AVOIDING EFFECTIVE PACKING DIMENSION 1 BELOW ARRAY NONCOMPUTABLE C.E. DEGREES.

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Abstract. Recent work of Conidis [Con12] shows that there is a Turing degree with nonzero effective packing dimension, but which does not contain any set of effective packing dimension 1.

This paper shows the existence of such a degree below every c.e. array noncomputable degree, and hence that they occur below precisely those of the c.e. degrees which are array noncomputable.

1. Introduction

Packing dimension was independently introduced by Tricot [Tri82] and Sullivan [Sul84] as a counterpart to the previously established notion of Hausdorff dimension. Both notions allow one to assign a (possibly noninteger) dimension to subsets of any metric space. The Hausdorff dimension of a set A is defined by considering how many open balls of small radius are required if they are to cover A entirely. The packing dimension of A is a closely related notion, but asks instead how many disjoint open balls of small radius can be placed so that each has its center in A.

Effective versions of both notions have been developed by Lutz, Staiger, Athreya et al. ([Lut03], [AHLM07], [Sta93]). For our purposes, the characterizations of Mayordomo [May02] and Lutz [Lut05] of, respectively, effective Hausdorff and packing dimension below can be taken as definitions.

Definition 1.1. Let A be a real (i.e. member of Cantor Space), then the effective Hausdorff dimension of A is

$$\dim(A) = \liminf_{n\to\infty} \frac{K(A \upharpoonright n)}{n},$$

and the effective packing dimension of A is

$$\mathrm{Dim}(A) = \limsup_{n\to\infty} \frac{K(A \upharpoonright n)}{n}.$$

The reader should note that we are ascribing a notion of dimension to a single real, in the same way that we can use computability theory to give meaning to randomness of a single real.

These effective notions of dimension have strong links to complexity and algorithmic randomness. Moreover, work of Simpson [Sim15] and Day [Dayep], for example, has shown that effective notions of dimension can be used to derive classical results in mathematics. In discussions with co-workers, Simpson [Sim15] proved that the classical dimension equals the entropy (generalizing a difficult result of Furstenburg 1967) using effective methods, which were much simpler. Recently Day used effective packing dimension to give a simple proof of the Kolmogorov-Sinai Theorem on Ergodic theory.

In many ways, effective packing dimension is quite well behaved on degrees. For example, we know that each Turing degree obeys a 0-1 Law for effective packing dimension. That is, complexity extraction procedures given independently by Bienvenu et al., and Fortnow et al. ([BDS09] and [FHP$^+$06], respectively) show that for any real X, $\sup\{\mathrm{Dim}(Y) \mid Y \leq_T X\}$ is either 0 or 1. These extraction processes both yield only that the supremum of the packing dimensions of the reals in the degree is 1, and hence authors wondered if the supremum of 1 was always achieved. Work of Conidis [Con12] shows that there are reals X for which the supremum is 1, but for which that supremum is not attained[1].

Conidis' construction was a direct forcing argument and resulted in a hyperimmune-free degree. The second author [Ste15] showed that the construction given by Conidis, which utilizes forcing with computable trees, can be modified to work below $\emptyset'$. That version may be interpreted as a limit-computable construction with permissions provided by $\emptyset'$. In light of this observation one might ask below which c.e. sets A the construction can be carried out; the obvious restriction is that A must provide appropriate permissions.

The array noncomputable degrees are a class introduced by Downey, Jockusch and Stob in [DJS96]. They are noted for their compatibility with constructions requiring multiple permissions (which we will see arise naturally when one carries out an approximation-based version of Conidis' construction). They have also been shown to form a natural cutoff in the Turing degrees for constructions involving reals with nonzero effective packing dimension (see for instance [DG08],[DN10],[DH10]). In our case, a result of Kummer [Kum96] is most relevant:

Theorem 1.2 (Kummer). *If A is an array computable c.e. set, any real $X \leq_T A$ has* $\mathrm{Dim}(X) = 0$.

[1]Any Martin-Löf random real X has $\dim(X) = 1$, and the computable reals all have $\mathrm{Dim}(X) = 0$, so an unattained supremum is the only difficult case to achieve.

Moreover, Downey and Greenberg [DG08] proved the 0-1 Law dichotomy held for array noncomputable degrees. If **a** is an array noncomputable c.e. degree, then **a** has effective packing dimension 1.

These results show that the only c.e. sets which can possibly provide the necessary permissions for a construction à la Conidis are the array noncomputable ones. In this paper, we show that every array noncomputable c.e. degree computes a set X with the desired properties:

Theorem 1.3. *Given any array non-computable c.e. set A, there is a real $X \leq_T A$ such that* $\mathrm{Dim}(X) > 0$ *and such that for each $Y \leq_T X$,* $\mathrm{Dim}(Y) < 1$.

In light of Kummer's result, this gives a full characterisation of the situation which follows the general pattern observed above:

Corollary 1.4. *A c.e. set A is array noncomputable if and only if there is a set $X \leq_T A$ such that* $\mathrm{Dim}(X) > 0$ *and for each $Y \leq_T X$,* $\mathrm{Dim}(Y) < 1$.

We remark that the array noncomputable degrees again show up as quite a ubiquitous class. Kummer's other result was that a c.e. degree contains a c.e. set A where the plain complexity $C(A \upharpoonright n) =^+ 2\log n$ for infinitely many n iff the degree was array noncomputable. There are other characterizations of this class. It is not yet understood how these combinatorial arguments all inter-relate.

We remark that the proof here is not a simple modification of the earlier work of the second author, but requires a reasonably delicate argument of some combinatorial complexity.

Before embarking on our construction, we should pause to note that effective Hausdorff dimension and effective packing dimension behave in quite distinct ways. There is no analogous computable extraction procedure which produces sets with higher effective Hausdorff dimension than a given input. Indeed a result of Miller confirms this fact directly:

Theorem 1.5 (Miller [Mil11])**.** *There is a set X with effective Hausdorff dimension $\frac{1}{2}$ but which cannot compute any set of higher effective Hausdorff dimension.*

The classification of reals with such fractional Hausdorff dimension is still open.

2. Strategy

Throughout this paper, we denote Turing functionals by upper-case Greek letters. We will let $\{\Phi_e\}_{e\in\omega}$ be a computable list of all Turing functionals. Other notation will be standard, and follows the conventions of Soare [Soa87]. We fix a single c.e. set $A = \lim_s A_s$ which is array noncomputable. The remainder of the paper is devoted to constructing a real $X \leq_T A$ which satisfies the requirements of Theorem 1.3.

The simplest characterisation of effective packing dimension is in terms of Kolmogorov complexity. If $\lambda \in 2^{<\omega}$, then we will denote the prefix-free Kolmogorov complexity of λ by $K(\lambda)$. As is conventional we fix a computable decreasing approximation K_s with limit K.

By creating a real X with nonzero effective packing dimension, we will automatically guarantee that for each $\varepsilon > 0$, there is some $Y \leq_T X$ such that $\mathrm{Dim}(Y) > 1 - \varepsilon$. The difficulty which arises in our construction is thus that we must prevent each $Y \leq_T X$ from having $\mathrm{Dim}(Y) = 1$.

This calls for us to maintain quite delicate control on complexity throughout our construction. In order to achieve this, we will work with pruned clumpy trees. Clumpy trees were introducted as a forcing notion by Downey and Greenberg [DG08], and will soon be defined.

Definition 2.1. For each n, we write $2^{=n}$ to mean the binary strings with length equal to n, and $2^{\leq n}$ to mean those with length less than or equal to n, respectively. If $\rho \in 2^{<\omega}, P \subseteq 2^{<\omega}$ then ρP is the strings formed by concatenating ρ with members of P. If $\sigma \in 2^{<\omega}, \tau \in 2^{<\omega} \cup 2^{\omega}$ write $\sigma \prec \tau$ to mean that σ is a proper initial segment of τ. $P \subset 2^{<\omega}$ then the $\prec$-maximal elements of P are called *leaves*.

A *pruned clump* is a downward closed subset of a set of the form $\rho 2^{\leq |\rho|}$, and which contains at least two leaves of $\rho 2^{\leq |\rho|}$. We will refer to ρ as the *root* of such a pruned clump.

If T is a tree we will say that a pruned clump D is on T if $\rho 2^{\leq |\rho|} \cap T = D$. We say that a tree $T \subseteq 2^{<\omega}$ is a *pruned clumpy tree* if every string τ on T which is an initial segment of a path through T has an extension ρ which is the root of some pruned clump on T.

A general intuition which may be useful to the reader is to expect that if T is a pruned clumpy tree which we consider in our construction, then it will have only occasional pruned clumps, which are spaced far apart from each other, but that these pruned clumps will be sites of rapid branching on T.

Our construction will be carried out within a prototypical tree T_{-1} which captures this idea nicely.

Definition 2.2. Let T_{-1} be the tree formed by taking the union of the following finite trees T^s_{-1}: T^{-1}_{-1} consists of the empty string together with the string consisting of a single 0. Let T^s_{-1} be given by the downward closure of the strings

$$\bigcup_{\substack{\lambda \in T^{s-1}_{-1} \\ \lambda \text{ a leaf of } T^{s-1}_{-1}}} \lambda 2^{=|\lambda|} 0^{2|\lambda|}.$$

Note that if T is a pruned clumpy tree, and we arrange that each pruned clump on T has a large enough number of leaves, then some of those leaves will be forced to have quite high complexity, simply because there are not many strings of low complexity of any given length. In particular, in our construction we will be able to ensure that every pruned clump we build has a leaf λ with $K(\lambda) \geq {}^{|\lambda|}/_4$. By arranging for $X \in 2^\omega$ to have such leaves among its initial segments, we will guarantee that $\mathrm{Dim}(X) \geq {}^1/_4$.

We will build a sequence $\{T_e\}_{e\in\omega}$ of c.e. pruned clumpy trees such that $T_e \subseteq T_{e-1}$ for each e. The real X which satisfies the hypotheses of theorem 1.3 will be the unique common path through all of the trees.

We will also make use of the fact that if X is a path through T_e, and Φ^X_e is total, then by choosing which leaves should be on each of the pruned clumps of T_e carefully, we can maintain some control on Φ^X_e; in particular, we will see that we are able to ensure that $\mathrm{Dim}(\Phi^X_e)$ is able to be bounded away from 1. The following lemma gives the precise conditions required to achieve this. It is a minor variation on a result given in [Ste15] (the proof is essentially unchanged), and is inspired by a similar computation given by Conidis in [Con12].

Lemma 2.3. *Let $e \in \omega$, and let $T \subseteq T_{-1}$ be a c.e. pruned clumpy tree given by a computable enumeration $T^1 \subseteq T^2 \subseteq \cdots$ such that:*

(1) For each s and each $\rho \in T^s$, if ρ is the root of a pruned clump on T^{s+1}, it is either the root of a pruned clump on T^s or a leaf of T^s, and that all branching in T^s occurs as part of some pruned clump on T^s

(2) If $\rho_0 \prec \rho$ are roots of pruned clumps on T, then $|\rho| \geq 4 \cdot 2^{2e+4}|\rho_0|$

(3) For each pruned clump P on T with root ρ, there is a string $\tau \in 2^{<\omega}$ with $|\tau| = 2^{-2e-4}|\rho|$ and such that:

  (a) for each leaf λ of P, and each $\widehat{\lambda} \in T$ such that $\lambda \preceq \widehat{\lambda}$, if $x < |\tau|$ and $\Phi^{\widehat{\lambda}}_e(x)\downarrow$, then $\Phi^{\widehat{\lambda}}_e(x) = \tau(x)$, and

  (b) for each leaf λ of P, there is some $\widehat{\lambda} \in T$ such that $\lambda \preceq \widehat{\lambda}$ and for each $x < 2^{-2e-4}|\rho|$, $\Phi^{\widehat{\lambda}}_e(x)\downarrow$.

If X is a path through T and Φ^X_e is total, then $\mathrm{Dim}(\Phi^X_e) < \alpha_e$ for some fixed $\alpha_e < 1$.

Although we will not prove the lemma in this paper, we will briefly discuss the intuition behind it. Suppose that T_e is a pruned clumpy tree which meets the conditions of the lemma, and that X is a path through T_e. Then Φ_e^X must be total. If P is a pruned clump on T_e, and λ is a leaf of P which is an initial segment of X, let ρ be the $\prec$-least extension of λ which is the root of another pruned clump on T_e. Then $\rho \prec X$, and *(3)* of the lemma ensures that all sufficiently long extensions $\widehat{\lambda} \in T_e$ of ρ have $\Phi_e^{\widehat{\lambda}}(x)\downarrow$ for each $x < 2^{-2e-4}|\rho|$; and furthermore that all of these computations agree. Thus, from the leaf λ alone, we are able to compute an initial segment of Φ_e^X of considerable length. This ensures that that initial segment cannot have particularly high complexity, which in turn will suffice to guarantee $\mathrm{Dim}(\Phi_e^X) < 1$.

3. Overview and terminology

We will be working on requirements for each $e \in \omega$, as follows:

$R_e:$ either Φ_e^X is nontotal, or
$\mathrm{Dim}(\Phi_e^X) < 1$, and for infinitely many $\xi \prec X$, $K(\xi) \geq |\xi|/4$.

Remark 3.1. If Φ_e is a total reduction, then to meet R_e we must meet the second of the conditions. Because such reductions exist, satisfying R_e for every e will ensure that $\mathrm{Dim}(X) \geq 1/4$.

For each e, we will guarantee that X satisfies the requirement R_e, either by ensuring that Φ_e^X is not total, or, if that is not possible, by attempting to make T_e satisfy the condition of Lemma 2.3. Because we will build X as a limit of a computable approximation, we will be unable to tell which of the two strategies succeeds for each e.

In addition, the approximate nature of the construction means that our attempt to build a tree T_e meeting the conditions of Lemma 2.3 is not immediately successful — to satisfy the lemma we make a minor modification to T_e after the construction.

At every stage s, we will let T_{-1}^s be as in Definition 2.2. At the start of stage s, we will be given trees T_e^{s-1} for each $e < s$ and a string ξ^{s-1} which is our current guess at an initial segment of X. We will then construct a tree T_e^s for each $e \leq s$, and define ξ^s to be some string in T_s^s. The trees we build will be nested in the sense that $T_{e-1}^s \subseteq T_e^s$ at every stage of the construction, but it will not always be the case that $T_e^{s-1} \subseteq T_e^s$.

Recall the definition of array noncomputability, as given in [DJS96].

Definition 3.2. A very strong array is a family $\mathcal{F} = \{F_k\}_{k\in\omega}$ of uniformly computable pairwise disjoint finite sets such that $|F_k| > |F_l|$

and $\max F_l < \min F_k$ whenever $k > l$, and for which $k \mapsto \max F_k$ is a computable function.

A c.e. set A is array noncomputable if there is some very strong array $\{F_k\}_{k\in\omega}$ such that for any c.e. set W, there is some k such that $W \cap F_k = A \cap F_k$.

We note that it follows easily from the definition of array noncomputability that if A is an array noncomputable c.e. set, then every very strong array meets the condition of the definition, and furthermore that for each very strong array $\{F_k\}_{k\in\omega}$, and each c.e. set W, there are infinitely many k for which such that $W \cap F_k = A \cap F_k$.

We will use the definition directly to set up permissions provided by A throughout our construction. To do so, we will first specify a particular very strong array $\mathcal{F} = \{F_k\}_{k\in\omega}$. We will then build a c.e. set W which will be used to request permission to make changes by challenging the array noncomputability of A. At each stage of the construction we will take action at a single pruned clump.

Definition 3.3. If $0 \leq e \leq s$ and $P \subseteq T^{s-1}_{e-1}$ is a pruned clump such that some leaf of P is an initial segment of ξ^{s-1}, we will say that R_e *is working on* P at stage s.

If R_e is working on a pruned clump P at stage s, we will say that one or more numbers are assigned to the root ρ of P at stage s.

At stage s, if we wish to make a change to our set X at the root ρ of a pruned clump, we will request permission to do so, by arranging that $W \cap F_k \neq A_s \cap F_k$ for each number k assigned to ρ.

Throughout the construction, we may sometimes wish to reassign a number k to a different string. When we do so, if k is currently assigned to some ρ, the new assignment will be to some $\rho_0 \prec \rho$. This will indicate that the permissions provided by F_k will now be used to request changes to X on extensions of ρ_0. This action will cause us to devote many boxes F_k to the same string ρ.

To meet the requirements R_e, it will be enough to show that there are infinitely many different roots $\rho \prec X$ of pruned clumps for which any request for permission is granted. This will be achieved in the following way: each time we are granted permission to make a change to ξ at the level of ρ, any permissions which are assigned to an extension of ρ will be reassigned to ρ. This is because we only know that $W \cap F_k = A \cap F_k$ for infinitely many k, but do not know for which k this is true. Therefore we must ensure that the permissions associated with any particular F_k are not "wasted". At the end of the construction, we will have assigned finitely many numbers k to each string $\rho \prec X$ which is

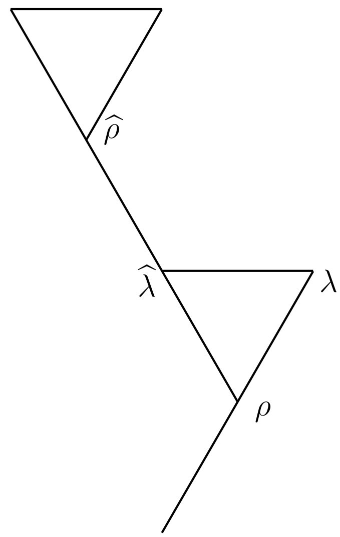

FIGURE 1. The two triangles represent pruned clumps in T_{-1}, with roots ρ and $\widehat{\rho}$; we have $\rho \prec \widehat{\rho} \prec \xi^{s-1}$. Two of the leaves on the clump with root ρ are λ and $\widehat{\lambda}$, the latter being an initial segment of $\widehat{\rho}$. Suppose at stage s we are permitted to makie a change at the level of ρ, and that $\lambda \preceq \xi^s$. Then at stage s, we reassign each k working on $\widehat{\rho}$ to instead work on ρ. We also declare that $\Gamma^A(k)[s] = \lambda$, with a large use.

the root of a pruned clump on one of our trees, and, if $W \cap F_k = A \cap F_k$, and k settles on ρ as its final assignment, every request for permission to make changes at the level of ρ will eventually be granted.

It is the process of reassignment of permissions which tells us what size the boxes F_k should be. The size of the set F_k is the number of times which we are able to use it to request permissions, so it will be important that we choose it to be large enough to accommodate any permissions which we might request throughout the construction. Each F_k must be large enough to provide enough permissions to successfully make any changes at the level of the string ρ on which k initially is working, but we must in addition include enough "spare" permissions to allow for the possibility that k could later be reassigned to work on shorter strings. In general, we can expect that many numbers k will be assigned to work on a particular string ρ. Because we are using a Turing reduction Γ to construct $X \leq_T A$, all of these numbers will be responsible for permitting changes at ρ; we will set $\Gamma^A(k)$ to be the leaf of the pruned clump on T_{-1} with root ρ which is an initial segment of ξ^s, so that all of the numbers assigned to ρ provide the same information, and will choose all of these computations to have the same use.

It will be enough if we arrange that $|F_k| \geq \sum_{i=0}^{4^k}(i+1)(2^{i+1}+1)$ for each k. This corresponds to the number of permissions needed to move through the leaves on a pruned clump with root ρ of length 4^k to try to find one which forces divergence of Φ_e, for each $e \leq 4^k$, and, if one of those searches fails, to look for a leaf of high complexity; as mentioned earlier, we also include enough permissions that the process can be repeated again on any number of initial segment of ρ, in case k is reassigned.

We will now introduce some definitions which we will use throughout our construction.

Definition 3.4. If Q and P are pruned clumps, we write $P \prec Q$ if the root of P is a proper initial segment of the root of Q, and $P \sim Q$ if P and Q have the same root. We will write $P \precsim Q$ if $P \prec Q$ or $P \sim Q$.

Notice that if $i < j$ then there will be be pruned clumps $P \subset T_i^{s-1}$, $Q \subset T_j^{s-1}$ such that $P \sim Q$. It will sometimes be convenient to ignore the distinction between such clumps, which we will do by referring to the root of a pruned clump rather than to the clump itself.

In the construction, we will build each tree T_e by attending to each pruned clump within the tree T_{e-1} individually. Our basic strategy for succeeding on a pruned clump P on T_{e-1} has two steps.

We first seek a leaf λ of P which forces divergence, i.e. to arrange that if Y is a path through T_e for which $\lambda \prec Y$, then Φ_e^Y is nontotal. We then ask for permission to make that leaf an initial segment of X; we may need to change our choice of λ several times as we discover additional halting computations.

If we later discover that every leaf λ of P can be extended to some $\widehat{\lambda} \in T_{e-1}$ for which $\Phi_e^{\widehat{\lambda}}$ halts on a large number of inputs, we switch our strategy to try to make $\mathrm{Dim}(\Phi_e^X) < 1$. We ask for permission to "thin" the pruned clump P to get a pruned clump $Q \subseteq P$ which we can place on T_e, which meets condition *(3)* of Lemma 2.3, and to choose some leaf λ of Q with $K(\lambda) \geq |\lambda|/4$ to be an initial segment of X. Once again, our choice of λ may need to change as we look for a leaf with high enough complexity, and we must seek permission to change X to match. We will later refer to condition *(3)* as the *e-majority vote criterion.*

Whether we achieve the goals outlined above will depend on whether we are granted a sufficient number of permissions by A.

Definition 3.5. If k is assigned to work on a string ρ at stage $s-1$, and $A_s \upharpoonright \max F_k \neq A_{s-1} \upharpoonright \max F_k$ then we will say that A *permits changes at* ρ at stage s.

We will be building a reduction Γ^A throughout the construction, as follows: at each stage s, if k is assigned to the root ρ of a pruned clump P on T^s_{-1}, we will set $\Gamma^{A_s}(k)$ to be the leaf λ of P for which $\lambda \preceq \xi^s$. The use $\gamma_s(k)$ for this computation will be $\max F_n$ for the largest n assigned to work on ρ. In this way, any time A permits changes at ρ, A_s will have changed on the use of that computation. This allows us to redefine $\Gamma^A(k)$ for every k assigned to ρ, any time any k meets the permitting condition defined above.

At each stage s of our construction, we make predictions about which strings will remain on the tree T^t_e at all stages $t > s$. For the root ρ of each pruned clump P on T^s_e, we will have a corresponding notion, called e-ρ-verification. Informally, we will say that a string $\sigma \succ \rho$ is e-ρ-verified if the only reason we will ever remove σ from T^t_e at some later stage t is if we take action to meet a requirement R_i for $i < e$ in a way which prevents P from being on T^t_e.

These predictions will tell us how to meet the conditions of Lemma 2.3 for $e+1$ as we build T_{e+1} inside T_e.

We will define e-ρ-verification by recursion on e. We will first define the base case of (-1)-ρ-verification, and defer $e \geq 0$ until after outlining other concepts used in the construction.

Definition 3.6. At any stage s of the construction and for any root ρ of any pruned clump on T^{s-1}_{-1}, every string $\sigma \succ \rho$ on T^{s-1}_{-1} is -1-ρ-verified.

In what follows, many of the definitions given depend on a stage s. Typically that stage will be clear throughout the construction and its verification, but we include it here to avoid ambiguity.

The next definitions formalize the e-majority vote criterion as well as some related notions which are key in satisfying Lemma 2.3. This is the point at which e-ρ-verification is first discussed. The notions of e-majority vote criterion and e-ρ-verification are defined in terms of each other, and we present the former first.

Definition 3.7. Suppose that P is a clump on T^{s-1}_{e-1} with root ρ, and $\tau \in 2^{<\omega}$. Let λ be a leaf of P.

We will say that λ is *e-τ-extendible at stage s* if there is an $(e-1)$-ρ-verified extension $\widehat{\lambda} \in T^{s-1}_{e-1}$ of λ with the property that $\Phi^{\widehat{\lambda}}_e[s] \upharpoonright |\tau| = \tau$, and such that $\widehat{\lambda}$ is the root of a pruned clump on T^{s-1}_{e-1} and $|\widehat{\lambda}| \geq 4 \cdot 2^{2e+4}|\rho|$. In this case we will say that $\widehat{\lambda}$ is an *e-τ-extension of λ at stage s*.

We will say that λ is *e-τ-extended at stage s* if there is an $(e-1)$-ρ-verified e-τ-extension $\widehat{\lambda}$ of λ on T_e^{s-1}, and furthermore that for any $\sigma \in T_e^{s-1}$ such that $\lambda \prec \sigma$, either $\widehat{\lambda} \prec \sigma$ or $\sigma \preceq \widehat{\lambda}$.

We will say that λ is *e-extendible at stage s* if λ is e-τ-extendible for some $\tau \in 2^{<\omega}$ of length $|\rho|2^{-2e-4}$ at stage s.

We will use e-τ-extendibility as the main tool to ensure $\mathrm{Dim}(\Phi_e^X) < 1$: if enough of the leaves of a clump P on T_{e-1}^s are e-τ-extendible for some fixed τ of appropriate length, we can use them to build a pruned clump $Q \sim P$ on T_e which meets the third condition of Lemma 2.3.

When searching for e-τ-extendible strings, we restrict our attention to $(e-1)$-ρ-verified strings, because these are the strings which we can safely assume actually will remain on T_{e-1}, unless we are interrupted by a higher priority requirement.

Definition 3.8. Suppose P is a pruned clump on T_{e-1}^{s-1} with root ρ.

We will say that P *meets the e-majority vote criterion at stage s* if $T_e^{s-1} \cap P$ is a pruned clump, and there is some string $\tau \in 2^{<\omega}$ of length $2^{-2e-4}|\rho|$ such that each leaf of $T_e^{s-1} \cap P$ is e-τ-extended at stage s.

We now introduce the conditions which tell us when a requirement R_e requires attention at a pruned clump in the tree T_{e-1}^s.

Definition 3.9. Suppose P is a clump on T_{e-1}^{s-1} with root ρ, where $|\rho| \geq e$, and $P \cap T_e^{s-1}$ is a pruned clump on which R_e is working.

Say that requirement R_e *requires attention due to halting at P at stage s* if the leaf λ of P which is an initial segment of ξ^{s-1} is e-extendible at stage s, but P does not meet the e-majority vote criterion.

If P is a pruned clump in T_{e-1}^{s-1} whose root ρ has $|\rho| \geq e$, say R_e *requires attention due to complexity at P at stage s* if P meets the e-majority vote criterion but the leaf λ of P which is an initial segment of ξ^{s-1} has $K_s(\lambda) < {}^{|\lambda|}/_4$.

If P does not require attention due to halting and does not meet the e-majority vote criterion, say that *the active leaf on P appears to force e-divergence* at stage s. Say that P is the *first witness to e-divergence* at stage s if P is the $\prec$-least clump on T_{e-1}^{s-1} with root of length at least e with an active leaf which appears to force e-divergence at stage s.

The restriction that $|\rho| \geq e$ given above ensures that there is a finite computable bound on the number of times we seek permission to make a change at the level of ρ.

We are now ready to complete our definition of e-ρ-verification.

Definition 3.10. Let $e \geq 0$, and $\sigma \in T_e^{s-1}$. Suppose $\rho \prec \sigma$ is the root of a pruned clump Q on T_{e-1}^{s-1}.

We say that σ is e-ρ-verified if σ is $(e-1)$-ρ-verified and either

(1) the active leaf on Q appears to force e-divergence at stage s, or
(2) For each $\rho_0 \prec \sigma$ which is the root of a pruned clump P on T_{e-1}^{s-1} such that $P \cap T_e^{s-1}$ is a pruned clump, P meets the e-majority vote criterion at stage s.

There are several intuitions behind this definition: the first is that before we believe that σ will stay on T_e, we should fisrt believe that it will stay on T_{e-1}. Thus e-ρ-verification implies $(e-1)$-ρ-verification.

The intuition behind the condition (1) of the definition is that if we believe that the active leaf on Q forces e-divergence, then we assume we have successfully met R_e by forcing divergence of Φ_e^X. Then we will not make any future attempts to restrict which strings are on T_e, and therefore verify all of them.

Condition (2) reflects the fact that each time we meet the e-majority vote criterion, we will attempt to protect the strings used to do so, and to keep them on T_e; thus they should also be e-ρ-verified.

We will only remove e-ρ-verified strings from T_e^s at a later stage if required to do so in order to attend to a requirement acting on an initial segment of ρ.

We are now ready to specify how we choose where to act at each stage of the construction. We will focus on a single pruned clump on which some requirement R_e is working, and which requires attention at stage s. If we identify such a pruned clump, we refer to it as our *target for action* at stage s. We will choose this target from a list of potential *candidates for action*.

We will say that a pair $\langle e, P\rangle$ consisting of a number $e < s$ and clump $P \subseteq T_{e-1}^{s-1}$ is a *candidate for action* at stage s if R_e is working on P at stage s, P requires attention at stage s, and furthermore A permits changes at the root of P at stage s.

A candidate for action $\langle e, P\rangle$ is the *target for action* at stage s if it meets each of the following conditions:

(1) there is no pruned clump $Q \prec P$ such that for some i, $\langle i, Q\rangle$ is a candidate for action at stage s,
(2) there is no requirement $i < e$ which requires attention on a pruned clump $Q \sim P$,
(3) there is no pruned clump $Q \succsim P$ and number $i < e$ such that Q is the first witness to i-divergence at stage s.

Note that the third condition may result in a situation where there is no target for action even though there are candidates for action.

In the next section, we will outline the construction proper.

We will build the trees T_e^s by attempting to find strings which force divergence of Φ_e, and, if that is not possible, will attempt to meet the e-majority vote criterion on the pruned clumps in T_{e-1}^s. If we meet the e-majority vote criterion on a pruned clump $Q \subseteq T_{e-1}^s$, we will want to preserve this at all future stages. However, it may be the case that at a later stage $t > s$ we have a target for action of form $\langle i, P \rangle$, where $P \precsim Q$. At such a stage, if R_i requires attention at P due to halting, then we will be forced to abandon our progress on Q. However, if R_i requires attention at P due to complexity, we will ensure that Q remains a pruned clump on T_e^t. This will assist us in meeting the enumerability criterion required by Lemma 2.3.

4. The Construction

Initialization

At stage 0, we set ξ^0 to be the string consisting of a single 0.

We will now describe how to use the situation at the end of stage $s-1$ of the construction to carry out stage s.

We will define our reduction Γ alongside the construction. The idea will be to ensure that if at stage s, a number k is working on some ρ, we have $\Gamma^A(k)[s] = \lambda$, where λ is a leaf of a pruned clump with root ρ, and λ is an initial segment of ξ^s.

After the construction we will give a clean-up process which assigns unused numbers k to work and makes initial commitments for $\Gamma^A(k)$'s use. This will be the same regardless of what kind of action we take at stage s.

Defining the trees T_e^s and approximation ξ^s

How we proceed at stage s depends on whether there is a target $\langle e, P \rangle$ for action, and, if so, the reason that R_e requires attention at P.

In the case that there is a target $\langle e, P \rangle$ for action, let ρ be the root of P. Then we will ensure ρ is an initial segment of ξ^s, but the leaf of P which is an initial segment of ξ^s may change. For this reason, we will want to redefine Γ to reflect that fact, and to reassign permissions.

If there is no target for action, then for each $\widehat{\rho}$, and each m assigned to work on ρ at stage s, assign m to work on ρ at stage $s+1$, and set $\Gamma^A(m)[s] = \Gamma^A(m)[s-1]$, with the use $\gamma_s(m) = \gamma_{s-1}(m)$.

We are now ready to see the various ways the construction should proceed, depending on the particular form of action required at stage s

Case 1: No target for action.

If there is no target for action, then for each $i < s$, define T_i^s as follows. If P is the $\prec$-least pruned clump on T_{i-1}^{s-1} on which R_i is working, but which does not meet the i-majority vote criterion at stage s, then let μ be the leaf of P which is an initial segment of ξ^{s-1}, and let

$$T_i^s = T_i^{s-1} \cup \{\tau \in T_{i-1}^s \mid \mu \prec \tau\}.$$

If every pruned clump P on T_{i-1}^s on which R_i is working meets the u-majority vote criterion at stage s, let

$$T_i^s = T_i^{s-1} \cup \{\tau \in T_{i-1}^s \mid \xi^{s-1} \prec \tau\}.$$

Define ξ^s to be some leaf λ of T_{s-1}^s such that $\xi^{s-1} \preceq \lambda$.

Case 2a: Target for action due to halting, and an apparently divergent computation is found.

Let $\langle e, P\rangle$ be the target for action. Suppose that R_e requires attention due to halting at P, and that the root of P is ρ. Suppose that there is a leaf λ of P which is not e-extendible at stage s.

Then we choose $\xi^s = \lambda$ (if there are several possible choices, choose the leftmost). For $i < e$, let $T_i^s = T_i^{s-1}$. For $e \leq i < s$, let

$$T_i^s = \{\sigma \in T_i^{s-1} \mid \neg(\rho \prec \sigma)\} \cup \{\sigma \in 2^{<\omega} \mid (\exists \mu \in P)[\sigma \preceq \mu]\}.$$

For each m which is assigned to work on a string $\widehat{\rho} \succeq \rho$ at stage s, assign m to work on ρ at stage $s+1$. Let n be the largest such number. For each m assigned to work on ρ at stage $s+1$, set $\Gamma^A(m)[s] = \lambda$ with use $\gamma_s(m) = \max F_n$.

Case 2b: Target for action due to halting, but every leaf is e-extendible.

Let $\langle e, P\rangle$ be the target for action. Suppose that R_e requires attention due to halting at P, and that the root of P is ρ. Suppose that each leaf λ of P is e-extendible at stage s.

For each $\tau \in 2^{<\omega}$ of length $|\rho| \cdot 2^{-2e-4}$, define $E(\tau)$ to be the set of leaves λ of P which are e-τ-extendible at stage s. From amongst these strings, choose τ for which $|E(\tau)|$ is maximal. Let $D(\tau)$ be a subset of $E(\tau)$ with exactly $2^{|\rho|\left(1-\sum_{j=0}^{e} 2^{-2j-4}\right)}$ leaves[2].

Choose a set $\widehat{D}(\tau)$ of strings on T_{e-1}^{s-1} consisting of one e-τ-extension of each $\lambda \in D(\tau)$. Define ξ^s to be the leftmost member of $\widehat{D}(\tau)$.

Define $T_i^s = T_i^{s-1}$ for $i < e$.

There is some $\prec$-least pruned clump $Q \preceq P$ on T_{e-1}^{s-1} on which R_e is working, and such that Q does not meet the e-majority vote criterion at stage s. Let ρ_0 be the root of Q, and define

$$T_e^s = \{\sigma \in T_{e-1}^{s-1} \mid \neg(\rho_0 \prec \sigma)\} \cup \{\sigma \in 2^{<\omega} \mid \exists \widehat{\lambda} \in \widehat{D}(\tau)[\sigma \preceq \widehat{\lambda}]\}.$$

For $e < i < s$, define

$$T_i^s = \{\sigma \in T_i^{s-1} \mid \neg(\rho_0 \prec \sigma)\} \cup \{\sigma \in 2^{<\omega} \mid \sigma \preceq \xi^s\}.$$

[2]We will later see that $E(\tau)$ has at least this many leaves.

For each m which is assigned to work on a string $\widehat{\rho} \succeq \rho$ at stage s, assign m to work on ρ at stage $s+1$. Let n be the largest such number. Let $\lambda \preceq \xi^s$ be a leaf of P. For each m assigned to work on ρ at stage $s+1$, set $\Gamma^A(m)[s] = \lambda$ with use $\gamma_s(m) = \max F_n$.

Case 3: Target for action due to complexity.

Finally, suppose that $\langle e, P\rangle$ is the target for action, that R_e requires attention due to complexity at P, and that the root of P is ρ.

For $0 \leq i < s$, let $T_i^s = T_i^{s-1}$.

In this case, P meets the e-majority vote criterion. For $e \leq i < s$ let $P_i = T_{i-1}^{s-1} \cap P$. Let D consist of the numbers i for which P_i is a pruned clump on T_{i-1}^{s-1} which meets the i-majority vote criterion. For each $i \in D$ let $\tau_i = \Phi_i^{\xi^{s-1}}[s] \upharpoonright 2^{-2i-4}|\rho|$. Let i_0 be the largest member of D. Let λ be an effectively chosen leaf of P_{i_0} with the property that $K_s(\lambda)$ is maximal amongst all such leaves.

Choose strings $\xi_e^s \succeq \xi_{e+1}^s \succeq \cdots \succeq \xi_{s-1}^s \succeq \lambda$ such that for each i, ξ_i^s is a leaf of T_i^{s-1}. Let $\xi^s = \xi_{s-1}^s$.

For each m which is assigned to work on a string $\widehat{\rho} \succeq \rho$ at stage s, assign m to work on ρ at stage $s+1$. Let n be the largest such number. For each m assigned to work on ρ at stage $s+1$, set $\Gamma^A(m)[s] = \lambda$ with use $\gamma_s(m) = \max F_n$.

In all of the cases 1-3, let T_s^s consist of ξ^s together with all of its initial segments.

If at stage s, m is assigned to work on a string ρ_0, and we did not yet specify how it should be assigned at stage $s+1$, assign it to work on ρ_0 again, and set $\Gamma^A(m)[s] = \Gamma^A(m)[s-1]$, with use $\gamma_s(m) = \gamma_{s-1}(m)$.

Requesting permissions

In each of the above cases, suppose that n is assigned to work on a string $\widehat{\rho}$ at stage $s+1$, some requirement R_e requires attention on a pruned clump with root $\widehat{\rho}$ at stage $s+1$, but there was no such requirement at stage s. Then enumerate a single element of F_n into W, in order to ensure that $W_{s+1} \cap F_n \neq A_s \cap F_n$; if $W_s \cap F_n \neq A_s \cap F_n$ already, then make no such enumeration.

Assigning new permissions

Let $\rho_1 \prec \rho_2 \prec \cdots \prec \rho_k$ be the roots of the clumps on T_{-1}^s of which ξ^s has a leaf as an initial segment, and on which we assigned no number to work at stage s. Let $n_1 < n_2 < \cdots < n_k$ be the least k numbers that were not assigned to work on any string at stage s. For $1 \leq i \leq k$, assign n_i to work on ρ_i at stage $s+1$. For $1 \leq i \leq k$, do as follows: if some requirement R_e requires attention on a pruned clump Q with root ρ_i on T_{e-1}^s at stage $s+1$, enumerate a single element of F_{n_i} into

W, in order to ensure that $W_{s+1} \cap F_{n_i} \neq A_s \cap F_{n_i}$; if $W_s \cap F_{n_i} \neq A_s \cap F_{n_i}$ already, then make no such enumeration.

This concludes the construction.

5. Verification of construction

For each e, let $T_e = \{\sigma \in 2^{<\omega} \mid \sigma \in T_e^s$ at cofinitely many stages $s\}$, and $X = \lim_s \xi^s$.

We will begin our analysis of the construction by establishing that some of its basic features function as intended. We will check that the strings ξ^s come to a limit X, and that the permission process behaves as intended.

Remark 5.1. Let $0 \leq i \leq s$. Then $T_i^s \subseteq T_{i-1}^s$, and for each pruned clump P on T_i^s, there is a pruned clump Q on T_{i-1}^s such that $Q \sim P$.

In addition, $\xi^s \in T_i^s$ for each s and $i \leq s$, so X is a path through T_i.

Each of these facts is easily verified by checking that they are preserved from one stage of the construction to the next.

Lemma 5.2. *For each s and each $i \leq s$, if P is a pruned clump on T_i^s with root ρ, then P has at least $2^{|\rho|\left(1-\sum_{j=0}^{i} 2^{-2j-4}\right)}$ leaves.*

Proof. If $i = -1$, then P has exactly $2^{|\rho|}$ leaves, since in that case $P = \rho 2^{\leq|\rho|}$.

Now, work by induction on i. Suppose that the result is true of every pruned clump Q on T_{i-1}^s for every s. Fix some s, and let P be some pruned clump on T_i^s. Consider the largest $t \leq s$ such that P is on T_i^t but not on T_i^{t-1}.

If the construction proceeds via case *1* at stage t, then there is some string $\mu \in T_{i-1}^{t-1}$ such that $T_i^t = T_i^{t-1} \cup \{\tau \in T_{i-1}^{t-1} \mid \mu \prec \tau\}$. Let $P \sim Q$, where Q is a pruned clump on T_{i-1}^{t-1}. The string μ must be an initial segment of the common root of P and Q, and therefore that every leaf of Q is also a leaf of P. But that implies that P has at least $2^{|\rho|\left(1-\sum_{j=0}^{i-1} 2^{-2j-4}\right)}$ leaves, by induction. This is more than the minimum required.

If the construction proceeds via case *2a* or *3* at stage t, then there are no pruned clumps on T_i^t that were not already on T_i^{t-1}, and there is nothing to prove.

If the construction proceeds via case *2b* at stage t, then it must be the case that $\langle i, Q\rangle$ is the candidate for action at stage t, where Q is the pruned clump on T_{i-1}^{t-1} with $P \sim Q$. In this case, there are at least $2^{|\rho|\left(1-\sum_{j=0}^{i-1} 2^{-2j-4}\right)}$ leaves on Q. But each such leaf λ is e-τ-extendible at stage t for some $\tau \in 2^{<\omega}$ with $|\tau| = 2^{-2i-4}|\rho|$, where the ρ is the root of

P. Since there are $2^{2^{-2i-4}|\rho|}$ many such τ, it follows that there is some particular τ such that at least $\dfrac{2^{|\rho|\left(1-\sum_{j=0}^{i-1} 2^{-2j-4}\right)}}{2^{2^{-2i-4}|\rho|}} = 2^{|\rho|\left(1-\sum_{j=0}^{i} 2^{-2j-4}\right)}$ of the leaves of Q are e-τ-extendible. So the construction builds a pruned clump with exactly this many leaves. Hence P has at least $2^{|\rho|\left(1-\sum_{j=0}^{i} 2^{-2j-4}\right)}$ leaves, as desired. □

Corollary 5.3. *For each s and each $i \leq s$, if P is a pruned clump on T_i^s, then some leaf λ of P has $K(\lambda) \geq |\lambda|/4$.*

Proof. Any prefix-free set of binary strings of length at most $|\lambda|/4$ can have at most $2^{|\lambda|/4}$ members. However,

$$\left(1 - \sum_{j=0}^{i} 2^{-2j-4}\right) = \left(1 - \frac{1}{12}(1 - 4^{-i-1})\right)$$
$$\geq \frac{11}{12}$$

so that $2^{|\rho|\left(1-\sum_{j=0}^{i} 2^{-2j-4}\right)} \geq 2^{11|\rho|/12} > 2^{|\rho|/2} = 2^{|\lambda|/4}$, and therefore P has too many leaves for them to all have such short descriptions. □

Lemma 5.4. *For each e and string ρ, there are only finitely many stages t at which there is a target for action of the form $\langle e, P\rangle$, where ρ is the root of a pruned clump P on T_{e-1}^s.*

In addition, the strings ξ^s approach a limiting real X. That is, for each k, there is some s such that $|\xi^s| \geq k$ and for each $t \geq s$, $\xi^s \restriction k = \xi^t \restriction k$.

Proof. We will prove the first result by induction on the length of ρ and (within that) by induction on e.

Fix a number e and string ρ which is the root of a pruned clump on T_{-1}. Applying the inductive hypothesis, choose t_0 such that for $s \geq t_0$, $\langle i, P\rangle$ is not the target for action at stage s for any P with root $\rho_0 \prec \rho$, nor for any $i < e$ and clump P with root ρ.

Suppose that for some $s_0 \geq t_0$, ξ^{s_0} has an initial segment λ which is a leaf of some pruned clump P on $T_{e-1}^{s_0}$ with root ρ.

Then P is also on T_{e-1}^s for each $s \geq s_0$ because after that stage there will never be a target for action which can cause P to be removed.

Now we check that amongst stages $t \geq s_0$, $\langle e, P\rangle$ can be the target for action at most finitely many times.

For each leaf λ of P there can be at most one stage t at which $\langle e, P\rangle$ is the target for action and at which R_e requires attention due to halting at P, since at such a stage, if λ is the leaf of P for which $\lambda \prec \xi^{t-1}$, we know that λ is e-extendible. But then we either are in case *2a* and define ξ^t in a way which guarantees that it extends a leaf λ_1 of P which is not e-extendible at stage t, or are in case *2b* and have verified that every leaf of P is e-extendible. In the latter case P will meet the majority vote criterion at the next stage, and R_e will never again require attention due to halting at P.

Likewise, $\langle e, P\rangle$ can be the target for action at a stage t where R_e requires attention due to complexity at P only finitely many times. At such a stage t we will note that the leaf λ of P such that $\lambda \preceq \xi^{t-1}$ has $K_t(\lambda) < {}^{|\lambda|}/4$. We will then will define ξ^t to be an extension of a leaf $\widetilde{\lambda}$ of a pruned clump $Q \sim P$ which is on a tree T_i^{t-1} for some $i < s$, and such that $K_t(\widetilde{\lambda})$ is maximal amongst such leaves. It follows that $K_t(\widetilde{\lambda}) \geq {}^{|\widetilde{\lambda}|}/4$, by Corollary 5.3. Once again, $\langle e, P\rangle$ can only be the target at a stage where R_e requires attention due to complexity once for each leaf of P.

Only finitely many requirements ever require attention on the pruned clump P (namely those R_e for which $e \leq |\rho|$). As has been seen, each $\langle e, P\rangle$ is a target for action at finitely many stages. So it follows that eventually $\xi^t \restriction |\rho|$ will remain constant.

We will now check that $\lim_s \xi^s$ exists as a member of 2^ω. Note that if ξ^s has the root of P as an initial segment and $\langle e, P\rangle$ is never a target for action after stage s, then ξ^t will still have that root as an initial segment at any stage $t \geq s$. Thus it suffices to show that for any given k, ξ^s eventually remains at least k in length.

Our proof will be by contradiction. Assume there is some longest string ρ which is the root of a pruned clump on T_{-1} and which is an initial segment of ξ^s at all stages $s \geq t$ of the construction. In addition, choose t large enough that for $s \geq t$, the target $\langle e, P\rangle$ for action will never have the property that P has a root $\rho_0 \preceq \rho$. Thus if $s \geq t$, a target $\langle e, P\rangle$ for action must have the property that the root ρ_1 of P satisfies $\rho \prec \rho_1 \preceq \xi^s$.

If such a target exists at a later stage t_0, then ρ_1 is an initial segment of ξ^{t_0}. Suppose ρ_1 is $\prec$-minimal amongst strings which are roots of pruned clumps P for which there is some stage $t_0 \geq t$ at which $\langle e, P\rangle$ is the target for action. Then ρ_1 will be an initial segment of ξ^{t_0} for all sufficiently large t_0. This contradicts that ρ is the longest such string.

Thus we may assume that there are no stages $s \geq t$ at which there is a target for action. So at each stage $s > t$, and for each $e < s$,

$$T_e^s \supseteq T_e^{s-1} \cup \{\tau \in T_{-1}^s \mid \xi^{s-1} \prec \tau\},$$

and ξ^s is always chosen to be a leaf of T_{s-1}^s which extends ξ^{s-1}. But then ξ^s an initial segment $\rho_1 \succ \rho$ which is the root of a pruned clump on T_{-1}, and ρ_1 is an initial segment of ξ^s at cofinitely many stages s. This gives the desired contradiction.

So $\lim_s \xi^s$ does exist as a member of 2^ω. $\square$

Lemma 5.5. *If m is assigned to work on ρ_1 and n to work on ρ_2 at some stage s, and $m < n$, then $\rho_1 \preceq \rho_2$.*

Proof. If s is the first stage at which we assign n to work on some string ρ_2, then for each $m < n$, m is assigned to work on a proper initial segment of ρ_2 at that stage.

If n is assigned to work on ρ_3 at stage $s-1$ and on ρ_2 at stage s, there is some $i < n$ such that for $i \leq m < n$, we also assign m to work on ρ_2 at stage s, and for $m \leq i$, we assign m to work on the same string $\rho_1 \prec \rho_2$ at stages $s-1$ and s. So the condition of the lemma is preserved from one stage to the next. $\square$

Lemma 5.6. *Let* $f(n) = |F_n| = \displaystyle\sum_{i=0}^{4^n} (i+1)(2^{i+1}+1)$. *For each n, there are at most $f(n)$ stages s at which we enumerate an element of F_n into W.*

Proof. Observe that if we assign n to work on a string ρ at some stage s, then at stage $s+1$, we must assign n to work on a string $\rho_0 \preceq \rho$.

Note that if s is the first stage at which we assign n to work on the root of some pruned clump, that root has length at most 4^n, since it is assigned to work on the shortest root of a pruned clump on T^s_{-1} which has no number already assigned to work on it.

Next, note that if we enumerate an element of F_n into W at stage s, then at that stage, n is assigned to work on the root of a pruned clump P on which a requirement R_e requires attention at stage s, and that furthermore either no requirement R_i required attention on a pruned clump $Q \sim P$ at stage $s-1$, or n was not assigned to work on ρ at stage $s-1$.

Definition 5.7. Suppose that at some stage s, we assign n to work on some string ρ. We will say that the interval $[t_0, t_1)$ is *dedicated to e on ρ* if for $t_0 \leq t < t_1$,

i.) we assign n to work on ρ at stage t, and
ii.) for $i < e$, if Q_i is a pruned clump on T^t_{i-1} with root ρ, then $\langle i, Q_i\rangle$ is not the target for action at stage t.

Note that if ρ is the root of a pruned clump Q on $T^{t_0}_e$ and $[t_0, t_1)$ is dedicated to e on ρ then Q is on T^t_e for $t_0 \leq t \leq t_1$.

Fix some number k, and suppose $|\rho| = k$, and that $[t_0, t_1)$ is dedicated to k on ρ. Recall that R_i can only require attention on a pruned clump with root ρ if $i \leq k$.

Then if $t_0 \leq t < t_1$ and we enumerate an element of F_n into W at stage $t+1$, it must be the case that R_k requires attention on a pruned clump Q with root ρ at stage $t+1$. In that case, Q is on T^t_{k-1} at each stage in $[t_0, t_1)$. We now count the number of stages $t \in [t_0, t_1)$ at which $\langle k, Q\rangle$ can be the target for action. For each leaf λ of Q, there is at most one such stage at which $\xi^t \succeq \lambda$ and the construction proceeds via case *2a*, and at most one such stage at which the construction proceeds via case *3* — as discussed in Lemma 5.4. The target for action may also be $\langle k, Q\rangle$ at one stage at which the construction proceeds via case *2b*. Thus we enumerate an element of F_n into W at at most $2^{k+1} + 1$ stages t such that $t_0 \leq t < t_1$ (since this is one more than double the maximum possible number of leaves on Q).

We now show that for each e, if $[t_0, t_1)$ is dedicated to e on ρ, there are at most $(k-e+1)(2^{k+1}+1)$ stages $t \in [t_0, t_1)$ at which we enumerate an element of F_n into W, by backward induction. The base case $(e = k)$ is given above.

Fix $e \leq k-1$, and assume that whenever $[t_0, t_1)$ is dedicated to $e+1$ on ρ, there are at most $(k-e)(2^{k+1}+1)$ many stages $t \in [t_0 < t < t_1)$ at which we enumerate an element of F_n into W.

Suppose that $[t_0, t_1)$ is dedicated to e on ρ. Let t_2 be the largest number in $[t_0, t_1]$ such that $[t_0, t_2)$ is dedicated to $e+1$ on ρ. There are at most $(k-e)(2^{k+1}+1)$ many stages t such that $t_0 < t < t_2$ and at which we enumerate an element of F_n into W.

If $t_2 < t_1$, then at stage t_2, the target for action is of form $\langle e, Q\rangle$, where Q has root ρ. Thus for $t_2 \leq t < t_1$, only $\langle e, Q\rangle$ can be the target for action at stage t. Applying the reasoning given above in the case $e = k$, we see that there are at most $2^{k+1}+1$ stages $t \in [t_2, t_1)$ at which we enumerate an element of F_n into W. So the total number of stages $t \in [t_0, t_1)$ at which we do so is at most $(k-e)(2^{k+1}+1) + 2^{k+1} + 1 = (k-e+1)(2^{k+1}+1)$, completing the induction.

Now we take account of the fact that n may be assigned to different strings throughout the construction. Of the stages at which we assign n to work on the root of P, there are at most $(k+1)(2^{k+1}+1)$ many at which we enumerate an element of F_n into W. Because we first assign n to work on a string ρ for which $|\rho| \leq 4^n$, and at later stages assign n to work on initial segments of ρ, there are at most $\displaystyle\sum_{i=0}^{4^n}(i+1)(2^{i+1}+1)$

stages at which we enumerate an element of F_n into W; this is of course the bound f that we specified. $\square$

Note that our assignment of each number n eventually settles on some string ρ; we now name that string.

Definition 5.8. If we assign n to work on ρ at all stages $t \geq s$, we will say that n *settles* on ρ by stage s. If n settles on ρ by some stage, then we will simply say that n *settles* on ρ.

We will now check that for each e the requirement R_e is met. To do so we must check that X is a path through each T_e, and that either $\mathrm{Dim}(\Phi_e^X) < 1$ and there is some string $\xi \prec X$ with $|\xi| \geq e$ and $K(\xi) \geq {}^{|\xi|}/_4$, or that Φ_e^X is a nontotal function. In the former case, the required inequality on the effective packing dimension of Φ_e^X will be verified indirectly using Lemma 2.3.

Lemma 5.9. *Suppose that n_0 is a number such that $W \cap F_{n_0} = A \cap F_{n_0}$, that n_0 settles on some string ρ by stage s with $|\rho| \geq e$ and that ρ is the root of a pruned clump P which is on T_{e-1}^s at every stage $t > s$. Suppose also that for each $i < e$ and pruned clump $Q \succsim P$, Q is not the first witness to i-divergence at any stage $t > s$.*

Then one of the following conditions holds:

(a) There is a leaf λ of P and stage t_1 such that for $t > t_1$, ξ^t has λ as an initial segment, and λ is not e-extendible at stage t.
(b) There is a leaf λ of P and stage t_1 such that for $t > t_1$, ξ^t has λ as an initial segment, P meets the e-majority vote criterion at stage $t+1$, and $K(\lambda) \geq {}^{|\lambda|}/_4$.

Proof. We proceed by induction on n_0. Fix n_0 such that $W \cap F_{n_0} = A \cap F_{n_0}$, and assume the result for $n < n_0$.

Suppose n_0 settles on some string ρ by stage s. Note that at stages $t \geq s$, if Q has a root which is a proper initial segment of ρ, then $\langle i, Q\rangle$ cannot be the target for action, since that would cause us to assign n_0 to a different string.

Fix some number e, and let $P \subset T_{e-1}^s$ be a pruned clump with root ρ. Suppose that for $t \geq s$ and $i < e$, R_i does not require attention on any clump $Q \sim P$ at stage t. Then P is a pruned clump on T_{e-1}^t at each stage $t \geq s$, since we have just ruled out all of the possible targets for action which could prevent that. If $t_0 \geq s$ is a stage at which R_e requires attention at P, then at a later stage $t \geq t_0$, $W_{t-1} \cap F_{n_0} = A_t \cap F_{n_0}$. At the first such stage, $A_t \cap F_{n_0} \neq A_{t-1} \cap F_{n_0}$, and either R_e no longer requires attention on P, or $\langle e, P\rangle$ is a target for action.

Suppose that for some $t_1 > s$, ξ^{t_1} has an initial segment which is a leaf λ of P which is not e-extendible at any stage $t \geq t_1$. If so, we may choose t_1 so that if P is the first witness to e-divergence at any stage $t > t_1$, then P is the first witness to e-divergence at stage t_1.

If so, P is the first witness to e-divergence at every stage $t \geq t_1$. In that case, if $t \geq t_1$ and Q has a root $\rho_0 \preceq \rho$, $\langle i, Q\rangle$ cannot be the target for action at stage t. Thus $\xi^t \succeq \lambda$ for all $t \geq t_1$.

If P is not the first witness to e-divergence at stage t_1, then no leaf of P is e-extendible at any stage $t > t_1$, and there is some leaf λ_0 of P and $t_2 > t_1$ such that for $t > t_2$, $\xi^t \preceq \lambda_0$.

Thus in this case the first of the two conditions is satisfied.

Otherwise there is some stage t_0 at which every leaf λ of P is e-extendible. Because $\lim_t \xi^t$ exists there is some $t_1 > t_0$ such that for $t \geq t_1$, ξ^t has some fixed leaf λ of P as an initial segment. But $W \cap F_{n_0} = A \cap F_{n_0}$, so if t_1 is large enough, R_e does not require attention at P at any stage $t \geq t_1$. This implies that at each stage $t \geq t_1$, P meets the e-majority vote criterion and that $K_t(\lambda) \geq {}^{|\lambda|}/_4$. □

Definition 5.10. If P is a pruned clump on T_{e-1} such that there is a leaf λ of P and stage t_1 such that for $t > t_1$, ξ^t has λ as an initial segment, and λ is not e-extendible at stage t, then we will say that λ *forces e-divergence of X*.

Note that in the preceding definition and lemma, P forcing e-divergence merely guarantees that we never find any $(e-1)$-ρ-verified extensions of λ which threaten to make Φ_e^X total. We will later see that our terminology is appropriate: if λ forces e-divergence, then Φ_e^X really is nontotal.

Lemma 5.11. *For each $n \in \omega$ let ρ_n be the string on which n settles. For each e, there are finitely many numbers n such that $W \cap F_n = A \cap F_n$ and ρ_n is not the root of a pruned clump on T_e.*

The finitely many exceptions to this assertion are numbers amongst those for which either $|\rho_n| < e$ or when there is some $i \leq e$ such that ρ_n is an initial segment of the root of a pruned clump on T_{i-1}^{s-1} which is the first witness to i-divergence at stage s for all sufficiently large s.

Proof. First, fix some number e. There are finitely many numbers n for which $|\rho_n| < e$. Likewise, for each $i \leq e$, there is at most one string ρ which is the root of a pruned clump on T_{i-1}^{s-1} that is the first witness to i-divergence at stage s for all sufficiently large s, and hence only finitely many n for which $\rho \succeq \rho_n$. So the list of purported potential problems is indeed finite.

Now, fix some n such that $W \cap F_n = A \cap F_n$. Fix some e and assume the result of the lemma for each $i < e$. We will show that it holds of e, too.

Assume that ρ_n does not satisfy either exceptional condition. Note that if either of the exceptional conditions discussed above holds of ρ_n and e, the same condition also applies to ρ_n and i, for each $i < e$.

There are two possible scenarios.

The first is as follows: n settles on ρ_n by some stage t, and ρ_n has a proper initial segment which is the root of a pruned clump P on T_{e-1}^{s-1} such that for $s \geq t$, P is the first witness to e-divergence at stage s. Assume t is large enough that for $s \geq t$, there is no target for action of the form $\langle j, Q\rangle$, where the root of Q is an initial segment of ρ_n. At stages $s > t$ at which there is no target for action, if μ is the leaf of P which is an initial segment of ξ^{s-1}, we have $T_e^s = T_e^{s-1} \cup \{\tau \in T_{e-1}^s \mid \mu \prec \tau\}$. By our inductive hypothesis, ρ_n is the root of a pruned clump on T_{e-1}^s for all sufficiently large s. Because $\mu \prec \rho_n$, it follows that ρ_n is also the root of a pruned clump on T_e^s. But then ρ_n is the root of a pruned clump on $T_e^{s_1}$ at all stages $s_1 \geq s$.

The second scenario is that ρ_n is the root of a pruned clump P on T_{e-1} with the property that there is a stage s at which R_e requires attention at P due to halting, and at which the target for action is $\langle e, P\rangle$. This stage may be assumed to be the last stage at which the target for action is of the form $\langle j, Q\rangle$, where $j \leq e$ and $\rho \preceq \rho_n$. In this case we add a pruned clump $Q \sim P$ to T_e^s, and never remove it again. $\square$

Lemma 5.12. *Fix e and t_1. Let $\rho \in T_e^{t_1-1}$ be a root of a pruned clump P on $T_{e-1}^{t_1-1}$ such that:*

1. *For each $i < e$ such that a leaf of some pruned clump P_i on T_{i-1} forces i-divergence, ρ is an extension of the root of the $\prec$-least such P_i,*
2. *Some number n settles on ρ by stage t_1, and furthermore for each $i < e$, there is no stage $t > t_1$ at which there is a target for action of the form $\langle i, Q\rangle$, where Q is a pruned clump on T_{i-1}^{t-1} such that $Q \sim P$, except in case R_i requires attention due to complexity at Q,*
3. *P meets the e-majority vote condition at stage t_1.*

Then P is on T_e^t and meets the e-majority vote criterion at each stage $t > t_1$.

Proof. The only targets for action which might cause P to not be on T_{e-1}^t for some first stage $t > t_1$ are those of the form $\langle i, Q\rangle$ where $Q \prec P$ or $\langle i, Q\rangle$ where $i \leq e$ and $Q \sim P$. In the case where R_i requires attention due to complexity no pruned clump will be removed. But our assumption rules out any other target for action. $\square$

We now introduce a new kind of verification, which is called e-permanence.

Definition 5.13. Let $e \geq 0$. Say that $\sigma \in T_e^{s-1}$ is *e-permanent at stage s* if for $0 \leq i < e$, σ is i-permanent at stage s and either:

(1) There is a pruned clump Q with root ρ and a leaf λ such that for each $t \geq s$, Q is on T_{e-1}^t, $\lambda \preceq \xi^t$, and the active leaf on Q appears to force e-divergence at stage t, and furthermore that either $\sigma \preceq \lambda$ or $\lambda \preceq \sigma$, or
(2) Each pruned clump P on T_{e-1}^s with root $\rho \prec \sigma$ for which $P \cap T_e^{s-1}$ is a pruned clump meets the e-majority vote criterion at stage s.

Lemma 5.14. *For each e, there is some stage t_1 such that for each $t > t_1$, any $\sigma \in T_e^t$ which is e-permanent at stage t is also in T_e^{t+1} and is e-permanent at stage $t+1$.*

Proof. Fix some e, and assume the result for all $i < e$.

Let $\rho \prec X$ be the root of some pruned clump on T_{e-1} such that for each $i \leq e$ for which a leaf of some pruned clump P forces i-divergence, one such P has a root which is a proper initial segment of ρ.

Let t_0 be large enough that

(1) t_0 meets the condition given by the lemma for each $i < e$
(2) some number n settles on ρ by stage t_1

Now suppose that σ is an e-permanent string on T_e^t at some stage $t \geq t_1$.

At stages $s > t$, if $i < e$ and Q is a pruned clump on T_{i-1}^{s-1} with a root which is a proper initial segment of σ, $\langle i, Q\rangle$ can only be the target if R_i requires attention at Q due to complexity (otherwise Q could not have met the i-majority vote criterion, and hence σ was not i-permanent at stage $s-1$).

If a leaf λ of some pruned clump P on T_{e-1} forces e-divergence, then for $s \geq t_1$, ξ^s has λ as an initial segment and λ appears to force e-divergence at stage t. So the strings which are e-permanent at a stage $s \geq t_1$ are precisely the $(e-1)$-permanent strings σ on T_e^s such that $\sigma \preceq \lambda$ or $\lambda \preceq \sigma$. No such string can be removed from T_e^s at a stage s at which there is no target for action, nor at a stage at which there is a target for action because some requirement R_i requires attention due to complexity. Thus σ remains e-permanent at all stages $s \geq t$.

Otherwise, there is no leaf of any pruned clump on T_e which forces e-divergence. Then every pruned clump P on T_{e-1}^{t-1} for which $P \cap T_e^{t-1}$ is a pruned clump with root ρ meets the e-majority vote criterion at stage t. At a stage s at which there is no target for action or at which there is a target for action chosen because some requirement requires attention due to complexity, a pruned clump on T_{e-1}^s cannot cease to meet the e-majority vote criterion. Once again, σ will remain e-permanent at all stages $s \geq t$. □

Lemma 5.15. *Suppose that a leaf of some pruned clump P on T_{e-1} forces e-divergence of X. Then Φ_e^X is nontotal.*

Proof. Let P have a leaf which forces e-divergence of X. Assume that P has root ρ such that $|\rho| \geq e$, and that for each $i < e$ such that there is some $\prec$-least pruned clump Q on T_{i-1} with a leaf which forces i-divergence of X, $Q \prec P$ (choose P to be a clump with a longer root, if necessary). Suppose that Φ_e^X is total. Choose some stage t_0 such that for $t > t_0$, P is on T_{e-1}^t, such that some n_0 settles on ρ by stage t_1, and such that there is a leaf λ of P such that $\xi^t \succeq \lambda$ for all $t > t_0$.

Let $\sigma \succ \lambda$ be an initial segment of X such that $\sigma \in T_{e-1}^t$ for all $t > t_0$. Choose $t_1 > t_0$ such that if $\rho_0 \preceq \sigma$ is the root of any pruned clump P_0, the target for action cannot be $\langle i, P_0\rangle$ at any stage $t > t_1$. For each $i < e$ such that there is some t_2 such that for each $t \geq t_2$, σ is i-verified at stage t, assume that $t_1 \geq t_2$.

If σ is not $(e-1)$-ρ-verified at every stage $t > t_1$, then there is some least $i < e$ such that σ is not i-ρ-verified at every stage $t > t_1$. We will show that this is impossible, by showing that σ is i-ρ-verified at all sufficiently large stages t. Thus it follows that σ is eventually $(e-1)$-ρ-verified, and since σ was an arbitrary extension of λ, P cannot have a leaf which forces e-divergence of X.

By our assumption on P, no leaf of the pruned clump $Q \sim P$ on T_{i-1} appears to force i-divergence at any stage $t \geq t_1$.

Find the $\prec$-least initial segment ρ_1 of X such that some number n settles on ρ_1 by a stage $t_2 > t_1$, and that t_2 is the largest stage at which there is a target for action of form $\langle i, P_1\rangle$, where ρ_1 is the root of a pruned clump P_1 on $T_{i-1}^{t_2-1}$, R_i requires attention due to halting at P_1, and the construction proceeds via case *2b*.

Note that for $j < i$ and $P_2 \sim P_1$ there is never a target for action of the form $\langle j, P_2\rangle$ at any stage $t > t_2$ except if R_j requires attention due to complexity (or t_2 would not be the last stage at which there is a target of form $\langle i, P_1\rangle$ as specified above).

At stage t_2, consider the $\prec$-least pruned clump $Q \preceq P_1$ on $T_{i-1}^{t_2-1}$ on which ξ^{t_2-1} is working, and such that Q does not meet the i-majority vote criterion at stage t_2.

If $Q = P_1$ then σ is i-ρ-verified at stage t_2, by definition.

Otherwise $Q \prec P_1$ and $\rho_0 \prec \sigma$. But in that case the definition of $T_i^{t_2}$ ensures that every pruned clump Q such that $P \precsim Q \precsim P$ meets the i-majority vote criterion at stage t_2+1, and so σ is i-ρ-verified at stage t_2+1.

By our choice of t_2 and Lemma 5.12, there is no stage $t>t_2$ at which any target for action could cause σ to cease being i-ρ-verified.

This contradicts the minimality of i, as promised. $\square$

Remark 5.16. For each e and s, the e-permanent strings on T_e^s are downward closed, and therefore form a tree.

Lemma 5.17. *Suppose that e is a number such that Φ_e^X is total. For each s, let $\widehat{T_e^s}$ consist of the strings on T_e^s which are e-permanent at stage s, and t_1 be a number satisfying the condition of Lemma 5.14.*

Then $\bigcup_{s\geq t_1}\widehat{T}_e^{\,s}$ satisfies the conditions of Lemma 2.3.

Proof. Every pruned clump P on T_{e-1}^s such that $P\cap T_e^s$ is a pruned clump on $\widehat{T_e^s}$ meets the e-majority vote criterion at stage $s+1$, because its leaves are e-permanent. This shows that $\widehat{T_e^s}$ meets conditions *(2)* and *(3)* of Lemma 2.3.

By Lemma 5.14, we have $\widehat{T_e^s}\subseteq\widehat{T_e^{s+1}}$ for each $s\geq t_1$. Determining which of the leaves of T_e^s is e-permanent is a computable procedure, and so $\bigcup_{s\geq t_1}\widehat{T_e^s}$ is a c.e. tree.

Finally, suppose ρ is the root of a pruned clump on $\widehat{T_e^{s+1}}$ for some $s\geq t_1$. Then each string in that pruned clump must be $(e-1)$-verified at stage s, because otherwise there is no way that ρ can be the root of a pruned clump on T_e^{s+1} which meets the e-majority vote criterion. Suppose that $\rho\in\widehat{T_e^s}$. If there is some $\tau\succ\rho$ in $\widehat{T_e^s}$, then we may deduce that ρ is the root of a pruned clump P on T_{e-1}^s which meets the e-majority vote criterion. Hence $P\cap T_e^s$ is a pruned clump, and furthermore every string in $P\cap T_e^s$ is e-permanent. Thus ρ is the root of a pruned clump on $\widehat{T_e^s}$. If no such τ exists, then ρ is a leaf of $\widehat{T_e^s}$.

Thus $\bigcup_{s\geq t_1}\widehat{T_e^s}$ satisfies the conditions of Lemma 2.3, as desired. $\square$

Lemma 5.18. $\mathrm{Dim}(X)\geq\frac{1}{4}$.

Proof. If Φ_e is total, then for infinitely many pruned clumps P on T_e with root $\rho\prec X$, condition *(b)* of Lemma 5.9 must be met. Thus there is a leaf λ of P such that $\lambda\prec X$ and $K(\lambda)\geq{}^{|\lambda|}/_4$. Therefore $\mathrm{Dim}(X)\geq{}^1/_4$. $\square$

Lemma 5.19. *$X\leq_T A$.*

Proof. We will first check that Γ really is a Turing functional.

To do this, it suffices to check that there are no strings $\sigma\prec\tau$ such that for some n, $\Gamma^\sigma(n)\downarrow\neq\Gamma^\tau(n)\downarrow$.

To this end, suppose that at some stage s, we set $\Gamma^{A_s}(m) = \lambda$ for some string λ. Then the use of that computation is $\gamma_s(n) = \max F_n$, where n is the largest number assigned to work on the root ρ of the pruned clump P on T_{-1} of which λ is a leaf.

We must check that we will not later define $\Gamma^{A_t}(m)$ to be a different string, unless $A_t \restriction \gamma_s(m) \neq A_s \restriction \gamma_s(m)$.

The next stage t at which we define $\Gamma^{A_t}(m)$ may be one at which we have a target for action of the form $\langle e, Q\rangle$, where $Q \precsim P$. If so, then at that stage t, let k be the largest number assigned to the root of Q. Then we must have $A_t \restriction \max F_k \neq A_{t-1} \restriction \max F_k$, since we are permitted to act. Since $Q \precsim P$, we have $k \leq m$ and hence $A_t \restriction \gamma_s(m) \neq A_{t-1} \restriction \gamma_s(m)$. Because A is c.e. it follows that $A_t \restriction \gamma_s(m) \neq A_s \restriction \gamma_s(m)$, as required.

Otherwise the next stage t at which we define $\Gamma^{A_t}(m)$ is one at which $A_t \restriction \gamma_s(m) \neq A_{t-1} \restriction \gamma_s(m)$, and at which we define $\Gamma^{A_t}(m) = \lambda$, with use $\gamma_t(m) = \gamma_s(m)$.

We now note that Γ does not explicitly compute X from A. Nonetheless, we can readily modify Γ to do so. It is enough to show that if $\alpha \prec A$ and $\Gamma^\alpha(m)\downarrow = \lambda$, then $\lambda \prec X$, and that given k, there is some n and sufficiently long $\alpha \prec A$ for which $\Gamma^\alpha(m)\downarrow = \lambda$ for a string λ of length greater than k.

At each stage s of the construction, $\Gamma^{A_s}(m)$ (if defined) is an initial segment of ξ^s.

Suppose that n settles on a string ρ by stage t, and that t is the last stage at which the target for action is of the form $\langle e, P\rangle$, where P has root ρ. Then at that stage we set $\Gamma^{A_t}(m)$ to be the leaf λ of P which ξ^t has as an initial segment, with use $\gamma_t(m) = \max F_n$, where n is the largest number assigned to ρ. At any future stage $t_0 > t$ at which $A_{t_0} \restriction \gamma_t(m) \neq A_{t_0-1} \restriction \gamma_t(m)$, we still set $\Gamma^{A_{t_0}}(m) = \lambda$, with use $\gamma_{t_0}(m) = \gamma_{t_0-1}(m) = \gamma_t(m)$ and at that stage we still have $\lambda \preceq \xi^{t_0}$. Thus we have $\Gamma^A(m) = \lambda$, and λ is indeed an initial segment of X.

Now, to compute a desired initial segment of X, simply search through all computations of the form $\Gamma^A(n)$ — any string output by this process is an initial segment of X, and sufficiently large n will output an initial segment greater than any desired length. $\square$

Combining the results of Lemmas 5.17, 5.18, and 5.19, we see that our real X satisfies the requirements of the main result given by Theorem 1.3, and thus suffices to prove both that result and Corollary 1.4, our characterization.

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