

# ASYMPTOTIC DENSITY AND THE ERSHOV HIERARCHY

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ABSTRACT. We classify the asymptotic densities of the  $\Delta_2^0$  sets according to their level in the Ershov hierarchy. In particular, it is shown that for  $n \geq 2$ , a real  $r \in [0, 1]$  is the density of an  $n$ -c.e. set if and only if it is a difference of left- $\Pi_2^0$  reals. Further, we show that the densities of the  $\omega$ -c.e. sets coincide with the densities of the  $\Delta_2^0$  sets, and there are  $\omega$ -c.e. sets whose density is not the density of an  $n$ -c.e. set for any  $n \in \omega$ .

## 1. INTRODUCTION

In computability theory, the complexity of sets  $A \subseteq \omega$  is often measured using Turing reducibility and the arithmetic hierarchy. In number theory, the size of a set  $A \subseteq \omega$  is often measured using its asymptotic density  $\rho(A) \in [0, 1]$ , if this density exists. It is natural to inquire about relationships between these measurements. In [3] it is shown that there is a very tight connection between the position of a set  $A$  in the arithmetic hierarchy and the complexity of its density  $\rho(A)$  as a real number, provided that  $A$  has a density. (These results are summarized in Theorem 2.1 below.) Here we measure the complexity of a real  $x_0$  in terms of the complexity of its left Dedekind cut; that is, the set of all rational numbers smaller than  $x_0$ . In the current paper we study the corresponding relationship when we classify  $A$  according to the Ershov hierarchy, that is, the number of changes in a computable approximation to  $A$ .

We identify sets with their characteristic functions. According to the Shoenfield Limit Lemma, the  $\Delta_2^0$  sets  $A$  are exactly those for which there is a computable function  $g$  such that, for all  $x$ ,  $A(x) = \lim_s g(x, s)$ . Roughly speaking, the Ershov hierarchy classifies  $\Delta_2^0$  sets by the number of  $s$  with  $g(x, s) \neq g(x, s + 1)$ . In particular, if  $f$  is a function and  $A \subseteq \omega$ , then  $A$  is called  $f$ -c.e. if there is a computable function  $g$  such that, for all  $x$ ,  $A(x) = \lim_s g(x, s)$ ,  $g(x, 0) = 0$ , and  $|\{s : g(x, s) \neq g(x, s + 1)\}| \leq f(x)$ .

Our goal here is to determine the relationship between the growth rate of  $f$  and the complexity of the asymptotic density of  $A$  as a real number, if it exists. We show that every real number which is the density of a  $\Delta_2^0$  set is the density of an id-c.e. set, where id is the identity function. In fact, we show that the identity function could be replaced here by any computable, non-decreasing, unbounded function  $f$ . Thus, for any such  $f$  the densities of the  $f$ -c.e. sets coincide with the densities of the  $\Delta_2^0$  sets. Since we consider only  $f$  which are computable and nondecreasing, it remains only to consider the densities of the  $f$ -c.e. sets in the special case where  $f$  is constant. A set  $A$  is called  $n$ -c.e. if  $A$  is  $c_n$ -c.e, where  $c_n$  is the constant function

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with value  $n$  on all arguments. Thus, for example, the 1-c.e. sets are precisely the c.e. sets and the 2-c.e. sets are precisely the *d.c.e.* sets; i.e. those sets that are differences of two c.e. sets.

It is shown in Theorem 5.13 of [3] that the densities of the c.e. sets are precisely the left- $\Pi_2^0$  reals in the interval  $[0, 1]$ . Thus one might expect that the densities of the d.c.e. sets are precisely the differences of left- $\Pi_2^0$  reals in  $[0, 1]$ . We prove that this is the case, but care is necessary because  $A \setminus B$  can have a density even though  $A$  and  $B$  do not have densities. The essential observation here is that if  $B \subseteq A$  and  $A \setminus B$  has a density, then this density is  $\bar{\rho}(A) - \bar{\rho}(B)$  (where  $\bar{\rho}(X)$  is the upper density of the set  $X$ ). Note that a difference of left- $\Pi_n^0$  reals is also a difference of left- $\Sigma_n^0$  reals. A difference of left- $\Sigma_1^0$  reals is also known as a *d.c.e.* real. Relativizing the proof of Corollary 4.6 of [1] shows that there is a real which is a difference of left- $\Pi_2^0$  reals but which is neither left- $\Pi_2^0$  nor left- $\Sigma_2^0$ . Combining this with our results and Theorem 5.13 of [3] shows that there is a real which is the density of a d.c.e. set but not the density of any c.e. or co-c.e. set.

We next consider the densities of  $n$ -c.e. sets for arbitrary  $n \geq 2$ . It is well known that every  $n$ -c.e. set is a finite disjoint union of d.c.e. sets. Also the reals which are differences of left- $\Pi_2^0$  reals are easily seen to be closed under addition. Indeed, these reals form a field, as may be seen by relativizing Theorem 3.7 of [1]. Thus one might expect that if a real  $r$  is the density of an  $n$ -c.e. set, then  $r$  is a difference of left- $\Pi_2^0$  reals. We prove this, but care is again necessary because a disjoint union of sets can have a density when the sets themselves fail to have densities. It follows that, for all  $n \geq 2$ , the densities of the  $n$ -c.e. sets coincide with the densities of the d.c.e. sets.

Say that a set  $A$  is  $\omega$ -c.e. if  $A$  is  $f$ -c.e. for some computable function  $f$ . This hierarchy has been extended to levels indexed by notations for arbitrary computable ordinals (see [4]), but there are some subtleties because for levels  $\alpha \geq \omega^2$  the sets occurring at level  $\alpha$  depend on the choice of a notation for  $\alpha$ . We show that if a  $\Delta_2^0$  set has a density  $r$  then  $r$  is also the density of an  $\omega$ -c.e. set. Thus, if  $\alpha$  is a notation for a computable ordinal greater than or equal to  $\omega$ , the densities of the  $\alpha$ -c.e. sets coincide with the densities of the  $\omega$ -c.e. sets and these in turn coincide with the densities of the  $\Delta_2^0$  sets.

We summarize some background and prior results needed in Section 2. In Section 3 we characterize the densities of d.c.e. sets, and in Section 4 we characterize the densities of  $n$ -c.e. sets. In Section 5 we show that the densities of  $\Delta_2^0$  sets coincide with the densities of the  $f$ -c.e. sets for any computable, nondecreasing, unbounded function  $f$ . Finally in Section 6 we show that with respect to upper and lower densities, the Ershov hierarchy collapses even further.

## 2. BACKGROUND

We begin with the basic definitions related to asymptotic density. Let  $X$  be a set of natural numbers. When  $n \in \mathbb{N}$ , let

$$X \upharpoonright n = \{j : j \in X \wedge j < n\}.$$

For  $n > 0$ , define

$$\rho_n(X) = \frac{|X \upharpoonright n|}{n}$$

The *upper density* of  $X$  is defined to be

$$\bar{\rho}(X) = \limsup_n \rho_n(X)$$

The *lower density* of  $X$  is defined to be

$$\underline{\rho}(X) = \liminf_n \rho_n(X)$$

If the upper and lower density of  $X$  coincide, then this common value  $\rho(X) = \lim_n \rho_n(X)$  is called the *asymptotic density* of  $X$ . If  $\mathcal{C}$  is a complexity class (such as  $\Pi_2^0$ ,  $\Delta_2^0$ , etc.), then a real number  $r$  is *left (right)- $\mathcal{C}$*  if and only if its left (right) Dedekind cut belongs to  $\mathcal{C}$ . So, for example, a real  $r$  is left- $\Pi_2^0$  if and only if the set

$$\{q \in \mathbb{Q} : q < r\}$$

is  $\Pi_2^0$ .

Theorem 2.21 of [8] shows that the densities of the computable sets are exactly the  $\Delta_2^0$  reals in the interval  $[0, 1]$ . It is shown in Theorem 5.13 of [3] that the densities of the c.e. sets are exactly the left- $\Pi_2^0$  reals in  $[0, 1]$ . By relativizing and dualizing these results, one easily obtains the following theorem.

**Theorem 2.1.** (*Downey, Jockusch, Schupp*) *Let  $r$  be a real number in the interval  $[0, 1]$ .*

- (1)  *$r$  is the density of a  $\Delta_n^0$  set if and only if  $r$  is  $\Delta_{n+1}^0$ .*
- (2)  *$r$  is the density of a  $\Sigma_n^0$  set if and only if  $r$  is left- $\Pi_{n+1}^0$ .*
- (3)  *$r$  is the density of a  $\Pi_n^0$  set if and only if  $r$  is left- $\Sigma_{n+1}^0$ .*

Soare [9] gives many examples of real numbers which are left- $\Sigma_1^0$  but which are not computable and hence not left- $\Pi_1^0$ . Another example of such a real is given in [2], Corollary 5.1.9. It follows by relativization that for each  $n \geq 1$  there is a real which is left- $\Sigma_n^0$  but not left- $\Pi_n^0$ . Since a real  $r$  is left- $\Sigma_n^0$  if and only if  $-r$  is left- $\Pi_n^0$ , it follows that for  $n \geq 1$  the left- $\Sigma_n^0$  reals are not closed under subtraction. To obtain closure under subtraction, we instead consider reals of the form  $r - s$  where the reals  $r$  and  $s$  are left- $\Pi_n^0$  reals. Let  $\mathcal{D}_n$  be the set of such reals. Study of the class  $\mathcal{D}_1$  was initiated in [1], where elements of  $\mathcal{D}_1$  are called *weakly computable reals*. That paper shows that  $\mathcal{D}_1$  is actually a field. It was further shown by Ng and independently by Raichev that  $\mathcal{D}_1$  is a real-closed field. (Proofs of these statements are also given in Chapter 5 of [2] which is a comprehensive source of information on the subject.) The cited results extend by relativization to  $\mathcal{D}_n$  for all  $n \geq 1$ .

By the remarks above, there are reals in  $\mathcal{D}_n$  which are neither left- $\Sigma_n^0$  nor left- $\Pi_n^0$ . Also, Ambos-Spies, Weihrauch, and Zheng ([1], Corollary 4.10) showed that there is a  $\Delta_2^0$  real which is not in  $\mathcal{D}_1$ . It again follows by relativization that for each  $n \geq 1$  there is a  $\Delta_{n+1}^0$ -real which is not in  $\mathcal{D}_n$ .

Let  $\mathcal{D}_n^A$  be the class of reals which are differences of left- $\Pi_n^{0,A}$  reals, so  $\mathcal{D}_n^A$  is simply the relativization of  $\mathcal{D}_n$  to  $A$ . Such relativized classes play a useful role in algorithmic randomness. Call a set  $A$  *low for  $\mathcal{D}_n$*  if  $\mathcal{D}_n^A = \mathcal{D}_n$ . It was shown by J. Miller (see Theorem 15.9.2 of [2]) that the  $K$ -trivial sets in the sense of algorithmic randomness are precisely the sets  $A$  which are low for  $\mathcal{D}_1$ .

### 3. DENSITIES OF D.C.E. SETS

It is shown in Theorem 5.13 of [3] that the densities of the c.e. sets are the left- $\Pi_2^0$  reals in  $[0, 1]$ . Hence if  $A, B$  are c.e. sets having densities and  $B \subseteq A$ , then

$\rho(A \setminus B) = \rho(A) - \rho(B)$  is a difference of left- $\Pi_2^0$  reals. This suggests that if  $r$  is the density of a d.c.e. set, then  $r$  should also be a difference of left- $\Pi_2^0$  reals, that is,  $r \in \mathcal{D}_2$ . However,  $A \setminus B$  can have a density even when  $B \subseteq A$ , and the sets  $A$  and  $B$  do not have densities. Nonetheless, we will prove in this section that if a d.c.e. set has density  $r$ , then  $r \in \mathcal{D}_2$ . Conversely, we show that every real in  $\mathcal{D}_2 \cap [0, 1]$  is the density of a d.c.e. set, thus characterizing the densities of the d.c.e. sets as the reals in  $\mathcal{D}_2 \cap [0, 1]$ . This implies that there is a real which is the density of a d.c.e. set but not of any c.e. or co-c.e. set.

The following proposition shows that we can use upper densities to avoid the above mentioned difficulty of nonexistent densities.

**Proposition 3.1.** *If  $M \geq a_n \geq b_n \geq L$  for all  $n$ , and if  $\lim_{n \rightarrow \infty} (a_n - b_n)$  exists, then  $\lim_{n \rightarrow \infty} (a_n - b_n) = \limsup_n a_n - \limsup_n b_n$ .*

*Proof.* Note that the result is clear if  $a_n - b_n$  is constant, since then  $\{a_n\}$  and  $\{b_n\}$  are near their respective lim sups simultaneously, and so respective lim sups must differ by the same constant. We show below that essentially this same argument works when we assume only that  $a_n - b_n$  has a limit.

Let  $a = \limsup_n a_n$  and  $b = \limsup_n b_n$ , where these are real numbers because the given sequences are bounded. Let  $d = \lim_n (a_n - b_n)$ , which exists by hypothesis. We must show that  $d = a - b$ , which we prove in the form  $b = a - d$ , i.e.  $\limsup_n b_n = a - d$ .

Let  $\epsilon > 0$  be given. Since  $\limsup_n a_n = a$ , we have  $a_n \leq a + \epsilon/2$  for all sufficiently large  $n$ . Since  $\lim_n (b_n - a_n) = -d$ , we also have  $b_n - a_n \leq -d + \epsilon/2$  for all sufficiently large  $n$ . Adding these inequalities, we have  $b_n \leq a - d + \epsilon$  for all sufficiently large  $n$ . Since  $\epsilon$  was arbitrary, we conclude that  $b = \limsup_n b_n \leq a - d$ .

To obtain the reverse inequality, again let  $\epsilon > 0$  be given. Since  $\limsup_n a_n = a$ , there are infinitely many  $n$  such that  $a_n \geq a - \epsilon/2$ . Let  $S$  be the set of such  $n$ . Since  $\lim_n (b_n - a_n) = -d$ , we have  $b_n - a_n \geq -d - \epsilon/2$  for all sufficiently large  $n$ . Adding these inequalities, we have that  $b_n \geq a - d - \epsilon$  for all sufficiently large  $n \in S$ , and hence for infinitely many  $n$ . Since  $\epsilon$  was arbitrary, we conclude that  $b = \limsup_n b_n \geq a - d$ , and hence, by the previous paragraph,  $b = a - d$ .  $\square$

**Corollary 3.2.** *If  $Y$  is a subset of  $X$ , and if  $X - Y$  has a density, then its density is the upper density of  $X$  minus the upper density of  $Y$ .*

**Corollary 3.3.** *If  $C$  is a d.c.e. set which has a density, then  $\rho(C) \in \mathcal{D}_2$ .*

*Proof.* Let  $C = A \setminus B$ , where  $A, B$  are c.e. and  $B \subseteq A$ . Then  $\rho(C) = \bar{\rho}(A) - \bar{\rho}(B)$  by the previous corollary and the reals  $\bar{\rho}(A)$  and  $\bar{\rho}(B)$  are each left- $\Pi_2^0$  by [3], Theorem 5.6.  $\square$

The next theorem will allow us to prove the converse: Every real in  $\mathcal{D}_2 \cap [0, 1]$  is the density of a d.c.e. set. In order to prove the theorem we need the following lemma which asserts a well-known fact about conditional densities.

**Lemma 3.4.** *Let  $h$  be a strictly increasing function and let  $X \subseteq \omega$ . Then  $\rho(h(X)) = \rho(\text{range}(h))\rho(X)$  provided that both the range of  $h$  and  $X$  have densities.*

*Proof.* Let  $R$  be the range of  $h$ , and for each  $u$ , let  $g(u)$  be the least  $k$  such that  $h(k) \geq u$ . Note that, for all  $u$ ,

$$|h(X) \upharpoonright u| = |X \upharpoonright g(u)| \quad \& \quad |R \upharpoonright u| = g(u)$$

via the bijections induced by  $h$ . It follows that

$$\rho_{g(u)}(X) \cdot \rho_u(R) = \frac{|X \upharpoonright g(u)|}{g(u)} \frac{|R \upharpoonright u|}{u} = \frac{|h(X) \upharpoonright u|}{g(u)} \frac{g(u)}{u} = \frac{|h(X) \upharpoonright u|}{u} = \rho_u(h(X))$$

for all  $u$ . Hence,  $\rho_u(h(X)) = \rho_{g(u)}(X) \cdot \rho_u(R)$ . As  $u$  tends to infinity,  $g(u)$  also tends to infinity, and the lemma follows.  $\square$

**Theorem 3.5.** *If  $a, b$  are left- $\Pi_2^0$  reals such that  $0 \leq b \leq a \leq 1$ , then there is a c.e. set  $A$  with density  $a$  and a c.e. set  $B \subseteq A$  with density  $b$ .*

*Proof.* It is shown in Theorem 5.13 of [3] that every left- $\Pi_2^0$  real in the interval  $[0, 1]$  is the density of a c.e. set, which is the case  $a = b$  of the current result. Thus, we may assume that  $b < a$ . Let  $q$  be a rational number such that  $b < q < a$ , and let  $C$  be a computable set of density  $q$ , which exists by Theorem 2.21 of [8]. We will obtain  $A$  by expanding  $C$  and obtain  $B$  by shrinking  $C$ . In more detail, we obtain  $A$  as  $C \sqcup A_0$ , where  $A_0 \subset \overline{C}$  is a c.e. set of density  $a - q$ .

Let  $h$  be a computable, strictly increasing function with range  $\overline{C}$ . Then let  $A_0 = h(A_1)$ , where  $A_1$  is a c.e. set of density  $(a - q)/(1 - q)$ . Such a set exists by Theorem 5.13 of [3] because  $(a - q)/(1 - q)$  is a left- $\Pi_2^0$  real in  $[0, 1]$ . Hence,

$$\rho(A_0) = \rho(h(A_1)) = (1 - q) \frac{a - q}{1 - q} = a - q$$

by the lemma, and thus

$$\rho(A) = \rho(C \sqcup A_0) = \rho(C) + \rho(A_0) = q + (a - q) = a$$

as desired. The c.e. set  $B \subseteq C$  of density  $b$  is obtained analogously, but working within  $C$  instead of  $\overline{C}$ . Namely  $B = h(B_1)$ , where  $h$  is now a strictly increasing computable function with range  $C$  and  $B_1$  is a c.e. set of density  $b/q$ . Since  $B \subseteq C \subseteq A$ , the proof is complete.  $\square$

**Corollary 3.6.** *The densities of the d.c.e. sets coincide with the reals in  $\mathcal{D}_2 \cap [0, 1]$ .*

*Proof.* The density of a d.c.e. set is in  $\mathcal{D}_2 \cap [0, 1]$  by Corollary 3.3. For the other direction, consider a real  $r \in [0, 1]$  which is a difference of left- $\Pi_2^0$  reals. Write  $r$  as  $a - b$ , where  $a, b$  are left- $\Pi_2^0$  reals and  $1 \geq a \geq b \geq 0$ . By Theorem 3.5, there are c.e. sets  $A, B$  such that  $B \subseteq A$ ,  $\rho(A) = a$ , and  $\rho(B) = b$ . Then  $A \setminus B$  is d.c.e. and  $\rho(A \setminus B) = a - b$ .  $\square$

**Corollary 3.7.** *There is a 2-c.e. set which has a density but whose density is not the density of any c.e. set or co-c.e. set.*

*Proof.* By Corollary 4.6 of [1], relativized to  $0'$ , there is a real  $r$  which is a difference of left- $\Pi_2^0$  reals but is not left- $\Pi_2^0$  or left- $\Sigma_2^0$ . We may assume that  $r \in [0, 1]$ , so  $r$  is the density of a 2-c.e. set. The real  $r$  is not the density of a c.e. or co-c.e. set, since the densities of c.e. sets are left- $\Pi_2^0$  and the densities of co-c.e. sets are left- $\Sigma_2^0$ .  $\square$

#### 4. THE DENSITIES OF $n$ -C.E. SETS

It is well known that if  $D$  is an  $n$ -c.e. set then, for some  $k$ ,  $D = D_1 \cup D_2 \cup \dots \cup D_k$  where  $D_1, D_2, \dots, D_k$  are pairwise disjoint d.c.e. sets. If each  $D_i$  has a density, then  $\rho(D) = \sum_{i \leq k} \rho(D_i)$ , where  $\rho(D_i) \in \mathcal{D}_2$  by Corollary 3.3. Since  $\mathcal{D}_2$  is closed under addition, it follows that  $\rho(D) \in \mathcal{D}_2$ . However, we again have the situation that a

disjoint union of sets can have a density when the sets themselves do not. This time, an algebraic trick will come to our rescue.

The following proposition is a well-known fact about  $n$ -c.e. sets.

**Proposition 4.1.** *Suppose  $A$  is an  $n$ -c.e. set where  $n$  is a positive integer.*

- (1) *If  $n = 2k$  where  $k \in \mathbb{N}$ , then  $A$  can be written in the form*

$$(A_1 - A_2) \cup \dots \cup (A_{2k-1} - A_{2k})$$

*where  $A_1, \dots, A_{2k}$  are c.e. and  $A_1 \supseteq \dots \supseteq A_{2k}$ .*

- (2) *If  $n = 2k + 1$  where  $k \in \mathbb{N}$ , then  $A$  can be written in the form*

$$(A_1 - A_2) \cup \dots \cup (A_{2k-1} - A_{2k}) \cup A_{2k+1}$$

*where  $A_1, \dots, A_{2k+1}$  are c.e. and  $A_1 \supseteq \dots \supseteq A_{2k+1}$ .*

**Theorem 4.2.** *If  $n \geq 1$ , and if  $A$  is an  $n$ -c.e. set that has a density, then the density of  $A$  is a difference of left- $\Pi_2^0$  reals.*

*Proof.* Without loss of generality, suppose  $n = 2k$  where  $k \in \mathbb{N}$ . By Proposition 4.1, there are c.e. sets  $A_1, \dots, A_{2k}$  such that  $A = (A_1 - A_2) \cup \dots \cup (A_{2k-1} - A_{2k})$  and  $A_1 \supseteq \dots \supseteq A_{2k}$ . Thus,  $A_1 - A_2, \dots, A_{2k-1} - A_{2k}$  are pairwise disjoint. Let  $a_{j,s} = \rho_s(A_j)$ . It follows that

$$\rho_s(A) = \left( \sum_{j \text{ odd}} a_{j,s} \right) - \left( \sum_{j \text{ even}} a_{j,s} \right).$$

Note that  $a_{2,s} \leq a_{1,s}$ ,  $a_{4,s} \leq a_{3,s}$ ,  $\dots$ ,  $a_{2k,s} \leq a_{2k-1,s}$ . So, by Proposition 3.1,  $\rho(A) = a - b$  where

$$a = \limsup_s \left( \sum_{j \text{ odd}} a_{j,s} \right), \text{ and where}$$

$$b = \limsup_s \left( \sum_{j \text{ even}} a_{j,s} \right).$$

It thus suffices to show that if  $\{q_n\}$  is a computable sequence of rational numbers and  $r = \limsup_n q_n$ , then  $r$  is a left- $\Pi_2^0$  real. This is obvious if  $r$  is itself rational. Otherwise, for every rational number  $q$ ,  $q < r$  if and only if there are infinitely many  $n$  with  $q < q_n$ , from which the claim follows.  $\square$

**Corollary 4.3.** *Let  $n \geq 2$ . The densities of the  $n$ -c.e. sets coincide with the reals in  $\mathcal{D}_2 \cap [0, 1]$  and hence with the densities of the 2-c.e. sets.*

## 5. DENSITIES OF $\omega$ -C.E. SETS

It is shown in [8] that the densities of the computable sets are precisely the  $\Delta_2^0$  reals in  $[0, 1]$ . By relativization, the densities of the  $\Delta_2^0$  sets are precisely the  $\Delta_3^0$  reals in  $[0, 1]$ . In this section, we show that the densities of the  $\omega$ -c.e. sets coincide with the densities of the  $\Delta_2^0$  sets and in fact prove the following much stronger result.

**Theorem 5.1.** *Let  $f$  be a computable, nondecreasing, unbounded function. If  $A$  is a  $\Delta_2^0$  set that has a density, then the density of  $A$  is that of an  $f$ -c.e. set.*

*Proof.* We must construct an  $f$ -c.e. set  $B$  such that  $\rho(B) = \rho(A)$ . Our definition of  $B$  uses an oracle for  $0'$  and also a computable approximation  $\{A_s\}$  to  $A$ . We will define an increasing modulus function  $m$  for  $A$ , and arrange that, for each  $x > 0$ ,

$$|\rho_{m(x)}(B) - \rho_x(A)| \leq 1/x$$

It then follows that  $\rho(B) = \rho(A)$  if  $B$  has density. We show that  $B$  can be defined on arguments not in the range of  $m$  in such a way that  $B$  does in fact have a density.

We now define  $m$  by recursion. Let  $m(0) = (\mu s)[f(s) > 0]$ . Given  $m(x)$ , let  $m(x+1)$  be the least  $s > (x+1) \cdot m(x)$  such that:

$$f(s) > x+1 \ \& \ (\forall t \geq s)[A_t \upharpoonright (x+1) = A_s \upharpoonright (x+1)]$$

Note that  $m$  is total because  $f$  is unbounded.

We now define  $B(y)$  by recursion on  $y$ . If  $y < m(0)$ , then  $y \notin B$ . Now suppose  $y \in [m(x), m(x+1))$ , and  $B(z)$  has been defined for all  $z < y$ , so  $\rho_y(B)$  is defined. Then put  $y$  into  $B$  if and only if  $\rho_y(B) < \rho_{x+1}(A)$ . The intuition is that we are increasing the density of  $B$  when it is less than or equal to its “target value”  $\rho_{x+1}(A)$  and otherwise we are decreasing it. Hence, as  $y$  increases toward  $m(x+1)$ ,  $\rho_y(B)$  should move in the direction of this target value, and not stray far from it once it gets close to it.

To make this argument more precise, consider first the case where  $\rho_{m(x)}(B) \leq \rho_{x+1}(A)$ . Let  $y_0$  be the least element  $y$  of  $\overline{B}$  in the interval  $[m(x), m(x+1))$ , or  $y_0 = m(x+1)$  if there is no such  $y$ . Then  $\rho_y(B)$  is increasing in  $y$  for  $y \in [m(x), y_0)$  and  $|\rho_{y_0}(B) - \rho_{x+1}(A)| \leq 1/(x+1)$ . Further, it is easy to see by induction on  $y$  that  $|\rho_y(B) - \rho_{x+1}(A)| \leq 1/(x+1)$  for  $y_0 \leq y \leq m(x+1)$ . Thus,  $|\rho_{m(x+1)}(B) - \rho_{x+1}(A)| \leq 1/(x+1)$ . In addition, for all  $y \in [m(x), m(x+1))$ , either  $\rho_{m(x)}(B) \leq \rho_y(B) \leq \rho_{x+1}(A)$  or  $|\rho_y(B) - \rho_{x+1}(A)| \leq 1/(x+1)$ , as can be seen by considering the cases  $y \leq y_0$  and  $y > y_0$ . Dual considerations show that if  $\rho_{m(x)}(B) \geq \rho_{x+1}(A)$ , then again  $|\rho_{m(x+1)}(B) - \rho_{x+1}(A)| \leq 1/(x+1)$ . Also, for all  $y \in [m(x), m(x+1))$ , either  $\rho_{m(x)}(B) \geq \rho_y(B) \geq \rho_{x+1}(A)$  or  $|\rho_y(B) - \rho_{x+1}(A)| \leq 1/(x+1)$ .

From the above, it follows at once that  $\lim_x \rho_{m(x)}(B) = \lim_x \rho_x(A) = \rho(A)$ . Further, if  $\rho_{m(x)}(B)$  and  $\rho_{m(x+1)}(B)$  are both within  $\epsilon$  of  $\rho(A)$ , then for all  $y \in [m(x), m(x+1))$ ,  $\rho_y(B)$  is within  $\epsilon + 1/(x+1)$  of  $\rho(A)$  by the above paragraph. Hence,  $\rho(B) = \lim_y \rho_y(B) = \lim_x \rho_{m(x)}(B) = \rho(A)$ .

We now show that  $B$  is  $f$ -c.e. First, observe that  $B$  is  $\Delta_2^0$  since  $A$  and  $m$  are  $\Delta_2^0$ ,  $f$  is computable, and the sets  $A_s$  are uniformly computable. Thus  $B$  has a computable approximation  $\{B_s\}$ . Further, if  $A_0 = \emptyset$  and we choose  $\{B_s\}$  in a natural way starting with our given approximation  $\{A_s\}$  to  $A$ , then  $B_0 = \emptyset$  and for each  $z$  there are at most  $f(z)$  values of  $s$  with  $B_{s+1}(z) \neq B_s(z)$ . This implies that  $B$  is  $f$ -c.e. The proof is a straightforward argument which we merely sketch. Call a function  $h$  *approximable from below* if there is a computable function  $g$  such that  $h(x) = \lim_s g(x, s)$  for all  $x$  and  $g(x, s) \leq g(x, s+1)$  for all  $x$  and  $s$ . It is easy to see that the function  $m$  defined above is approximable from below. Let  $h(z)$  be the least  $x$  with  $z < m(x)$ . Define “approximable from above” analogously. Then  $h$  is approximable from above because  $m$  is approximable from below. Further, note that  $z < m(f(z))$  for all  $z$ , by the definition of  $m$ . It follows, by the definition of  $h$ , that  $h(z) \leq f(z)$  for all  $z$ . Hence  $h$  is approximable from above via a function  $g$  with  $g(z, 0) = f(z)$  for all  $z$ . It follows that for each  $z$  there are at most  $f(z)$

values of  $s$  with  $g(z, s+1) \neq g(z, s)$ . Crucially, if we define the approximation  $g$  in a natural way, and if  $g(z, s+1) = g(z, s)$ , then  $B_{s+1}(z) = B_s(z)$ . This is because, if  $z \in [m(x), m(x+1))$  (so  $h(z) = x+1$ ) then  $B(z)$  is determined by  $A \upharpoonright (x+1)$ , so if our approximation to the value of  $x+1$  does not change, our approximation to  $m(x)$  does not change either, and so our approximation to  $A \upharpoonright (x+1)$  does not change either. Since  $B(z)$  is determined by our approximations to  $h(z)$  and  $A \upharpoonright h(z)$ , it follows that our approximation to  $B(z)$  does not change if our approximation to  $g(z)$  does not change. Since our approximation to  $g(z)$  changes at most  $f(z)$  times, our approximation to  $B(z)$  changes at most  $f(z)$  times, and hence  $B$  is  $f$ -c.e.  $\square$

In the above proof, we assumed that  $A$  had a density. However, the same proof establishes the following stronger result, where we make no such assumption.

**Corollary 5.2.** *(to proof) For any computable, nondecreasing, unbounded function  $f$  and any  $\Delta_2^0$  set  $A$ , there is an  $f$ -c.e. set  $B$  such that  $\bar{\rho}(B) = \bar{\rho}(A)$  and  $\underline{\rho}(B) = \underline{\rho}(A)$ .*

**Corollary 5.3.** *For any computable, nondecreasing, unbounded function  $f$  there is an  $f$ -c.e. set that has a density, but its density is not the density of any  $n$ -c.e. set,  $n \in \omega$ .*

*Proof.* By Corollary 4.10 of [1], relative to  $0'$ , there is a  $\Delta_3^0$  real  $r$  in the interval  $[0, 1]$  which is not a difference of left- $\Pi_2^0$  reals. Thus, by Theorem 4.2,  $r$  is not the density of any  $n$ -c.e. set for any  $n$ . On the other hand, by Theorem 2.21 of [8], relativized to  $0'$ , there is a  $\Delta_2^0$  set  $A$  with density  $r$ . Then by Theorem 5.1, there is an  $f$ -c.e. set  $B$  of density  $r$ .  $\square$

## 6. UPPER AND LOWER DENSITY

Since, with respect to density, the Ershov hierarchy collapses to levels 0, 1, 2, and  $\omega$ , it is natural to ask if there is any more separation with respect to upper and lower densities. The following observations show that in some sense we get even more collapse.

**Proposition 6.1.** *Let  $r \in [0, 1]$ . Then, the following are equivalent.*

- (1)  $r$  is the upper density of a  $\Sigma_2^0$  set.
- (2)  $r$  is left- $\Pi_3^0$ .
- (3)  $r$  is the upper density of a  $\Pi_1^0$  set.

*Proof.* Without loss of generality, let us assume  $r$  is irrational.

Suppose  $r$  is the upper density of a  $\Sigma_2^0$  set. Then, by Theorem 5.7 of [3] relativized to  $\emptyset'$ ,  $r$  is left- $\Pi_3^0$ .

At the same time, if  $r$  is left- $\Pi_3^0$ , then  $1-r$  is left- $\Sigma_3^0$ . So, by Theorem 5.8 of [3],  $1-r$  is the lower density of a c.e. set. It follows that  $r$  is the upper density of a co-c.e. set. The remaining implication is immediate.  $\square$

**Corollary 6.2.** *Let  $r \in [0, 1]$ . Then, for all  $n \geq 2$ ,  $r$  is the upper density of an  $n$ -c.e. set if and only if  $r$  is the upper density of a co-c.e. set.*

The above results can be dualized to show that the lower densities of the  $\Pi_2^0$  sets coincide with the left- $\Sigma_3^0$  reals in  $[0, 1]$  and with the lower densities of c.e. sets and we thus have a similar collapse for lower densities. In particular, if  $A$  is any  $\Delta_2^0$  set,



there is a c.e. set with the same lower density as  $A$  and a co-c.e. set with the same upper density as  $A$ .

## 7. SUMMARY

We have shown that the densities of the 2-c.e. sets coincide with the reals in  $[0, 1]$  which are differences of left- $\Pi_2^0$  reals, and hence there is a real which is the density of a 2-c.e. set but not of any c.e. or co-c.e. set. We have also proved that, for  $n \geq 2$  the densities of the  $n$ -c.e. sets coincide with the densities of the 2-c.e. sets. Finally, we have shown that if  $A$  is a  $\Delta_2^0$  set that has a density, then its density is the density of an  $\omega$ -c.e. set, and in fact the density of an  $f$ -c.e. set for each computable, nondecreasing, unbounded function  $f$ . It follows that for each such  $f$  there is a real number which is the density of an  $f$ -c.e. set but not of any  $n$ -c.e. set,  $n \in \omega$ .

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