty) = L_1(\nu\widehat{\ }f) = L_1(\nu) - \{\langle e, i\rangle\}$. Let $L_2(\nu\widehat{\ }f) = L_2 - \{\langle e, i, k\rangle : k \in \omega\}$. ν is a τ node, $e(\nu) = e$ and $i(\nu) = i$.

Finally, if $|\nu| \equiv 2 \bmod 3$, find the least $\langle e, i, k\rangle$ in $L_2(\nu)$ such that $\langle e, i\rangle \notin L_1(\nu)$. Assign $N_{e,i,k}$ to α. Put $\nu\widehat{\ }s$ and $\nu\widehat{\ }g$ on the priority tree. Let $L_2(\nu\widehat{\ }s) = L_2(\nu) - \{\langle e, i, k\rangle\}$. Let $L_2(\nu\widehat{\ }g) = L_2(\nu) - \{\langle e, i', k'\rangle : i', k' \in \omega\}$. Let $L_1(\nu\widehat{\ }g) = L_1(\nu) - \{\langle e, i'\rangle : i' \in \omega\}$. ν is an α-node, $e(\nu) = e$, $i(\nu) = i$ and $k(\nu) = k$.

2.3 The Construction

Step 1.

At each stage of the construction, we put at most one number into A. We determine this number by approximating TP by TP_s as follows. We begin at λ and say that s is a λ-stage. Suppose that s is a ν-stage. There are 3 cases.

Case 1. ν is a β-node.

If $\max\{q : \varphi_{e(\nu)}(q') \downarrow [s] \text{ for all } q' \leq q\} > \max\{q : \varphi_{e(\nu)}(q') \downarrow [t] \text{ for all } q' \leq q \wedge t \text{ a } \nu \text{ stage}\}$, declare that s is a $\nu\hat{\ }\infty$ stage and that ν *desires the largest number less than or equal to* all $\gamma_\nu(e(\nu), x)$, if any, with $\gamma_\nu(e(\nu), x) > \max\{r(\delta, s) : \delta < \nu\}$, and $\varphi_{e(\nu)}(x') \downarrow$ for all $x' \leq x$, but $\Gamma^A(e(\nu), x) = 0$ to enter A.

Otherwise say that s is a $\nu\hat{\ }f$ stage.

Case 2. ν is a τ-node.

Let $e = e(\nu)$. Determine the ν-correct length of convergence. $\ell(\nu, s) = \max\{x : \forall y \leq x(W_{e,s} \sqcup V_{e,s} = A_s \restriction \phi_{i,s}(y) \wedge \Phi_i^{W_e}(y) \downarrow [s])\}$ where the computations are ν-correct. That is, (with the understanding that all objects below have $[s]$ appended) for all $\beta\hat{\ }\infty \subseteq \nu$, with $|\beta| \equiv 0 \bmod 3$, and any x, if
(i) $\gamma(e(\beta), x) < u(\Phi_i^{W_e}(y))$
(ii) $\gamma(e(\beta), x) > \max\{r(\delta, s) : \delta < \beta\}$,
then $\Gamma^A(e(\beta), x) = 1$.

If the stage is ν-expansionary we say that s is a $\nu\hat{\ }\infty$-stage. We require that the k^{th} expansionary stage have ν-correct length of agreement exceeding

- $\max\{m(\alpha, s) + 1 : \alpha \supseteq \nu\hat{\ }\infty$, such that $e(\alpha) = e$, $i(\alpha) = i$ and α is devoted to $N_{e,i,k'}$ for some $k' \leq k\}$.

If s is not ν-expansionary, we say that s is a $\nu\hat{\ }f$ stage. Let $r(\nu\hat{\ }f, s)$ be the last $\nu\hat{}\infty$-stage (or 0 if there is no such stage).

Case 3. ν is an α-node.

Let $e = e(\alpha), i = i(\alpha), k = k(\alpha)$, and $\tau = \tau(\alpha)$. If $\ell(\tau, s) \leq m(\nu, s)$, set $TP_s = \nu$. Otherwise, see if for all $\beta\hat{}\infty \subseteq \nu$, with $|\beta| \equiv 0 \bmod 3$, and any x, if
(i) $\gamma(e(\beta), x) < u(\Phi_i^{W_e}(k'))$ for $k' \leq k$,
(ii) $\gamma(e(\beta), x) > \max\{r(\delta, s) : \delta < \beta\}$,
then $\Gamma^A(e(\beta), x) = 1$,

If so, let $r(\nu, s) = u(\Phi_i^{W_e}(k))$. Declare s to be a $\nu\hat{}s$-stage.

If not then declare s to be a $\nu\hat{}g$-stage, and reset $m(\nu, s+1) = m(\nu, s)+1$.

Step 2.

Having determined TP_s, we initialize all α-nodes σ to the right of TP_s. This entails returning $m(\sigma, s+1)$ to $m(\sigma, 0)$, and setting $r(\sigma, s+1) = 0$.

Step 3.

Finally, put into A the smallest number z, if any, that any β node σ (such that s is a σ-stage) desires to put into A. For $\beta \subset TP_s$, reset $\gamma(e(\beta), x)[s+1]$ for all $e(\beta), x$ with $\gamma(e(\beta), x)[s] > z$, and some $\Gamma(e(\beta), x')[s]$ with $x' \leq x$ causes β to desire a number to enter A at stage s. For such $e(\beta), x$, set $\Gamma^A(e(\beta), x) = 1[s+1]$ if $\varphi_{e(\beta)}(x') \downarrow$ for all $x' \leq x$.

Step 4.

For each τ-node μ with $\mu\hat{}\infty \subseteq TP_s$, set $r(\mu\hat{}f, s+1) = s+1$. For each α-node μ with $\mu\hat{}g \subseteq TP_s$, set $r(\mu\hat{}g, s+1) = s+1$. (Note that if $\mu\hat{}s \subseteq TP_s$ then $r(\mu\hat{}s, s)$ was set in step 1, Case 3. Of course, $r(\mu\hat{}s, s+1) = r(\mu\hat{}s, s)$.)

End of Construction.

3 The Verification

We verify the following by simultaneous induction on $\nu \subset TP$:

(i) $\lim_{\{s:s \text{ is a } \nu\text{-stage}\}} r(\nu', s) < \infty$ exists for all $\nu' \leq \nu$.

(ii) If ν is a β-node, then $\lim_x \Gamma^A(e, x) = Tot(e)$. Moreover, $\nu\hat{\ }\infty \subseteq TP$ iff $Tot(e) = 1$.

(iii) If ν is a τ-node, then $\nu\hat{\ }\infty \subset TP$ iff there are infinitely many τ-correct τ-expansionary stages and hence $\nu\hat{\ }\infty \subseteq TP$ iff $W_e \sqcup V_e = A$ and for all k and almost all $\tau\hat{\ }\infty$ stages s, $\Phi_{i(\nu)}^{W_{e(\nu)}}(k)\downarrow$.

(iv) If ν is an α node then $\nu\hat{\ }g \subset TP$ implies that $V_{e(\nu)}$ is computable. If $\nu\hat{\ }s \subset TP$ then $\Phi_{i(\nu)}^{W_{e(\nu)}}(k(\nu))\downarrow$.

We assume (i)-(iv) for all $\sigma \subset \nu$. Let s_0 be a stage at which the hypotheses apply to all such σ and we are never again to the left of ν. There are 3 cases to consider.

Case 1. ν is a β-node. Then there is no restraint associated with ν and hence (i) holds and (iii) and (iv) are irrelevant. Let $e = e(\nu)$. To see that (ii) holds suppose first that $Tot(e) = 0$. Then, after some stage, ν will stop desiring to put numbers into A in accordance with the first case of the construction. Hence $\nu\hat{\ }f \subset TP$. Next, suppose that $Tot(e) = 1$. In this case, infinitely often when we reach ν there will have been a change in $Tot(e)[s]$ since the last ν-stage t (i. e. $\max\{q : \varphi_{e(\nu)}(q')\downarrow [s] \text{ for all } q' \leq q\} > \max\{q : \varphi_{e(\nu)}(q')\downarrow [t] \text{ for all } q' \leq q \wedge t \text{ a } \nu \text{ stage}\}$). According to case 1 of the construction, all such stages will be $\nu\hat{\ }\infty$ stages. Furthermore, since the higher priority restraints come to a limit, for sufficiently large x, if s is a $\nu\hat{\ }\infty$ stage and ν desires a number below $\gamma_\nu(e(\nu), x)$, to enter A since $\gamma_\nu(e(\nu), x) > \max\{r(\delta) : \delta < \nu\}$, and $\varphi_{e(\nu)}(x')\downarrow$ for all $x' \leq x$, but $\Gamma^A(e(\nu), x) = 0$, then this desire cannot be restrained by any σ. Therefore at step 3 of the construction, either $\gamma_\nu(e(\nu), x)$ itself, or some $z < \gamma_\nu(e(\nu), x)$ will be enumerated into A. Finally, to see that the $\gamma(e(\nu), x)[s]$ come to a limit, note that we only gratuitously change $\gamma(e(\nu), x)[s]$ in step 3 of the construction when ν desires to correct $\Gamma_{e(\nu)}$ on some $x' \leq x$. But each time such a change is desired and made for some x', we will set $\Gamma^A(e(\nu), x') = 1$ during that stage. Of course, this can happen only finitely often.

Case 2. ν is an τ node. Straightforward.

Case 3. ν is a α-node. Let $e = e(\nu), i = i(\nu)$, and $k = k(\nu)$. By the construction of the priority tree, we can suppose that for all α-nodes $\nu' \subset TP$, devoted to $N_{e,i,k'}$ for $k' < k$, $\nu'\hat{\ }s \subset \nu$. It follows that after some stage $s_1 > s_0$ each time we have a $\tau(\nu)\hat{\ }\infty$-stage, we must have the τ-correct length of agreement above $m(\nu, s)$. We argue as in the intuitive discussion. First, suppose that at some stage s_2 after s_1, ν imposes restraint. Now we see that s_2 must be a ν-stage at which, for all $\beta\hat{\ }\infty \subseteq \nu$ with $|\beta| \equiv 0 \bmod 3$ and for any x, if
(i) $\gamma(e(\beta), x) < u(\Phi^{W_e}(k'))$ for $k' \leq k$,
(ii) $\gamma(e(\beta), x) > \max\{r(\delta, s) : \delta < \beta\}$,
then $\Gamma^A(e(\beta), x) = 1$. It follows that no number of higher priority can injure the $\Phi_i^{W_e}(k)[s_2]$-computation. By ν's restraint, step 3 resetting, and the step 2 initialization, no δ of lower priority can injure the $\Phi^{W_e}(k)[s_2]$-computation. Therefore $\Phi_i^{W_e}(k)[s_2] \downarrow = \Phi_i^{W_e}(k)[s_2]$, $m(\nu, s_2) = m(\nu)$, and $r(\nu, s_2) = r(\nu)$.

Thus we can suppose that there is no stage s_2 after s_1 where restraint is imposed by ν. In this case, we claim that V_e is computable. We reason by induction on stages after s_1. Suppose that no number below $m(\nu, s)$ will ever again enter V_e, and that s is some $\tau\hat{\ }\infty$ stage after s_1. Now the computation up to $\ell(\tau, s)$ is τ-correct and we know that the length of agreement exceeds $m(\nu, s)+1$ τ-correctly. We will not reset $m(\nu, s')$ until a stage $s' \geq s$ at which we visit ν. Suppose that s' is such a stage. At stage s' we will increment $m(\nu, s'+1)$ to be $m(\nu, s) + 1$. At stage s', at most one number will enter A and, by construction, every β with $\beta\hat{\ }\infty \subseteq \nu$ desires to put a number into A.

The action of putting z into A in case 3 will clearly lift all the $\gamma(e(\beta), x)$ for $\beta\hat{\ }\infty \subseteq \nu$ which are not permanently restrained and have $\gamma(e(\beta), x) < m(\nu, s')$ above $m(\nu, s')+1$. Therefore no β with $\beta\hat{\ }\infty \subseteq \nu$ can ever later desire to put a number below $m(\nu, s'+1)$ into V_e. Furthermore no $\delta \geq_L \nu$ can put a number below $m(\nu, s'+1)$ by ν restraint. Finally, at most one number below $m(\nu, s'+1)$ can enter A from nodes below $\nu\hat{\ }g$, and since no later stage is a $\nu\frown s$ stage, this small number must enter W_e and not V_e. Therefore, V_e is now fixed on $m(\nu, s') + 1$ and hence by induction, V_e is computable.

To complete the proof, note that if V_e is noncomputable and $\nu\frown\infty$ is a τ-node on the true path, then it can only be that $\Phi_{i(\nu)}^{W_{e(\nu)}}$ is total, since by (iv), for all α below $\nu\frown\infty$ on the True Path, and $\tau(\alpha) = \nu$, $\alpha\frown s$ is on the True Path. $\square$

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