ARITHMETICAL SACKS FORCING

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Abstract. We answer a question of Jockusch by constructing a hyperimmune-free minimal degree below a 1-generic one. To do this we introduce a new forcing notion called arithmetical Sacks forcing. Some other applications are presented.

1. introduction

Two fundamental construction techniques in set theory and computability theory are forcing with finite strings as conditions resulting in various forms of Cohen genericity, and forcing with perfect trees, resulting in various forms of minimality. Whilst these constructions are clearly incompatible, this paper was motivated by the general question of "How can minimality and (Cohen) genericity interact?". Jockusch [5] showed that for $n \geq 2$, no n-generic degree can bound a minimal degree, and Haught [4] extended earlier work of Chong and Jockusch to show that that every nonzero Turing degree below a 1-generic degree below $\mathbf{0}'$ was itself 1-generic. Thus, it seemed that these forcing notions were so incompatible that perhaps no minimal degree could even be comparable with a 1-generic one.

However, this conjecture was shown to fail independently by Chong and Downey [1] and by Kumabe [7]. In each of those papers, a minimal degree below $\mathbf{m} < \mathbf{0}'$ and a 1-generic $\mathbf{a} < \mathbf{0}''$ are constructed with $\mathbf{m} < \mathbf{a}$.

The specific question motivating the present paper is one of Jockusch who asked whether a *hyperimmune-free* (minimal) degree could be below a 1-generic one. The point here is that the construction of a hyperimmune-free degree by and large directly uses forcing with perfect trees, and is a much more "pure" form of Spector-Sacks forcing [10] and [9]. This means that it is not usually possible to use tricks such as full approximation or forcing with partial computable trees, which are available to us when we only wish to construct (for instance) minimal degrees. For instance, minimal degrees can be below computably enumerable ones, whereas no degree below $\mathbf{0}'$ can be hyperimmune-free. Moreover, the results of Jockusch [5], in fact prove that for $n \geq 2$, if $\mathbf{0} < \mathbf{a} \leq \mathbf{b}$ and $\mathbf{b}$ is n-generic, then $\mathbf{a}$ bounds a n-generic degrees and, in particular, certainly is *not* hyperimmune free. This contrasts quite strongly with the main result below.

In this paper we will answer Jockusch's question, proving the following result.

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Theorem 1.1. *There are hyperimmune-free degrees below* $\mathbf{0}''$ *which are below 1-generic degrees below* $\mathbf{0}''$.

Analysis of the original proof of our result allowed us to extract a new forcing notion which we call arithmetical Sacks forcing. We show that each arithmetical Sacks generic real is hyperimmune free and of degree below a 1-generic.

The proofs here are relatively straightforward, but will filter through the characterization of degrees computable in 1-generic ones of Chong and Downey.

2. Notation

Recall that a tree T is a subset of $2^{<\omega}$ so that for every $\sigma \in T$, $\tau \preceq \sigma$ implies $\tau \in T$. A perfect tree T is a tree so that for every $\sigma \in T$, there exists a $\tau \succeq \sigma$ so that $\tau^\frown 0 \in T$ and $\tau^\frown 1 \in T$. Given a tree T, define $[T] = \{G \subseteq \omega | \forall n(G \upharpoonright n \in T)\}$.

We recall that a set $C \subset \omega$ is n-generic iff it is Cohen generic for n-quantifier arithmetic. An equivalent formulation due to Jockusch and Posner (see Jockusch [5]) is given by the following.

Definition 2.1. Given a string $\sigma \in 2^{<\omega}$, a set $A \subseteq \omega$ and a set $S \subseteq 2^{<\omega}$.

(1) $\sigma \Vdash A \in S$ if $\sigma \prec A$ and $\sigma \in S$.
(2) $\sigma \Vdash A \notin S$ if $\sigma \prec A$ and $\forall \tau \succeq \sigma(\tau \notin S)$.

Definition 2.2. Given sets $A, B \subseteq \omega$ and a number $n \geq 1$, A is n-B-generic if for every $\Sigma^0_n(B)$ set $S \subseteq 2^{<\omega}$, there is a string $\sigma \prec A$ so that either $\sigma \Vdash A \in S$ or $\sigma \Vdash A \notin S$.

3. Arithmetical Sacks Forcing

Define $\mathcal{S} = \{T \mid T$ is a computable perfect tree in $2^{<\omega}\}$. A set $\mathcal{D} \subseteq \mathcal{S}$ is dense if for every $T \in \mathcal{S}$, there is a set $T' \in \mathcal{D}$ for which $T' \subseteq T$. Fix an effective enumeration $\{W_e\}_{e\in\omega}$ of c.e. sets $W \subseteq 2^{<\omega}$. Then there is an arithmetical (actually, Σ^0_3) definable predicate P so that $P(e)$ iff W_e is a computable perfect tree. Hence there is an arithmetical enumeration $\{T_e\}_{e\in\omega}$ of $\mathcal{S}$ so that if $P(e)$ then $T_e = W_e$; otherwise, $T_e = 2^{<\omega}$. We say a set $\mathcal{D} \subseteq \mathcal{S}$ is arithmetical if the index set $\{e | T_e \in \mathcal{D}\}$ is arithmetical. If $\mathcal{D} \subseteq \mathcal{S}$, define $[\mathcal{D}] = \{[T] | T \in \mathcal{D}\}$.

Define arithmetical Sacks forcing notion:

$$\mathbb{S} = \langle \subseteq, \mathcal{S} \rangle.$$

A filter $\mathcal{F} \subset \mathcal{S}$ is a set so that if $T \in \mathcal{F}$ and $T' \supseteq T$ in $\mathcal{S}$, then $T' \in \mathcal{F}$ and if $T_0, T_1 \in \mathcal{F}$, then there is $T_2 \in \mathcal{F}$ so that $T_2 \subseteq T_0 \cap T_1$. A generic set $\mathcal{G}$ is a filter for which $\mathcal{G} \cap \mathcal{D} \neq \emptyset$ for every arithmetical dense set $\mathcal{D}$. We say a set G is an *arithmetical Sacks set* if for every n, $G \upharpoonright n \in \bigcap_{T \in \mathcal{G}} T$.

The following lemma collects some well known facts.

Lemma 3.1. *For any* $e \in \omega$*,*

(1) $\mathcal{P}_W = \{T \in \mathcal{S} \mid \exists i \forall \sigma \in T(|\tau| > i \implies \sigma(i) \neq W(i)).\}$ *is dense where* W *is an arithmetical set.*
(2) $\mathcal{Q}_e = \{T \in \mathcal{S} \mid$ *One of the following cases is true:*
   (a) $\exists i \forall \sigma \in T(\Phi^\sigma_e(i) \uparrow)$,
   (b) $\forall i \exists n \forall \sigma_0 \in 2^n \cap T \forall \sigma_1 \in 2^n \cap T(\Phi^{\sigma_0}_e(i) \downarrow = \Phi^{\sigma_1}_e(i) \downarrow)$,
   (c) $\forall i \exists n \forall \sigma \in 2^n \cap T(\Phi^\sigma_e(i) \downarrow)$ *and* $\forall \sigma \exists j \forall \tau_0 \succeq \sigma^\frown 0 \forall \tau_1 \succeq \sigma^\frown 1(\tau_0 \in T \;\&\; \tau_1 \in T \implies \Phi^{\tau_0}_e(j) \neq \Phi^{\tau_1}_e(j))$.

} *is dense.*

(3) $\mathcal{R}_e = \{T \in \mathcal{S} \mid$ *One of the following cases is true:*
   (a) $\exists i \forall \sigma \in T(\Phi_e^\sigma(i)\uparrow)$,
   (b) $\forall i \exists n \forall \sigma \in 2^n \cap T(\Phi_e^\sigma(i)\downarrow)$ *and there is a computable function f so that* $\forall i \forall \sigma \in T(\Phi_e^\sigma(i)\downarrow \implies \Phi_e^\sigma(i) < f(i))$.

} *is dense.*

(4) $\mathcal{M}_e = \{T \in \mathcal{S} \mid \exists \sigma \in 2^e \forall \tau \in T(|\tau| > |\sigma| \implies \tau \succeq \sigma).\}$ *is dense.*

Proof. All of the statement above are well known. (1) says no arithmetical Sacks set is arithmetical, (2) is a minimality requirement and (3) is a hyperimmune-freeness requirement. (4) says arithmetical Sacks sets are well-defined. □

The following corollary is immediate.

Corollary 3.2. *If G is an Arithmetical Sacks set, the* $\mathbf{G}$ *is a hyperimmune-free minimal degree.*

In [2], Chong and Downey introduced the following notation.

Definition 3.3. Given a set $G \subseteq \omega$,

(1) A set $T \subseteq 2^{<\omega}$ is *dense* in G if for any $n \in \omega$, there is a finite string $\sigma \in T$ so that $G \upharpoonright n \preceq \sigma$.
(2) A set $W \subseteq 2^{<\omega}$ is Σ_1-dense in G if
   (a) For every $\sigma \in W$, $\sigma \not\prec G$ and
   (b) For any c.e. set $T \subseteq 2^{<\omega}$ which is dense in G, there are finite strings $\tau_0 \in T$ and $\tau_1 \in W$ so that $\tau_1 \preceq \tau_0$.

Lemma 3.4. *Given an arithmetical set W. The set $\mathcal{N}_W = \{T \in \mathcal{S} \mid$ One of the following cases is true:*

(1) $\exists \sigma \in W \forall \tau \in T(|\tau| > |\sigma| \implies \tau \succeq \sigma)$,
(2) $\forall \sigma \in T \forall \tau \in W(\sigma \not\succeq \tau)$.

} *is dense.*

Proof. Given a tree $T \in \mathcal{S}$, if there is a σ in $T \cap W$, then define $T' = \{\tau \in T \mid \tau \succeq \sigma.\}$. Otherwise, define $T' = T$. Since T' is computable perfect tree, $T' \in \mathcal{S}$ and satisfies (1) or (2). □

Corollary 3.5. *If G is an arithmetical Sacks set, then there is no arithmetical set W Σ_1-dense in G.*

Proof. Suppose G is an arithmetical Sacks set. Given an arithmetical set $W \subseteq 2^{<\omega}$, by Lemma 3.4, there are two cases:

(1) There is a $T \in \mathcal{N}_W$ so that $\exists \sigma \in W \forall \tau \in T(|\tau| > |\sigma| \implies \tau \succeq \sigma)$ and $G \in [T]$. Then by (a) in Definition 3.3, W is not Σ_1-dense in G

(2) There is a $T \in \mathcal{N}_W$ so that $\forall \sigma \in T \forall \tau \in W(\sigma \not\succeq \tau)$ and $G \in [T]$. Then T is a c.e. dense set in G. By (b) in Definition 3.3, T is a witness that W is not Σ_1-dense in G. □

It is not hard to see that a set $G \subseteq \omega$ is computable if and only if there is no set W Σ_1-dense in G. For c.e. sets Σ_1-dense in G, Chong and Downey proved the following interesting theorem.

Theorem 3.6 (Chong and Downey [2])**.** *Suppose G has Turing degree* $\mathbf{G}$. $\mathbf{G}$ *is below a 1-generic degree if and only if there is no c.e. Σ_1-dense set in G.*

Corollary 3.7. *Every arithmetic Sacks set is bounded by a 1-generic real. Hence there is a 1-generic degree* $\mathbf{C}$ *and a hyperimmune-free minimal degree* $\mathbf{G}$ *so that* $\mathbf{C} > \mathbf{G}$.

Proof. By Corollary 3.5, for any arithmetic Sacks set G, there is no is c.e. Σ_1-dense set in G. By Theorem 3.6, there is a 1-generic set C so that $C \geq_T G$. But obviously, $C \not\equiv_T G$. By Corollary 3.2, $\mathbf{G}$ is a hyperimmune-free and minimal degree. □

We remark that, in the same way as the kind of forcing used in algorithmic randomness, these proofs don't use the full strength of arithmetical Sacks forcing. Only that it works for Δ^0_3 collections of trees. Thus the hyperimmune free minimal degree we construct can be low$_2$ and below $\mathbf{0}''$. The full statement of the Chong-Downey result is that if M has no Σ_1-dense set of strings, then M is computable in a 1-generic set $G \leq_T M''$. In particular, we can choose here a minimal hyperimmune-free degree $\mathbf{G}$ computable in a 1-generic degree $\mathbf{C}$ below $\mathbf{0}''$.

A reasonable possible generalization of Theorem 3.6 is whether $\mathbf{G}$ is below a 2-generic degree if and only if there is no Σ^0_2 set Σ_1-dense in G. We give a negative answer.

Proposition 3.8. *There is a set* G *so that there is no arithmetical set* Σ_1*-dense in* G *but* $\mathbf{G}$ *is not below any 2-generic degree.*

Proof. By Corollary 3.5, for every arithmetical Sacks set G, there is no arithmetical set Σ_1-dense in G. By Corollary 3.2, every arithmetical Sacks set G has a minimal degree. But no 2-generic degree bounds a minimal degree as showed in [5]. So every arithmetical Sacks set G is not below any 2-generic set. □

Hyperimmune-freeness looks a stronger forcing notion than minimality forcing since there is no a Δ^0_2 hyperimmune-free degree. So it is natural to ask whether every hyperimmune-free degree is computable in a 1-generic degree. This possible result is not true since we have the following proposition.

Proposition 3.9. *There is a hyperimmune-free degree computable in no 1-generic degree.*

We need a lemma. This lemma is known in the folklore, and probably implicit in the work of Kučera and of Jockusch.

Lemma 3.10. *No 1-generic set computes a DNR-set.*

Proof. Recall that a DNR set A is a subset of ω so that $A \neq \Phi_e(e)$ for every $e \in \omega$. Suppose g is a 1-generic set and Ψ^g is a DNR-set. Define a Σ^0_1 set $M = \{\sigma | \exists e(\Phi_e(e) \downarrow \ \& \ \Psi^\sigma(e) \downarrow \ \& \ \Psi^\sigma(e) = \Phi_e(e))\}$. Since Ψ^g is DNR, there is a $\sigma \prec g$ so that for every $\tau \succ \sigma$, for every e, $\Phi_e(e) \downarrow$ and $\Psi^\tau(e) \downarrow$ implies $\Phi_e(e) \neq \Psi^\tau(e)$. Define a computable function Φ so that $\Phi(e) = \Psi^\mu$ where μ is the first $\tau \succ \sigma$ for which $\Psi^\tau(e) \downarrow$ at stage $|\tau|$. Since Ψ^g is total, Φ must be total. Φ has an index e. Then $\Phi_e(e) = \Phi(e) = \Psi^\tau(e)$ for some $\tau \succ \sigma$. A contradiction. □

Proof. (of proposition 3.9) By Jockusch-Soare [6], there is a hyperimmune-free DNR-degree. But, by Lemma 3.10, no 1-generic degree can compute a DNR degree. □

Perhaps it might be the case that every *minimal* hyperimmune-free degree is computable in a 1-generic one. Again this attractive suggestions fails. However, this time the proof is now not quite so straightforward. Since the result is only of marginal interest, we will only sketch its proof.

Theorem 3.11. *There is a minimal hyperimmune-free degree below* $\mathbf{0}''$ *with an* Σ_1*-dense set of strings, and hence one not computable in any 1-generic degree.*

Proof. (sketch) In Chong and Downey [1], using a full approximation argument, a minimal degree $\mathbf{m}$ is constructed with a Σ_1-dense set of strings, and hence one not computable in a 1-generic degree. In Downey [3], it is shown how to construct a (minimal) hyperimmune free degree below $\mathbf{0}''$ using a full approximation argument. The point is that these two constructions are compatible, with great detail and no real new insight. □

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