Diagram 0

\[ \begin{align*}
N & \\
\end{align*} \]

Distributive lattices are not embeddable into \( L \). Modular lattices are embeddable into \( L \). A lattice \( L \) is distributive if \( \forall \ x, \ y, \ z \in L \), then \( x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \).

Theorem 1.1 (a) A countable modular lattice \( L \) is embeddable into \#877/11 1/0.

We can show that there are lattices \( L \) that are not embeddable into \#877/11 1/0. For example, the lattice \( R \) is not distributive, and hence not embeddable into \#877/11 1/0.

Introduction

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Part of the proof of Theorem (1.1) is a straightforward embedding result of Ambos-Spies [1] which we give in Section 2.

(1.2) Theorem (Ambos-Spies [1]). All countable distributive lattices are embeddable into all nontrivial initial segments of \( \mathbb{R} \).

The difficult half of Theorem (1.1) is a very strong nonembedding result we prove in Section 3. This result actually gives quite a bit more. We need the following definition.

(1.3) Definition. Let \( a_0, a_1, a_2 \in \mathbb{R} \). We say \( a_0, a_1, a_2 \) form a critical triple if \( a_0 \cup a_1 = a_1 \cup a_2, a_0 \not\leq a_1 \) and \( \forall c (c \leq a_0, a_2 \rightarrow c \leq a_1) \).

For example, in Diagram 1, critical triples are identified in some typical lattices.

![Diagram 1](image)

The main result of the paper is

(1.4) Theorem. There exists an r.e. degree \( a \neq 0 \) that bounds no critical triple. Indeed each nonzero r.e. degree has a predecessor with this property.

Of course, Theorem (1.1) follows from (1.2) and (1.4) since \( M_3 \) is embeddable in all modular nondistributive lattices. At this stage we would like to put on record the fact that we feel the converse of (1.4) holds, viz:

(1.5) Conjecture. If \( L \) is a lattice that contains no critical triple, then \( L \) is embeddable below any nonzero r.e. degree.
$$\mathcal{D} \oplus \mathcal{D} \cup \mathcal{A} \cup \mathcal{B} = (\mathcal{D} \cup \mathcal{A}) \cup (\mathcal{D} \cup \mathcal{B}) \cup (\mathcal{A} \cup \mathcal{B})$$

The well-known representation trick gives $\mathcal{D} \cup \mathcal{A} \cup \mathcal{B}$. This will sometimes denote an infinite set and sometimes a corresponding index.

Here $\mathcal{A}$ will denote a dense-in-itself set and sometimes a corresponding

$$\{ x \in \mathcal{A} \text{ and } i \in \mathcal{I} : (x, i) \} = \mathcal{A}$$

construct $\mathcal{E}$. sets $\mathcal{A}$ and define

Let $\mathcal{A}$ be any uniformly recurrent sequence of recurrent sets forming $\mathcal{D} \cup \mathcal{A} \cup \mathcal{B}$. The $\mathcal{A}$ is any uniform recursive enumeration of $\mathcal{A}$. We refer to those sets which contain $\mathcal{A}$ as the construction of $\mathcal{A}$.

Theorem. The (constructible) nonmeasurable Boolean algebra $\mathcal{D}$ embeds into any nonmeasurable Boolean algebra $\mathcal{D}$.

We establish (1.7) by proving

2. Embeddings

usual, but not essential, is the reader to familiarize with that account.


Both of these are well known, but not particularly elegant.

We point out that all of the lattices in our examples above are embeddable into


distributivity mod 2

If $L$ is an latticable and all nontrivial intervals of $L$ is a

Lattice nonembeddings and initial segments of $L$'s.

For any $a > 0$. It does seem quite probable that the distributivity theorem (1.1) will hold. We have proven the analogous result for the distribution of a measureable

If we even be possible to show that $L$ is such a lattice, then $L$ embeds into

Lattices nonembeddings and initial segments of $L$'s. Definable.
As usual (see e.g. [13, Ch. IX, §2]) this guarantees \( \alpha \to \deg(A_\alpha) \) is the desired isomorphism. For \( e = \langle \alpha, \beta, i \rangle \), let

\[
l(e, s) = \max\{ x : (\forall y < x)(\Phi_{i,s}(A_\alpha, \beta, \omega ; y) = \Phi_{i,s}(A_{\alpha, \beta, \omega} ; y)) \}.
\]

As in the (tree version of the) minimal pair type argument of Lachlan–Lerman–Thomason, define the notion of \( \sigma\text{-stage} \) by induction on \( \lh(\sigma) \) for \( \sigma \in 2^{<\omega} \) via

(i) Every stage \( s \) is a \( \lambda \)-stage (where \( \lambda \) denotes the empty string).

(ii) If \( s \) is a \( \tau \)-stage and \( \lh(\tau) = \langle \alpha, \beta, i \rangle \) \( = e \), then if \( l(e, s) > \max\{ t + 1 : t \text{ is a } \tau^\omega \text{-stage and } t < s \} \), then \( s \) is a \( \tau^\omega \)-stage. (Here 0 is a \( \tau^0 \)-stage.) Otherwise \( s \) is a \( \tau^1 \)-stage.

We let \( \sigma^y \) denote the unique string \( y \) such that \( \lh(y) = s \) and \( s \) is a \( \gamma \)-stage. As usual let \( \leq_L \) denote lexicographic ordering with \( 0 \leq_L 1 \).

Now let

\[
L(e, i, s) = \max\{ x : (\forall y < x)(\Phi_{e,s}(Q_x ; y) = A_i(y)) \}.
\]

We attempt to meet the requirements \( P_{(e,i)} \) by followers \( y \). These will be of the form \( y(\sigma, x, s) \) to indicate they have guessed \( \sigma \) (where \( \lh(\sigma) = \langle e, i \rangle \)) and have \textit{permitting number} \( x \). Such a follower can only be appointed at \( \sigma \)-stages when all smaller followers are ‘realized’ (i.e. ready to be permitted). The definition of realized is deferred until later.

Once realized \( y(\sigma, x, s) \) will enter \( A_i \) only if \( E \) permits \( x \) at \( s \). The difference between this construction and a minimal pair type of construction is that we must be able to enumerate \( y \) whenever \( E \) so permits, whereas in (e.g.) a minimal pair argument we must await a \( \sigma \)-stage to so enumerate \( x \). As we know (Lachlan [10]) such waiting is a strong enough obstacle that in fact not all r.e. degrees bound minimal pairs. The additional trick we shall use will be a \( Q \)-marker \( q(y, s) \) \textit{tied to} \( y \) which we shall use to allow \( Q \) to recover ‘both sides’ of a changed computation. (Similar ideas were used in [3].)

The reader should note that in our construction to follow the definition of \( \sigma^\omega \)-stage above has the following consequence. Suppose \( y \) is a follower appointed at a \( \sigma^\omega \)-stage \( s \). Suppose \( \delta \) is a \( \sigma^\omega \)-stage larger than \( s \) and \( \tau^\omega = \sigma^\omega \) with \( \lh(\tau) = e \). Then \( l(e, \delta) > y \). This will follow as we can only appoint \( s \) as a follower at stage \( s \) as we will see. Now we give the formal details.

We say that \( P_{(e,i)} \) \textit{requires attention} at stage \( s + 1 \) if one of the following options holds.

(2.2) There is a follower \( y = y(x, \alpha, s) \) (say) of \( P_{(e,i)} \) with \( \lh(\sigma) = \langle e, i \rangle \) for some \( \sigma \leq_L \sigma_x \) such that

(i) \( E \) permits \( x \) at \( s \), and

(ii) \( y \) is realized at stage \( s \).

(2.3) \( P_{(e,i)} \) has an unrealized follower \( y = y(x, \alpha, s) \) with \( \sigma < \sigma_x \) such that \( L(e, i, s) > y \) where \( \lh(\sigma) = \langle e, i \rangle \) and (2.2) does not hold.
Now we argue that all N_i are non-empty. Let o \in N_i with \( (\lambda)^i \). Then:

\[
(\lambda)^i \lambda = 1 \neq 0 = (\lambda)^i \phi = (\lambda)^i (\lambda^2)^i \phi
\]

Suppose the least \( \phi \) such that there is a stage \( i < \lambda \) with \( \phi \) and \( (\lambda)^i \) will get an unremarkable follower \( r \).

Each of these has the property that \( b < (\lambda)^i (\lambda)^j \phi \) for some standard \( b \).

By induction we can suppose that for some \( \lambda \), the follower with \( \lambda \) has no followers with \( \lambda \) as \( \phi \).

If there exist a \( \lambda \) such that \( \lambda \), then \( \lambda \) and \( \lambda \) are \( \lambda \)-non-empty, otherwise \( \lambda \) is a leader and there is a path \( (\lambda)^i \lambda \in \lambda \).

Case 1: \( \lambda \) is a leader. Set \( \lambda \) as reached, cancel all numbers \( \lambda \) and all followers of the form \( \lambda \) together with their \( \lambda \) numbers and all followers of the form \( \lambda \).

Case 2: \( \lambda \) holds. Set \( \lambda \) as reached, cancel all numbers \( \lambda \) and all followers of the form \( \lambda \).

Case 3: \( \lambda \) holds. Set \( \lambda \) as reached, cancel all numbers \( \lambda \) and all followers of the form \( \lambda \).

Case 4: \( \lambda \) holds. Set \( \lambda \) as reached, cancel all numbers \( \lambda \) and all followers of the form \( \lambda \).

Case 5: \( \lambda \) holds. Set \( \lambda \) as reached, cancel all numbers \( \lambda \) and all followers of the form \( \lambda \).

Case 6: \( \lambda \) holds. Set \( \lambda \) as reached, cancel all numbers \( \lambda \) and all followers of the form \( \lambda \).

End of construction. The proof that each \( \phi \) becomes an unremarkable follower is straightforward so we only sketch the details. Let \( \phi \) denote the unremarkable follower with \( \phi \).
\( \Phi_{t}(A_{\alpha} \oplus Q) = \Phi_{t}(B_{\alpha} \oplus Q) = f \) total, and so \( \sigma^{n}0 \subseteq \gamma \). Let \( z \) be given. To compute \( f(z) \) find the least \( \sigma^{n}0 \)-stage \( t > s_{0} \) such that \( l(e, t) > z \). Now compute the least \( \sigma^{n}0 \)-stage \( \nu > t \) such that

\[
(2.5) \quad Q_{\nu}[t] = Q[t] \quad \text{and} \quad A_{\alpha \cap \beta, \nu}[t] = A_{\alpha \cap \beta}[t].
\]

We claim that \( f(z) = f_{\nu}(z) \). To see this, we show by induction that for all \( s > \nu \)

\[
(2.6) \quad \text{one of } \Phi_{t,s}(A_{\alpha,s} \oplus Q_{s}; z) = \Phi_{t,\nu}(A_{\alpha,\nu} \oplus Q_{\nu}; z) \quad \text{or} \quad \Phi_{t,s}(A_{\beta,s} \oplus Q_{s}; z) = \Phi_{t,\nu}(A_{\beta,\nu} \oplus Q_{\nu}; z) \quad \text{holds.}
\]

If (2.6) is to fail, then at some least \( \sigma^{n}0 \)-stages \( s_{1} > s_{2} \geq \nu \)—with \( s_{2} \) the preceding \( \sigma^{n}0 \)-stage before \( s_{1} \)—there must be two numbers \( y \) and \( \hat{y} \) which enter respectively the \( A_{\alpha} \)-side and the \( A_{\beta} \)-side below the \( s_{2} \) uses at stages \( r_{1} \) and \( r_{2} \) with \( s_{2} \ll r_{1} \), \( r_{2} < s_{1} \) respectively. The reason we can take \( s_{1} \) and \( s_{2} \) to be \( \sigma^{n}0 \)-stages is that—as with a minimal pair argument—if only one side changes between \( \sigma^{n}0 \)-stages, then as the computations of both sides hold at \( \sigma^{n}0 \)-stages, we must get (2.6). We shall claim this is impossible.

First we claim that in fact two numbers must enter as followers between stages \( s_{1} \) and \( s_{2} \) (and so not as \( Q \)-markers). To see that this is the case, if a number is enumerated as a \( Q \)-marker it must enter \( Q \).

We claim that if \( q \) is a \( Q \)-marker with \( q \) entering \( Q_{n} \) and \( q \ll u(\Phi_{t,s}(A_{\alpha,s} \oplus Q_{s}; z)) \) at any stage \( n > s \geq t \) for any \( \sigma^{n}0 \)-stage \( s \), then \( q \) was already appointed at stage \( t \). This will then contradict (2.5).

The point is that if \( q \) is appointed at or after stage \( t \), then \( q \) is appointed at a \( \sigma^{n}0 \)-stage \( t_{i} \leq s \) after stage \( t \). (Else it would die at \( s \).) Thus, by convention and definition of \( \sigma^{n}0 \)-stage, \( q = t_{1} \) and exceeds both uses. It is easy to see that when \( q \) is appointed then for \( q = q(\hat{y}) \), \( \hat{y} \) is the largest follower defined at \( t_{1} \). Now if \( q \) enters it can only be at the same stage \( n \) as \( \hat{y} \). No follower \( t_{1} \) can have entered \( A_{\alpha} \) or \( B_{\alpha} \) at any stage \( t_{2} \) with \( t_{1} \ll t_{2} \ll n \) lest it cancel \( \hat{y} \) and \( q \). Thus as \( q \) is still alive we see that \( q \) still exceeds both uses since the computations are unchanged since stage \( t_{1} \). The claim then follows.

Thus we may take \( y \) and \( \hat{y} \) to be followers. Note that by the same argument as above (cancellation and appointment at \( \sigma^{n}0 \)-stages) we can see that both \( y \) and \( \hat{y} \) must have been already appointed at stage \( t \). Note that our assumption (2.5) on \( A_{\alpha \cap \beta} \) means that \( y \neq \hat{y} \) and so without loss of generality \( y < \hat{y} \). Now both \( y \) and \( \hat{y} \) must be realized when they enter, as only realized followers enter \( A_{\alpha} \) or \( A_{\beta} \). Realization can only occur at \( \sigma^{n}0 \)-stages and hence both must be realized at stage \( s_{2} \). Now since the realization of \( y \) at stage \( r \) would have cancelled all followers \( g > y \) defined at stage \( r \), it must have been the case that \( y \) was realized at stage \( t \) (since \( \hat{y} \) is alive). It therefore follows (and this is the whole point) that \( q(y) \ll t \) and hence by (2.5), \( q(y) \in Q \) iff \( q(y) \in Q_{\nu} \). Thus \( y \) cannot enter \( A_{\alpha} \) or \( A_{\beta} \) after stage \( \nu \) after all, and the claim (2.6) follows, concluding the proof of (2.1). \( \square \)
Theorem. There exists an R that bounds no critical point.

\[ \exists R \]
are

\[ \hat{R}_e: \quad \text{for some } j \in \{0, 1, 2\} \text{ or } k \in \{0, 2\} \text{ either} \]

\[ \Lambda_e(A) \neq W_e^j \text{ or } \Phi_e^k(W_e^{k+1} \oplus W_e^{k+2}) \neq W_e^k. \]

To test \( \hat{R}_e \) we need to see if \( l(e, s) \to \infty \). We remark that as in the Slaman/Soare account of the \( \Theta^* \)-method (Soare [13]), a node \( \tau \) devoted to \( \hat{R}_e \) will be the top of 'links' to the subrequirements \( R_{e,i} \), below.

If \( \hat{R}_e \) fails to be met, we must build \( Q_e \leq_T W_e^0, W_e^2 \) and attempt to meet for all \( i \in \omega \) the subrequirements

\[ R_{e,i}: \quad \Psi_i^1(W_e^i) \neq Q_e. \]

Finally if for some least \( i \), we fail to meet \( \hat{R}_e \) and \( R_{e,i} \), we must ensure that \( W_e^0 \leq_T W_e^1 \). This will be the outcome if \( R_{e,i} \) receives attention infinitely often, and we will say \( R_{e,i} \) has outcome \( g \).

The rough idea is this. Assume \( l(e, s) \to \infty \). We aim to ensure \( \Psi_i^1(W_e^i) \neq Q_e \) by followers \( x = \gamma^{\geq}(x, e, i, s) \) and \( Q_e \leq_T W_e^0, W_e^2 \) by 'permitting'. Strictly speaking permitting is certainly not accurate. But it will do as a first approximation which we will later modify. The actual reductions \( J^0(W_e^0) = W_e \) and \( J^0(W_e^2) = Q \) will have uses \( \gamma \gamma(x, e, s) \) and \( \gamma^{\geq}(x, e, s) \) respectively. Our first attempt is to pick a follower \( x \) targeted for \( Q_e \) and wait till we see \( l(e, s) > x \). (We call this a confirmation stage.) When this occurs we cancel all followers \( y \) targeted for \( A \) with \( y > x \) (the reason for this becomes clear later) and restrain \( A \) on \( u(x, e, s) \) preserving the current e-computations. Our first 'permitting' attempt is to set

\[ \gamma^0(x, e, s) = x \quad \text{and} \quad \gamma^{\geq}(x, e, s) = \lambda^2(e, s), \]

and ask that \( x \) be allowed to enter \( Q_e \) only if both \( W_e^0[x] \) and \( W_e^2[\lambda^2(e, s)] \) change between e-expansionary stages (i.e. where \( l(e, s) > m(l(e, t)) = \max\{l(e, t) : t < s\} \)).

Now, we don’t drop the restraint \( r(e, i, s) \) on \( A[u(x, e, s)] \) until we see a stage \( s_1 > s \) where \( L(e, i, s_1) > x \). Clearly, should \( s_1 \) not occur, we win (since \( \Psi_e(W_e^i, y) \neq Q_e(y) \) for some \( y \leq x \)) with finite effect. Should we see \( s_1 \) occur, we then open an \( (e, i) \)-gap by setting \( r(e, i, s_1) = 0 \), potentially allowing \( A[u(x, e, s)] \) to change. Our main hope is that when we close our \( (e, i) \)-gap, nice enough conditions will have occurred to allow us to enumerate \( x \) into \( Q_e \) in such a way as to create a provable disagreement at \( x \). As \( Q_e \leq_T W_e^0, W_e^2 \) is predicated on \( l(e, s) \to \infty \) only, we must close our \( (e, i) \)-gap at the next e-expansionary stage \( s_2 > s_1 \). Of course, if \( s_2 \) does not occur we win \( \hat{R}_e \). When \( s_2 \) occurs if none of the uses have changed we will need only re-impose restraint and pick a new follower. When \( s_2 \) occurs the desirable conditions are: that \( W_e^0 \) has permitted \( x \), \( W_e^2 \) has permitted \( \gamma^{\geq}(x, e, s) \) and furthermore

\[ W_e^1 \equiv_{e, e_1 \ast} u(\Psi_i^1(W_e^1, x)) = W_e^2 \equiv_{e, e_1 \ast} u(\Psi_i^1(W_e^1, x)). \]

(After all we also need to know that at \( s_2 \), \( L(e, i, s_2) > x \).) Should these conditions occur we enumerate \( x \) into \( Q_e \), and raise \( r(e, i, s_2) = s_2 \) to preserve the disagreement \( 0 = \Psi_i^1(W_e^1, x) \neq Q_e(x) = 1 \).
that if we have a situation where $x$ permits $y$, then

Our solution to this problem is to define $\phi$ in such a way as to ensure

This obviously creates a very serious problem since we have a situation with a

permitted by the relevant change occurs only on $x$. Thus, we cannot use the relevant

of a subsequent attack at $x$, but we can—yet we cannot with a

The inequality of $d$ can be seen as follows. We need to argue that $d$ is ever

and $d$ are simply not satisfied. We need the conditions of the construction

The above is a general outline of the overall shape of the construction. We now

needer complete an exponential stage $\gamma$ where $x$ permits on $y$. In this way we can

permits $y$, we only need to complete $\gamma$, and so on. Furthermore, whenever $y$ is

that $x$ is to be a follower of $y$. Should we close the base case because a was reached before stage $s$ and

The rough idea is this: $\gamma$ fails, we close all followers of

Although we will also need another follower $x$ to attack $x$, this does not

$W^1_e$ doesn't permit $\rho(x, e, i, t)$ we will win on some follower $y \geq x$ (although we may not win at $x$). To do this, we delay the definition of $\rho(x, e, i, t)$ until a stage $t$ is found where $\rho(x, e, i, t)$ 'covers' a new follower. We do this as follows. In the situation above, the idea is that at stage $s_2$ although we close the $(e, i)$-gap we don't define $\rho(x, e, i, s_2)$ at all but appoint a new follower $y > s_2$ (setting $r(e, i, s_2) = s_2$).

We then wait until a stage $s_1 > s_2$ where $L(e, i, s) > y$ and only then define

$$\rho(x, e, i, s_1) = \rho(y, e, i, s_1) = \max\{\gamma^2(y, e, s_1), u(W_{s_1}(s_1))\}.$$

The reader should realise that this delay in the definition of $\rho$ is fine from $R_{e,i}$'s point of view as we only need $\rho$ if $W^2_e = Q_e$ so that $L(e, i, s) \to \infty$. By monotonicity this specifically ensures that also

$$\rho(x, e, i, s_1) > \max\{u(s_2), u(W_{s_1}(s_1))\}.$$

Note that this inequality follows as $\gamma^2(y, e, s_1) = \lambda^2(e, s_1) \equiv L(e, s_1) > u(s_1) = u(s_2)$, the last equality by restraints. Now we open an $(e, i)$-gap at $y$ (and $x$) as we did at stage $s_1$.

The crucial observation is that if we close our $(e, i)$-gap at stage $s_2 > s_1$ then if $W^0_e[x]$ changes, $W^2_e[u(s_2)]$ changes and $W^1_e[\rho(x, e, i, s_1)]$ does not, then it also must be that $W^0_e[y]$ changes, $W^2_e[\gamma^2(y, e, s_1)]$ changes and $W^1_e[\rho(y, e, i, s_1)]$ does not. (The point is that $y$ is in its 'initial attack' phase.) Hence we can cause a disagreement at $y$. Obviously if at $s_2$ we have the same problem with $y$ as we did with $x$ we will delay both $\rho(x, e, i, t)$ and $\rho(y, e, i, t)$ for some $z > y$. It is essential to this process that such resetting occurs only finitely often. This is achieved by a cancellation process; essentially we cancel all followers $z$ targeted for $A$ with $x < z < s_1$ when we set $\rho(x, e, i, s_1)$ and furthermore when we enumerate $\hat{x}$ into $A$ we always cancel all numbers $q$ targeted for $A$ with $q > \hat{x}$. (As usual these have lower priority.) The reader should note that this won't cancel followers targeted for $Q$.

(3.2) This cancellation process helps us in the following way. If we have an unsuccessful closure where no permissions occur, then there is no reason to, (and nor would we be able to) change $\gamma^0$, $\gamma^2$ and $\rho$. Thus the only reason we need to delay the definition of $\rho$ as above is that some follower $z$ with $z \leq s_1$ entered $A$. Now the cancellation process at stage $s_1$ means that there are now no followers left alive below $s_1$ except those $\leq s_1$ (that were already appointed at stage $s_1$ as we will see). Thus again we have ensured that the only numbers that would cause a change in $\gamma^0$, $\gamma^2$ or $\rho$ for $x$ (or $y$) are those numbers $\leq s_1$. This cancellation occurs each time we delay $\rho$, and hence we ensure $\rho$ is delayed (and changed) at most $s_1$ times.

The way other $P_i$ requirements live with this is that there will be infinitely many 'no permission' outcomes. In particular if at $s_2$ there was no change we would
Our solution to this last problem is to use a process we call a "hop"

In some situations, parts of the system may be in an un

needed state. For example, if we have a state with an unneeded

component, we may have some progress on it,

\( \langle x, x', \ldots, x_n \rangle \), where

\( \langle x, x', \ldots, x_n \rangle \)

is the state of the system. In this case, we need to consider the state of the system at some point in the future.

The problem is that we need to know the state of the system at some point in the future, but we do not know what the state will be. This is because the state of the system is not deterministic.

To address this issue, we can use a process called a "hop". A hop is a transition from one state to another. When we perform a hop, we move from one state to another, but we do not know what the new state will be. This is because the state of the system is not deterministic.

To illustrate this process, consider the following diagram:

- The initial state is represented by the node labeled \( x \).
- The next state is represented by the node labeled \( x' \).
- The transition from \( x \) to \( x' \) is represented by the edge labeled \( \langle x, x' \rangle \).

The hop process is performed by moving from one state to another, but we do not know what the next state will be. This is because the state of the system is not deterministic.

Here is a more detailed explanation of the hop process:

1. We start with the initial state, labeled \( x \).
2. We perform a hop, moving from \( x \) to another state, labeled \( x' \).
3. We do not know what the new state will be, so we label the transition as \( \langle x, x' \rangle \).
4. We repeat this process, moving from one state to another, but we do not know what the next state will be.

This process continues until we reach a state where we do know what the next state will be.

Here, we assume no change at all except for being helped by a follower. When we see a change, we just ignore it. This is because we do not know what the next state will be, and we do not want to make any assumptions about it.

Note that there are no negative numbers and initial segments of the reals appear.
observation we need make is this: at stage $s_2$ if there were no followers $z < u(x, e, s_2)$ left alive targeted for $A$ this problem can’t occur since the $W^0_c[x]$ computation is final. (As $A_{c}[u(x, e, s_2)] = A(u(x, e, s_2))$.) Now, again if we see this situation (i.e. a bad unsuccessful closure) occurs, we will cancel all new followers (since $s_1$) so the comment above pertains if at stage $s_1$ there was only one follower $\leq x$ targeted for $A$. Suppose there are only two followers $z_1 \leq z_2 \leq x$ (and so only two $\leq u(x, e, s_1)$ by cancellation when we set $\gamma^0$ and $\gamma^2$). In this case we really are in trouble. Between stages $s_1$ and $s_2$, $z_2$ may enter $A$, killing $\gamma^2$. Then between stages $s_2$ and $s_3$ above $z_1$ may enter $A$ putting us in the no win situation.

Our idea is to add one further layer to $\gamma^0$ and $\gamma^1$ to cope with $z_1$. That is, we don’t define $\gamma^0$ and $\gamma^2$ until we see a stage $s$ where not only is $l(e, s) > x$ but also $l(e, s) > u(x, e, s)$. Then as before we cancel all followers $> x$ targeted for $A$, to get the situation in Diagram 3.

![Diagram 3](image)

We freeze this by setting (again) $r(e, i, s) = s$ but now the twist is to set $\gamma^0 = \lambda^0(x, s)$ and $\gamma^2 = \lambda^2(e, s)$. And at the stage $s_1$ where $L(e, i, s_1) > x$ we set $\rho(x, e, i, s_1) = \max\{u(W^0_{e,i}(W^1_{e,s_1}; x)), \gamma^0(x, e, s_1), \gamma^2(x, e, s_1)\}$. The point of this procedure is this. Suppose at stage $s_2$ we get our bad outcome: that $W^0_{e}[\gamma^0]$ and $W^2_{e}[\rho]$ both don’t change but $W^2_{e}[\gamma^2]$ does. The only way this can occur is for $W^2_{e}[\gamma^2]$ to change only on the ‘outer layer’. That is $W^2_{e}$ can only change on

$$\{y : \gamma^2(x, e, s) \geq y > u = u(\Phi^0_{e}(W^1_{e,s}; x))\},$$

since if $W^2_{e}$ changes below $u$ then $W^2_{e} \oplus W^1_{e}$ must change below $\lambda^0(e, s)$. As $W^1_{e}[\rho]$
(3.3)

\[
\begin{align*}
\text{Let } y' = x' \text{ for all } x' \in X & \text{ and } y = \alpha(x) \text{ for some } x \in X, \\
\text{and let } \gamma = \beta(x) & \text{ for some } x \in X.
\end{align*}
\]

This creates a barrier with \( x' \) between \( y' \) and \( y \). Now define \( \gamma' = \gamma(x) \text{ for } x \in X \).

The basic module implemented the following steps:

1. Pick an initial follower \( x = \alpha(x) \) for \( x \in X \).
2. Set \( r(x) = \text{true} \).
3. For all \( y \in Y \), set \( r(y) = \text{false} \).
4. For all \( x' \in X \), set \( r(x') = \text{true} \).

Finally, if only one \( W \) is satisfied, the changes and \( M \) do not change.

The basic module below is inapplicable only \( d \) changes, that is, it is

\[
\text{true if and only if } \{ y \} := \emptyset \text{ or } \emptyset.
\]

In summary, the close of an \( \varepsilon \)-gap at stage \( 2 \) is unsuccessful. It is not

\[
(\varepsilon^d)\text{.}
\]

The point is that this sort of infinity only strips one layer per

uneffectual

possible infinity instead of two—we can see one only can obtain the same

unsuccessful

definition. Theorem 3 (5.5), \( \gamma \) is a follower, \( \alpha(x) = \gamma(y) \text{ for any one }

unsuccessful

layer, \( \gamma \) on \( \gamma \text{ for any one layer, or } \gamma \text{ for any one layer} \).

Theorem 4 (5.5), \( \gamma \) is a follower, \( \alpha(x) = \gamma(y) \text{ for any one }

unsuccessful

layer, \( \gamma \) on \( \gamma \text{ for any one layer} \).

Theorem 5 (5.5), \( \gamma \) is a follower, \( \alpha(x) = \gamma(y) \text{ for any one layer} \).

Hence, we know there is only one more possible change to

unsuccessful

change. \( \gamma \) is an unsuccessful this can only mean that the re-definition. Hence

unsuccessful
Step 4. At the least stage $s_2 > s_1$ where $l(e, s_2) > ml(e, s_2)$ and $f^v(e, s_2) > x_j$ close the $(e, i)$-gap by setting $r(e, i, s_2) = s_2$. Adopt the first case below to pertain.

Case 1 (successful closure). For some $k \leq i$, $W^0_{e, s_2}[\gamma^q(x_k, e, s_1)] = W^0_{e, s_1}[\gamma^q(x_k, e, s_1)]$ and $W^2_{e, s_2}[\gamma^q(x_k, e, s_1)] = W^2_{e, s_1}[\gamma^q(x_k, e, s_1)]$ but $W^1_{e, s_2}[\rho(x_k, e, i, s_1)] = W^1_{e, s_1}[\rho(x_k, e, i, s_1)]$.

Action. Set $Q_{e, s_2+1} = Q_{e, s_2} \cup \{x_k\}$. Put the basic module into state $f$ (for 'finish').

Case 2 (unsuccessful closure). Otherwise. This is outcome $g$ and is where we possibly have appointed new followers to $P_j$ for $j > (e, i)$ before we appoint followers to $R_{e, i}$. (More on the coherence later.) For each $k \leq i$ adopt the first case to pertain.

For convenience, set $\gamma^q(x_k) = \gamma^q(x_k, e, s_1)$ for $q \in \{0, 2\}$ and $\rho(x_k) = \rho(x_k, e, i, s_1)$.

Case 2(a). $W^q_{e, s_2}[\gamma^q(x_k)] = W^q_{e, s_1}[\gamma^q(x_k)]$ for $q \in \{0, 2\}$ and $W^1_{e, s_2}[\rho(x_k)] = W^1_{e, s_1}[\rho(x_k)]$.

Action. No change.

Case 2(b). $W^q_{e, s_2}[\gamma^q(x_k)] = W^q_{e, s_1}[\gamma^q(x_k)]$ for at least one of $q = 0$ or $q = 2$ but $W^1_{e, s_2}[\rho(x_k)] = W^1_{e, s_1}[\rho(x_k)]$.

Action. If $W^q_{e, s_2}[\gamma^q(x_k)]$ has changed, then set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$. Declare $\rho(x_k, e, i, s_2)$ as undefined (pending for the new follower $x_{j+1}$) unless there is some follower $x_k$ for $k > k$ to which case 2(b) below pertains. In that case we set $\rho(x_k, e, i, s_2) = \rho(x_k, e, i, s_2)$ for the least such $k$. In any case cancel all followers $y$ targeted for $A$ with $y > x_k$.

Case 2(c). $W^q_{e, s_2}[\gamma^q(x_k)] \neq W^q_{e, s_1}[\gamma^q(x_k)]$ for both $q = 0$ and $q = 2$ and $W^1_{e, s_2}[\rho(x_k)] \neq W^1_{e, s_1}[\rho(x_k)]$ but $l(e, i, s_2) > x_k$.

Action. Set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$ and $\rho(x_k, e, i, s_2) = \max\{\lambda^1(e, s_1), \lambda^1(e, s_2)\}$. Cancel all followers $y$ targeted for $A$ with $y > x_k$.

Case 2(d). As in case 2(c) but $L(e, i, s_2) \neq x_k$.

Action. Set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$ but $\rho(x_k, e, i, s_2)$ is declared undefined (pending $x_{j+1}$). Cancel as in case 2(c).

Case 2(e). For some $q = 0$ or $q = 2$, $W^q_{e, s_2}[\gamma^q(x_k)] \neq W^q_{e, s_1}[\gamma^q(x_k)]$ but $W^{q+2}_{e, s_2}[\gamma^{q+2}(x_k)] = W^{q+2}_{e, s_1}[\gamma^{q+2}(x_k)]$ and case 2(b) did not pertain so that $W^1_{e, s_2}[\rho(x_k)] = W^1_{e, s_1}[\rho(x_k)]$.

Action. Set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$ for this $q$. Cancel as in case 2(c). This case involves 'layer injury'.

(3.4) Remark. The reader should note that in the construction to follow, $P_j$ (for $j > (e, i)$) cooperates with $R_e$ by assigning followers during the gap. However, these new followers will be cancelled should any case except 2(a) pertain. The point is that if $g$ is the correct outcome, case 2(a) pertains infinitely often due to our cancellation process we described earlier (as we shall see). If case 2(a) does not pertain, the $R_{e, i}$ module only has finite effect. (More on this later.) The remaining details in the full construction contain no surprises and fit into the
For every \( (x, y) \in R \times R \) such that \( x \neq y \), let \( x \prec y \) or \( y \succ x \) denote the relation of precedence. A relation \( R \) is said to be a preorder if for every \( (x, y) \in R \times R \), either \( x \prec y \) or \( y \succ x \) holds. A preorder that is also transitive is called a partial order.

Let \( R \) be a preorder on a set \( X \). Then \( R \) has a transitive closure \( R^* \), which is the smallest preorder containing \( R \) that is also transitive. The transitive closure is defined as follows:

\[
R^* = \{(x, y) \in X \times X : \text{there exists a finite sequence } x_0, x_1, \ldots, x_n \in X \text{ such that } x_0 \prec x_1 \prec \cdots \prec x_n \text{ and } x_0 \prec x_n\}
\]

The transitive closure can also be constructed as the union of the transitive closure of \( R \) with \( R \) itself, iterated through all finite sequences of \( R \). This can be formalized as:

\[
R^* = \bigcup_{n \geq 0} (R^0)^n
\]

where \( R^0 \) is the base preorder and \( (R^0)^n \) denotes the \( n \)-fold composition of \( R^0 \) with itself.

To illustrate the construction of the transitive closure, consider the preorder \( R \) defined on the set \( X = \{a, b, c\} \) by the relations \( a \prec b \) and \( b \prec c \). The transitive closure of \( R \) is then given by the relation \( R^* = \{(a, b), (b, c), (a, c)\} \).

In conclusion, the transitive closure of a preorder is a way to extend the preorder to a larger relation by adding all possible transitive chains, ensuring that the resulting relation is also transitive.

---

For completeness, we give some formal details below:

1. The preorder is the collection of all strings \( w \) with \( \prec^{\mathcal{B}} \) such that it
2. The framework of Shram/Sesar in Sear [I2]
various guesses. If $e(\alpha) = e$, then a follower of $P_e$ at guess $\alpha$ (i.e. a follower of $P_\alpha$) is denoted by $y(\alpha, s)$. If $\lhd(\alpha) \equiv 2 \pmod{3}$, we will define a restraint $r(\alpha, s)$, and a use $\rho(x, \alpha, s) = \rho(x, e, i, s)$ at $\alpha$ for $x = x(\alpha, j, s)$ and also for either $\eta = \alpha$ or $\eta = \tau(\alpha)$ we simultaneously define $\gamma^p(x, \eta, s)$ for $p = 0, 2$. These are the guessed versions of the uses of the basic module. To initialize node $\alpha$ at stage $s$ we mean, as usual, to cancel all followers, restraints etc. associated with $\alpha$, reset the current state of the $\alpha$-module at node $\alpha$ (denoted by $F(\alpha, s)$) to $F(\alpha, s) = w$ if $\lhd(\alpha) = 2 \pmod{3}$, cancel $Q_\alpha$ (if defined) to $Q_\alpha = \emptyset$ and cancel any links (to be defined) with top or bottom $\alpha$.

(3.8) **Definition.** Let $\alpha \in T$.

(i) We say $s + 1$ is an $\alpha$-stage if $\alpha \in \sigma_{s+1}$ where $\sigma_{s+1}$ is to be defined later. In addition $0$ is an $\alpha$-stage.

(ii) We say $s + 1$ is a genuine $\alpha$-stage if $\sigma(t, s + 1) = \alpha$ for some substage $t$ of stage $s + 1$. We let $G^\alpha$ denote the collection of genuine $\alpha$-stages.

(iii) Suppose $\lhd(\alpha) = 0 \pmod{3}$ with $\lhd(\alpha) = 3e$ so $e = e(\alpha)$. We say that a stage $q$ is $\alpha$-expansionary if $q = 0$ or $q = s + 1$ where

(a) $s + 1$ is a genuine $\alpha$-stage,

(b) $l(e, q) > \max\{l(e, \hat{q}) : \hat{q} \text{ is an } \alpha \text{-expansionary stage and } \hat{q} < q\}$, and

(c) For all followers $x$ of the form $x = x(\gamma, j, \hat{q} - 1)$ with $\gamma \supset \alpha$, $\hat{q} \leq q$ and $\tau(\gamma) = \alpha$, $l^\gamma(e, q) > x$ (where $l^\gamma(e, q)$ is defined as in (3.3)).

(iv) Suppose that $\lhd(\alpha) = 3e + 1$. We say that $\alpha$ requires attention at substage $t$ of stage $s + 1$ (which we write as stage $(t, s + 1)$) if $W_{e,s} \cap A_\alpha = \emptyset$ and one of the following options holds:

(a) $\sigma(t, s + 1) = \alpha$ and $y(\alpha, s)$ is undefined,

(b) $\sigma(t, s + 1) = \alpha$ and $y(\alpha, s) \in W_{e,s}$.

(v) Suppose that $\lhd(\alpha) = 2 \pmod{3}$. Let $e = e(\alpha)$ and $i = i(\alpha)$. We say that a stage $q$ is $\alpha$-expansionary if $q = 0$ or $q = s + 1$ and

(a) $s + 1$ is a genuine $\alpha$-stage,

(b) $l(e, i, q) > \max\{l(e, i, \hat{q}) : \hat{q} \text{ is an } \alpha \text{-expansionary stage } < q\}$.

(c) For all followers $x(\alpha, i, s)$ currently defined, $L(e, i, q) > x(\alpha, i, s)$.

(vi) Suppose that $\lhd(\alpha) = 2 \pmod{3}$. We say that $\alpha$ requires attention at stage $s + 1$ if $s + 1$ is a genuine $\alpha$-stage and $\alpha$ has no follower (i.e. $x(\alpha, 0, s)$ is undefined).

(3.9) **The Construction**

*Stage 0.* Initialize all $\alpha \in T$. Define $\sigma_0 = \lambda$.

*Stage $s + 1$.* The value of a parameter $p \neq \sigma$ at stage $(t, s + 1)$ is denoted by $p_t$.

Substage $t = 0$. Define $\sigma(0, s + 1) = \lambda$.

Substage $t + 1$. We are given $\sigma(t, s + 1)$ and for all $\alpha \in T$ with $\lhd(\alpha) = 2 \pmod{3}$, $F_\alpha = F_\alpha = F(\alpha, s + 1)$, with $F_\alpha(\alpha) \in \{w, g, f\}$. Adopt the first case below to pertain. Let $\alpha = \sigma(t, s + 1)$. 
\[
\{ (1 + s, \varphi) \} \chi = (1 + s, \varphi) x \phi = (1 + s, \varphi) x (1 + s, \varphi) x \phi
\]

For each \( (1 + s, \varphi) \) there exists a unique \( \varphi \) such that \( (1 + s, \varphi) x = x \). Let \( x = (1 + s, \varphi) x \) for all \( (1 + s, \varphi) \) define a function \( x = x \).

Define the set of all \( (1 + s, \varphi) x \) for all \( (1 + s, \varphi) \) open on \( \varphi \). Create a function \( x = x \) of \( 1 + s, \varphi \) x = (1 + s, \varphi) x.

Note that \( S \) is not an exponential. Define \( x = x \) of \( 1 + s, \varphi \) x = (1 + s, \varphi) x.

And go to stage \( s + 1 \).

Case 1: If \( x \) does not require attention, suppose \( x \) does not exist.

Case 2: If \( x \) does not require attention, suppose \( x \) does not exist.

Case 3: If \( x \) requires attention, suppose \( x \) exists.

Case 4: If \( x \) requires attention, suppose \( x \) exists.

Case 5: If \( x \) requires attention, suppose \( x \) exists.

Case 6: If \( x \) requires attention, suppose \( x \) exists.

Case 7: If \( x \) requires attention, suppose \( x \) exists.

Case 8: If \( x \) requires attention, suppose \( x \) exists.

Case 9: If \( x \) requires attention, suppose \( x \) exists.

Case 10: If \( x \) requires attention, suppose \( x \) exists.

Case 11: If \( x \) requires attention, suppose \( x \) exists.

Case 12: If \( x \) requires attention, suppose \( x \) exists.

Case 13: If \( x \) requires attention, suppose \( x \) exists.

Case 14: If \( x \) requires attention, suppose \( x \) exists.

Case 15: If \( x \) requires attention, suppose \( x \) exists.

Case 16: If \( x \) requires attention, suppose \( x \) exists.

Case 17: If \( x \) requires attention, suppose \( x \) exists.

Case 18: If \( x \) requires attention, suppose \( x \) exists.

Case 19: If \( x \) requires attention, suppose \( x \) exists.

Case 20: If \( x \) requires attention, suppose \( x \) exists.

Case 21: If \( x \) requires attention, suppose \( x \) exists.

Case 22: If \( x \) requires attention, suppose \( x \) exists.

Case 23: If \( x \) requires attention, suppose \( x \) exists.

Case 24: If \( x \) requires attention, suppose \( x \) exists.

Case 25: If \( x \) requires attention, suppose \( x \) exists.

Case 26: If \( x \) requires attention, suppose \( x \) exists.

Case 27: If \( x \) requires attention, suppose \( x \) exists.

Case 28: If \( x \) requires attention, suppose \( x \) exists.

Case 29: If \( x \) requires attention, suppose \( x \) exists.

Case 30: If \( x \) requires attention, suppose \( x \) exists.

Case 31: If \( x \) requires attention, suppose \( x \) exists.

Case 32: If \( x \) requires attention, suppose \( x \) exists.

Case 33: If \( x \) requires attention, suppose \( x \) exists.

Case 34: If \( x \) requires attention, suppose \( x \) exists.

Case 35: If \( x \) requires attention, suppose \( x \) exists.

Case 36: If \( x \) requires attention, suppose \( x \) exists.

Case 37: If \( x \) requires attention, suppose \( x \) exists.

Case 38: If \( x \) requires attention, suppose \( x \) exists.

Case 39: If \( x \) requires attention, suppose \( x \) exists.

Case 40: If \( x \) requires attention, suppose \( x \) exists.

Case 41: If \( x \) requires attention, suppose \( x \) exists.

Case 42: If \( x \) requires attention, suppose \( x \) exists.

Case 43: If \( x \) requires attention, suppose \( x \) exists.

Case 44: If \( x \) requires attention, suppose \( x \) exists.

Case 45: If \( x \) requires attention, suppose \( x \) exists.

Case 46: If \( x \) requires attention, suppose \( x \) exists.

Case 47: If \( x \) requires attention, suppose \( x \) exists.

Case 48: If \( x \) requires attention, suppose \( x \) exists.

Case 49: If \( x \) requires attention, suppose \( x \) exists.

Case 50: If \( x \) requires attention, suppose \( x \) exists.
at stage $s$. Note that the next $\lambda$-stage will (be genuine) and there will be no links with top $\lambda$.

Now suppose the lemma for $n \geq 0$ and let $\gamma \subset \beta$ with $\text{lth}(\gamma) = n$. Let $\alpha = \gamma^{\gamma q}$ for $\alpha \subset \beta$. Let $s_1$ be a stage such that for all $s > s_1$:

(a) $\sigma_s \leq_\lambda \alpha$ implies $\sigma_s \subset \alpha$.

(b) If $\text{lth}(\gamma) = 1 \pmod{3}$ and $\gamma \subset \gamma^s \subset \gamma$, then $\gamma$ does not receive attention at stage $s$.

(c) If $\text{lth}(\gamma) = 2 \pmod{3}$ and $\gamma^{\gamma f} \subset \gamma$, then $\gamma$ does not receive attention at stage $s$.

(d) If $q = 1$ and $\text{lth}(\gamma) = 0 \pmod{3}$, then $s$ is not $\gamma$-expansionary.

(e) If $q = w$ and $\text{lth}(\gamma) = 2 \pmod{3}$, then $s$ is not $\gamma$-expansionary.

(f) If $q = f$ and $\text{lth}(\gamma) = 2 \pmod{3}$, then $\gamma$ does not require attention at stage $s$.

Now suppose the lemma for all $\gamma \subset \gamma$. To see (iii) and (iv) let $s_2 > s_1$ be any stage. By hypothesis, there is a genuine $\gamma$-stage $s_2 > s_2$ say $\gamma = \sigma(t, s_2)$. Now if $s_3$ is not a genuine $\alpha$-stage there are only two possibilities: either $\sigma_{s_3} = \gamma$ or there is a link $(\gamma, \rho)$. In the second case if $(\gamma, \rho)$ is not removed or cancelled at stage $s_3$, then it will be at stage $s_3 > s_3$, the least genuine $\gamma$-stage after $s_3$, by the same reasoning as we used for $\lambda$. In any case it follows that at the first genuine $\alpha = \gamma^{\gamma 0}$-stage exceeding $s_3$ there must be no link with top $\gamma$ and this will be a genuine $\alpha$-stage. Thus we get (ii) and (iv).

Finally to get (i) we need only establish (iii), since the only possible way $\beta$ might not be infinite is if some $\alpha$ receives attention infinitely often for $\text{lth}(\alpha) \equiv 1 \pmod{3}$. Thus suppose $\text{lth}(\alpha) \equiv 1 \pmod{3}$. Now the only way $\alpha$ might need attention infinitely often is if its follower keeps getting cancelled. Cancellation of such a follower $y$ only happens if, for some $\gamma \subset \gamma$ with $\text{lth}(\gamma) \equiv 2 \pmod{3}$, at the close of a $\gamma$-gap we have seen $\gamma^p(x, \gamma^s, s) \ (p \in \{0, 2\})$ or $\rho(x, \gamma, s)$ need changing, for $x \leq y$. By choice of $s_1$ and initialization of $\alpha$ when some $\alpha < \alpha$ receives attention we might as well suppose that $\alpha$ has no follower at stage $s_1$. Furthermore we may suppose (by (iv)) a stage $s_3$ such that for every link $(\tau, \eta)$ with $\eta \subset \alpha$ or $\tau \subset \alpha$ that existed at stage $s_3$, there has been a stage $s_2 = s_2(\tau, \eta) < s_3$ where $(\tau, \eta)$ has been removed. By cancellation we can suppose that $\alpha$ has no follower at stage $s_3$. Now at the least genuine $\alpha$-stage after $s_3$, $\alpha$ gets a follower $y = y(\alpha, s_3)$.

We claim that $y$ cannot be cancelled. The only way $y$ could be cancelled is if for some $\gamma^\gamma g \subset \alpha$ and follower $x < y$ we have seen $\gamma^p(x, \gamma^s, s_3)$ or $\rho(x, \gamma, s_3)$ need changing at the stage $s_4 > s_3$ where the $\gamma$-gap is closed. However by choice of $s_3$ and induction the only followers $z < s_3$ left alive and targeted for $A$ at stage $s_3$ are followers of $\alpha \subset \alpha$, which will never enter $A$.

It follows that at stage $s_4$, case 3, subcase 6, case 2(b) pertains to $x$ and $\gamma$. This case specifically protects $y$ from cancellation since no $x$-computations have changed since stage $s_3$. This gives the claim. Therefore $\alpha$ can receive attention at most once more (meeting $P_{e(\alpha)}$) and this gives the lemma. □

(3.13) **Lemma.** Let $\alpha \subset \beta$ with $\text{lth}(\alpha) \equiv 1 \pmod{3}$ and let $y$ be a follower of $\alpha$ where $y = \lim_{s} y(\alpha, s)$ (i.e. the last follower of $\alpha$ appointed at stage $s_1$, say,
This point follows that \( \phi \) can be reset at most \( n \) times, giving the lemma.

Consider computations, and followers applied after state \( s \) must exceed \( (s \phi) \). For this reason, we discuss in the basic module: since the reset interval processes these \( \exists \phi \) let another \( \phi \) become caused only by multiple \( \eta \) resetting a. 

However for the \( \phi \), \( \phi \) is reset only if 

This means that it suffices to show that \( (x, a) \) can be reset at most \( n \) times.

Proof. \( \phi \) is the collection of \( x, a \). 

Suppose \( \phi \subset (s, a) \). Then \( \phi \neq 0 \) and \((s \mod 0) = (s) \).

Also we need that 

Suppose \( \phi \neq 0 \). Then \( \phi \neq 0 \).

Similarly we see that 

Proof. This follows from \( \phi \) and \( \phi \).

Suppose \( \phi \neq 0 \). Then \((s \mod 0) = (s) \).

The following holds for some \( t \). For all \( a \), \( \phi \neq 0 \).

Hence one of 

Theorem. Suppose \( \phi \subset (s, a) \). Then one of 

Proof. This follows from \( \phi \) and \( \phi \).

Suppose \( \phi \neq 0 \). Then \((s \mod 0) = (s) \).

Hence \( (s \phi) \) is reset after \( n \) times.
(3.17) **Lemma** (Truth of outcome). Suppose $\alpha \subset \beta$ with $\text{lh}(\alpha) = 2$ (mod 3) and $\alpha^\text{a} \subset \beta$.

(i) If $a = f$, then $\Psi^1_{(\alpha)}(W^1_{e(\alpha)}) \neq Q_{\tau(\alpha)}$.

(ii) If $a = g$, then $W^0_{e(\alpha)} \leq_T W^1_{e(\alpha)}$.

**Proof.** (i) Let $s_1$ be a stage as in (3.12). Then if $a = f$ some follower $x_k$ is enumerated into $Q_{\tau(\alpha)}$ at some stage $s$ to create a disagreement and $r(\alpha, s) = s$. By choice of $s_1$ this restraint is not violated.

(ii) Suppose that $a = g$ and let $s_1$ be the relevant stage which is good for $\alpha$ as in (3.12). We show how to compute arbitrarily long initial segments of $W^0_{\tau(\alpha)}$. Note that choice of $s_1$ means that after stage $s_1$, $\alpha$ cannot be initialized. Moreover, at the close of every unsuccessful $\alpha$-gap, $\alpha$ gets a new follower hence $x_k = \lim x(\alpha, k, s)$ exists. Indeed, if $s > s_1$ and $x(\alpha, k, s)$ is defined at stage $s$, then $x(\alpha, k, s) = x_k$. Now we reason virtually exactly as for the basic module and we simply sketch to compute $W^0_{\tau(\alpha)}[z]$. Find a stage $s_2 > s_1$ where $x = x_k > z$ is appointed to $\alpha$. At this stage we create a link $(\tau, \alpha)$ where $\tau = \tau(\alpha)$. When this link at $s_2 > s_1$ is removed, we cancel all followers $m$ targeted at $A$ with $z$ and define $\gamma^p(x, s_3) = \gamma^p(x, \alpha, s_3)$ for $p \in \{0, 2\}$, and set $r(\alpha, s_3) = s_3$. These restraints are not violated by choice of $s_3$ until we open an $\alpha$-gap (where we define $\rho(x, \alpha, s_3)$). Since this gap is closed unsuccessfully at some $\alpha$-stage $s_4 > s_3$, and so case 3, subcase 6 pertains. If case 1 of this subcase holds, then nothing has changed and all the $\gamma^p$, $\rho$-computations are the same as they were at stage $s_3$. In any of the other cases of subcase 6 (except case 2(e)) we get to reset $\rho$ but cancel all followers $y > x$ targeted for $A$. In case 2(e) we only lose the outer layer but again get to cancel. Again the cancellation procedure ensures that $\rho$ gets reset at most $x$ times. Of course to decide what $\rho$ is reset to we need only go to the next $\alpha$-stage exceeding $s_4$. The whole point is that like the basic module $W^0_{\tau}[x]$ can only change between expansionary stages if $W^1_{\tau}[\rho(x, s)]$ does. Hence $W^0_e \leq_T W^1_e$. □

(3.18) **Variations and comments.** What is really important here is closing gaps at $\alpha$-stages and 'confirming' followers to define $\gamma^0$, $\gamma^2$ and $\rho$. With a little more care, one can combine the argument above with permitting to show:

(3.19) **Every nonzero r.e. degree has a predecessor that bounds no critical triple.**

The reader may note that what we also ensure in our construction is that for all $x$, for all functionals $\Phi$ and r.e. sets $W \leq_T A$ there is a recursive enumeration of $W$ and $\Phi$ such that if $\Phi(W)$ is r.e. then

$$\{s : W^1_e[u(\Phi_e(W_s; x))] \neq W^1_{e+1}[u(\Phi_e(W_s; x))]\} \leq x.$$  

It is unclear if some condition like this on $A$ is sufficient. It would seem to be related to the Mohrherr/Bickford–Mills superlow degrees (see e.g. Mohrherr [11]) and the array recursive degrees of Downey–Jockusch–Stob [5].
References


Remark. The technical difference between, say, $A_0$ and $A_0$ above is that $A_0$ is not bounded number. This is why we can use the repeated technique of Section 3.

Let $A$ be the embedding of $A$. Then $A$ has the degrees $e_A$. Therefore the degrees $e_A$ form a recursively enumerable set and multiple remaining and noncomputable degrees are dense (proof [6]).

Let $x > y$ denote that the degrees are dense (proof [5]) in the sense that for all $e < f$, if $e$ is a recursively enumerable set, then $f$ is a recursively enumerable set (proof [6]).

We remark here that is possible to use the definitions of [5] to create the non-r.e. degrees.