AVOIDING EFFECTIVE PACKING DIMENSION 1 BELOW ARRAY NONCOMPUTABLE C.E. DEGREES.

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Abstract. Recent work of Conidis [Con12] shows that there is a Turing degree with nonzero effective packing dimension, but does not contain any real of effective packing dimension 1.

This paper shows the existence of such a degree below every c.e. array noncomputable degree, and hence that they occur below precisely those of the c.e. degrees which are array noncomputable.

1. Introduction

Packing dimension was independently introduced by Tricot [Tri82] and Sullivan [Sul84] as a counterpart to the previously established notion of Hausdorff dimension. Both notions allow one to assign a (possibly noninteger) dimension to subsets of any metric space. Hausdorff dimension is defined in terms of outer measures, whereas packing dimension is based on inner measures.

Effective versions of both notions have been developed by Lutz, Staiger, Athreya et al. ([Lut03], [AHLM07], [Sta93]). For our purposes, the characterizations of Mayordomo [May02] and Lutz [Lut05] of, respectively, effective Hausdorff and packing dimension below can be taken as definitions.

Definition 1.1. Let A be a real (i.e. member of Cantor Space), then the effective Hausdorff dimension of A is

$$\dim(A) = \liminf_{n\to\infty} \frac{K(A \upharpoonright n)}{n},$$

and the effective packing dimension of A is

$$\mathrm{Dim}(A) = \limsup_{n\to\infty} \frac{K(A \upharpoonright n)}{n}.$$

The reader should note that we are ascribing a notion of dimension to a single real, in the same way that we can use computability theory to give meaning to randomness to a single real.

These effective notions of dimension have strong links to complexity and algorithmic randomness. Moreover, work of Simpson [Sim15] and Day [Dayep], for example, have shown that effective notions of dimesnion can be used to derive classical results in mathematics. In discussions with co-workers, Simpson proved that the classical dimension equals the entropy (generalizing a difficult result of Furstenburg 1967) using effective methods, which were much simpler. Recently Day used effective packing dimension to give a simple proof of the Kolmogorov-Sinai Theorem on Ergodic theory.

In many ways, effective packing dimension is quite well behaved on degrees. For example, we know that a Turing degree will obey a 0-1 Law for effective packing dimension. That is complexity extraction procedures given independently by Bienvenu et al., and Fortnow et al. ([BDS09] and [FHP$^+$06], respectively) show that for any real X, $\sup\{\mathrm{Dim}(Y) \mid Y \leq_T X\}$ is either 0 or 1. The extraction processes both yielded only that the supremum of the packing dimensions of the reals in the degree was 1, and hence authors wondered if the supremum of 1 was always achieved. Work of Conidis [Con12] shows that there are reals X for which the supremum is 1, but for which that supremum is not attained[1].

Conidis' construction was a direct forcing argument and resulted in a hyperimmune-free degree. The second author [Ste15] showed that the construction given by Conidis, which utilizes forcing with computable trees, can be modified to work below $\emptyset'$. This version may be interpreted as a limit-computable construction with permissions provided by $\emptyset'$. In light of this observation one might ask below which c.e. sets A the construction can be carried out; the obvious restriction is that A must provide appropriate permissions.

The array noncomputable degrees are a class introduced by Downey, Jockusch and Stob in [DJS96]. They are noted for their compatibility with constructions requiring multiple permissions (which we will see naturally arise in an approximation-based version of Conidis' construction). They have also been shown to form a natural cutoff in the Turing degrees for constructions involving reals with nonzero effective packing dimension (see for instance [DG08], [DN10], [DH10]). In our case, a result of Kummer [Kum96] is most relevant:

Theorem 1.2 (Kummer)**.** *If A is an array computable c.e. real, any real $X \leq_T A$ has* $\mathrm{Dim}(X) = 0$.

Moreover, Downey and Greenberg [DG08] proved the 0-1 Law dichotomy held for array noncomputable degrees. If **a** is array noncomputable c.e. degree, then **a** has effective packing dimension 1.

[1] Any Martin-Löf random real X has $\dim(X) = 1$, and the computable reals all have $\mathrm{Dim}(X) = 0$, so an unattained supremum is the only difficult case to achieve.

These results show that the only c.e. reals which can possibly provide the necessary permissions for a construction à la Conidis are the array noncomputable ones. In this paper, we show that every array noncomputable c.e. degree computes a real X with the desired properties:

Theorem 1.3. *Given any array non-computable c.e. real A, there is a real $X \leq_T A$ such that $\mathrm{Dim}(X) > 0$ and such that for each $Y \leq_T X$, $\mathrm{Dim}(Y) < 1$.*

In light of Kummer's result, this gives a full characterisation of the situation which follows the general pattern observed above:

Corollary 1.4. *A c.e. real A is array noncomputable if and only if there is a real $X \leq_T A$ such that $\mathrm{Dim}(X) > 0$ and for each $Y \leq_T X$, $\mathrm{Dim}(Y) < 1$.*

We remark that the array noncomputable degrees again show up as quite a ubiquitous class. Kummer's other result was that a c.e. degree contains a c.e. set A where the plain complexity $C(A \upharpoonright n) =^+ 2 \log n$ for infinitely many n iff the degree was array noncomputable. There are other characterizations of this class. It is not yet understood how these combinatorial arguments all inter-relate.

We remark that the proof here is not a simple modification of the earlier work of the second author, but requires a reasonably delicate argument of some combinatorial complexity.

Before embarking on our construction, we should pause to note that effective Hausdorff dimension and effective packing dimension behave in quite distinct ways. There is no analogous computable extraction procedure which produces sets with higher effective Hausdorff dimension than a given input. Indeed a result of Miller confirms this fact directly:

Theorem 1.5 (Miller [Mil11])**.** *There is a real number X with effective Hausdorff dimension $\frac{1}{2}$ but which cannot compute any real of higher effective Hausdorff dimension.*

The classification of reals with such fractional Hausdorff dimension is still open.

2. Strategy

Throughout this paper, we will assume all Turing functionals are partial maps on $2^{<\omega}$, and denote them by upper-case Greek letters. We will let $\{\Phi_e\}_{e\in\omega}$ be a computable list of all such functionals. Other notation will be standard, and follows the conventions of Soare [Soa87]. We fix a single c.e. real $A = \lim_s A_s$ which is array noncomputable. The remainder of the paper is devoted to constructing a real $X \leq_T A$ which satisfies the requirements of Theorem 1.3.

The simplest characterisation of effective packing dimension is in terms of Kolmogorov complexity. If $\lambda \in 2^{<\omega}$, then we will denote the prefix-free Kolmogorov complexity of λ by $K(\lambda)$. As is conventional we fix a computable decreasing approximation K_s with limit K.

By creating a real X with nonzero effective packing dimension, we will automatically guarantee that for each $\varepsilon > 0$, there is some $Y \leq_T X$ such that $\mathrm{Dim}(Y) > 1 - \varepsilon$. The difficulty which arises in our construction is thus that we must prevent each such Y from having $\mathrm{Dim}(Y) = 1$.

In order to achieve the delicate level of control on complexity implicit in that requirement, we will work with pruned clumpy trees. Clumpy trees were introducted as a forcing notion by Downey and Greenberg [DG08], and will soon be defined.

Definition 2.1. For each n, we write $2^{=n}$ to mean the binary strings with length equal to n, and $2^{\leq n}$ to mean those with length less than or equal to n, respectively. If $\rho \in 2^{<\omega}, P \subseteq 2^{<\omega}$ then ρP is the strings formed by concatenating ρ with members of P. If $\sigma \in 2^{<\omega}, \tau \in 2^{<\omega} \cup 2^{\omega}$ write $\sigma \prec \tau$ to mean that σ is a proper initial segment of τ. $P \subset 2^{<\omega}$ then the $\prec$-maximal elements of P are called *leaves*.

A *pruned clump* is a downward closed subset of a set of the form $\rho 2^{\leq |\rho|}$, and which contains at least two leaves of $\rho 2^{\leq |\rho|}$. We will refer to ρ as the *root* of such a pruned clump.

If T is a tree we will say that a pruned clump D is on T if $\rho 2^{\leq |\rho|} \cap T = D$. We say that a tree $T \subseteq 2^{<\omega}$ is a *pruned clumpy tree* if every string τ on T which is an initial segment of a path through T has an extension ρ which is the root of some pruned clump on T.

Definition 2.2. Let T_{-1} be the tree formed by taking the union of the following finite trees T^s_{-1}: T^{-1}_{-1} consists of the empty string together with the string consisting of a single 0. Let T^s_{-1} be given by

$$\bigcup_{\substack{\lambda \in T^{s-1}_{-1} \\ \lambda \text{ a leaf of } T^{s-1}_{-1}}} \lambda 2^{=|\lambda|} 0^{2|\lambda|}.$$

Definition 2.3. If Q and P are pruned clumps, we write $P \prec Q$ if the root of P is a proper initial segment of the root of Q, and $P \sim Q$ if P and Q have the same root. We will write $P \precsim Q$ if $P \prec Q$ or $P \sim Q$.

We will build a sequence $\{T_e\}_{e \in \omega}$ of pruned clumpy trees such that $T_{e+1} \subseteq T_e$ for each e. The real X which satisfies the hypotheses of theorem 1.3 will be the unique common path through all of the trees.

By ensuring that there are sufficiently many leaves on every pruned clump of a tree T, we can guarantee that one such leaf has complexity high enough to help us build a real with nonzero dimension (in particular, each pruned clump built in our construction will have a leaf λ with $K(\lambda) \geq {}^{|\lambda|}/_{4}$).

We will also make use of the fact that if X is a path through a pruned clumpy tree T, and Φ_e behaves appropriately on the leaves of the pruned clumps on T, then $\mathrm{Dim}(\Phi_e^X)$ is able to be bounded away from 1. The following lemma gives the precise conditions required. It is a variation on a result given in [Ste15] (the proof is essentially unchanged), and is inspired by a similar computation given by Conidis in [Con12].

Lemma 2.4. *Let $e \in \omega$, and let $T \subseteq T_{-1}$ be a c.e. pruned clumpy tree given by a computable approximation $T^1 \subseteq T^2 \subseteq \cdots$ such that:*

(1) For each s and each $\rho \in T^s$, if ρ is the root of a pruned clump on T^{s+1}, it is either the root of a pruned clump on T^s or a leaf of T^s

(2) If $\rho_0 \prec \rho$ are roots of pruned clumps on T, then $|\rho| \geq 4{\cdot}2^{2e+4}|\rho_0|$

(3) For each pruned clump P on T with root ρ, there is a string $\tau \in 2^{<\omega}$ with $|\tau| = 2^{-2e-4}|\rho|$ and such that:

(a) for each leaf λ of P, and each $\widehat{\lambda} \in T$ such that $\lambda \preceq \widehat{\lambda}$, if $x < |\tau|$ and $\Phi_e^{\widehat{\lambda}}(x)\downarrow$, then $\Phi_e^{\widehat{\lambda}}(x) = \tau(x)$, and

(b) for each leaf λ of P, there is some $\widehat{\lambda} \in T$ such that $\lambda \preceq \widehat{\lambda}$ and for each $x < 2^{-2e-4}|\rho|$, $\Phi_e^{\widehat{\lambda}}(x)\downarrow$.

If X is a path through T and Φ_e^X is total, then $\mathrm{Dim}(\Phi_e^X) < \alpha_e$ for some fixed $\alpha_e < 1$.

Achieving the agreement between computations specified in condition *(3)* of the lemma is the most prominent feature of our construction.

3. Overview and terminology

We will be working on requirements for each $e \in \omega$, as follows:

R_e : either Φ_e^X is nontotal, or
$\mathrm{Dim}(\Phi_e^X) < 1$, and for infinitely many $\xi \prec X$, $K(\xi) \geq {}^{|\xi|}/_{4}$.

Remark 3.1. If Φ_e is a total reduction, then to meet R_e we must meet the second of the conditions. Because such reductions exist, satisfying R_e for every e will ensure that $\mathrm{Dim}(X) \geq \frac{1}{4}$.

For each e, we will guarantee that X satisfies the requirement R_e, either by ensuring that Φ_e^X is not total, or, if that is not possible, by attempting to make T_e satisfy the condition of Lemma 2.4. Because we will build X as a limit of a computable approximation, we will be unable to tell which of the two strategies succeeds for each e.

In addition, the approximate nature of the construction means that our attempt to build a tree T_e meeting the conditions of Lemma 2.4 is not immediately successful — to satisfy the lemma we make a minor modification to T_e after the construction.

At every stage s, we will let T^s_{-1} be as in Definition 2.2. At the start of stage s, we will be given trees T^{s-1}_e for each $e < s$ and a string ξ^{s-1} which is our current guess at an initial segment of X. We will then construct a tree T^s_e for each $e \leq s$, and define ξ^s to be some string in T^s_s. The trees we build will be nested in the sense that $T^s_{e-1} \subseteq T^s_e$ at every stage of the construction, but it will not always be the case that $T^{s-1}_e \subseteq T^s_e$.

At each stage $s \geq 0$ we will choose at most one requirement R_e to attend to, and attend to it on a single pruned clump. We will do so by referring to functions g and h which determine when we have permission to change our string ξ^s. The function g will be built as we carry out the construction. We will define it by giving an approximation g_s at each stage s of the construction.

After the construction, we will give a computable bound $f(n)$ on the number of times $g_s(n) \neq g_{s+1}(n)$, which shows that $g \leq_{wtt} \emptyset'$. Because A is array noncomputable we may fix in advance an A-computable function h with the property that for infinitely many n, $h(n) > g(n)$. We fix a Turing functional Θ such that $\Theta^A = h$, and let $\Theta^{A_s}(n) = h_s(n)$ for each s. If necessary, we speed up our approximation to A in order to ensure that $h_s(n)$ is defined for every n and s. We assume that for each n and s, $0 < h_s(n) \leq h_{s+1}(n)$.

Definition 3.2. If $0 \leq e \leq s$ and $P \subseteq T^{s-1}_{e-1}$ is a pruned clump such that some leaf of P is an initial segment of ξ^{s-1}, we will say that e *is working on* P at stage s.

Notice that if $i < j$ then there will be be pruned clumps $P \subset T^{s-1}_i$, $Q \subset T^{s-1}_j$ such that $P \sim Q$. It will sometimes be convenient to ignore the distinction between such clumps, and we can do so by referring to the root of a pruned clump rather than to the clump itself.

If e is working on a pruned clump P at stage s, we will say that the function g *assigns* one or more particular numbers to work on the root ρ of P at stage s.

Definition 3.3. If g assigns n to work on a string ρ at stage $s-1$, and $g_{s-1}(n) < h_s(n)$, then we will say that g *permits changes at* ρ at stage s.

During our construction, we will sometimes want to fix some string ρ which is the root of a pruned clump P on T_e^s, and assume that we will never again act to satisfy a requirement R_i for $i < e$ in a way which affects P. Having made this assumption, we may fix some string $\sigma \in T_e^{s-1}$ such that $\rho \prec \sigma$, and ask whether is also a member of T_e^t at all stages $t \geq s$. We will refer to σ as *e-ρ-verified* in cases where we have reason to believe so. This concept will be defined by recursion on e, and will implicitly depend on the stage of the construction.

We will first define (-1)-ρ-verification, and defer the case $e \geq 0$ until after outlining the main concepts used in the construction.

Definition 3.4. At any stage s of the construction and for any root ρ of any pruned clump on T_{-1}^{s-1}, every string $\sigma \succ \rho$ on T_{-1}^{s-1} is -1-ρ-verified.

In what follows, many of the definitions given depend on a stage s. Typically that stage will be clear throughout the construction and its verification, but we include it here to avoid ambiguity.

The next definitions are key in satisfying Lemma 2.4.

Definition 3.5. Suppose that P is a clump on T_{e-1}^{s-1} with root ρ, and $\tau \in 2^{<\omega}$. Let λ be a leaf of P.

We will say that λ is *e-τ-extendible at stage s* if there is an $(e-1)$-ρ-verified extension $\widehat{\lambda} \in T_{e-1}^{s-1}$ of λ with the property that $\Phi_e^{\widehat{\lambda}}[s] \upharpoonright |\tau| = \tau$, and such that $\widehat{\lambda}$ is the root of a pruned clump on T_{e-1}^{s-1} and $|\widehat{\lambda}| \geq 4 \cdot 2^{2e+4}|\rho|$. In this case we will say that $\widehat{\lambda}$ is an *e-τ-extension of λ at stage s*.

We will say that λ is *e-τ-extended at stage s* if there is an $(e-1)$-ρ-verified e-τ-extension $\widehat{\lambda}$ of λ on T_e^{s-1}, and furthermore that for any $\sigma \in T_e^{s-1}$ such that $\lambda \prec \sigma$, either $\widehat{\lambda} \prec \sigma$ or $\sigma \preceq \widehat{\lambda}$.

We will say that λ is *e-extendible at stage s* if λ is e-τ-extendible for some $\tau \in 2^{<\omega}$ of length $|\rho|2^{-2e-4}$ at stage s.

Definition 3.6. Suppose P is a pruned clump on T_{e-1}^{s-1} with root ρ.

We will say that P *meets the e-majority vote criterion at stage s* if $T_e^{s-1} \cap P$ is a pruned clump, and there is some string $\tau \in 2^{<\omega}$ of length $2^{-2e-4}|\rho|$ such that each leaf of $T_e^{s-1} \cap P$ is e-τ-extended at stage s.

We now introduce the conditions which tell us when a requirement R_e requires attention at a particular point in the tree T_{e-1}^s.

Definition 3.7. Suppose P is a clump on T_{e-1}^{s-1} with root ρ, where $|\rho| \geq e$, and $P \cap T_e^{s-1}$ is a pruned clump on which R_e is working.

Say that requirement R_e *requires attention due to halting at P at stage s* if the leaf λ of P which is an initial segment of ξ^{s-1} is e-extendible at stage s, but P does not meet the e-majority vote criterion.

If P is a pruned clump in T^{s-1}_{e-1} whose root ρ has $|\rho| \geq e$, say R_e *requires attention due to complexity at P at stage s* if P meets the e-majority vote criterion but the leaf λ of P which is an initial segment of ξ^{s-1} has $\dfrac{K_s(\lambda)}{|\lambda|} < \dfrac{1}{4}$.

If P does not require attention due to halting and does not meet the e-majority vote criterion, say that P *appears to force e-divergence* at stage s. Say that P is the *first witness to e-divergence* at stage s if P is the $\prec$-least clump on T^{s-1}_{e-1} with root of length at least e which appears to force e-divergence at stage s.

The restriction that $|\rho| \geq e$ given above ensures that only finitely many requirements will ever require attention on the pruned clumps with root ρ.

We are now ready to complete our definition of e-ρ-verification.

Definition 3.8. Let $e \geq 0$, and $\sigma \in T^{s-1}_{e}$. Suppose $\rho \prec \sigma$ is the root of a pruned clump Q on T^{s-1}_{e-1}.

We say that σ is e-ρ-verified if σ is $(e-1)$-ρ-verified and either

1. Q appears to force e-divergence at stage s, or
2. For each $\rho_0 \prec \sigma$ which is the root of a pruned clump P on T^{s-1}_{e-1} such that $P \cap T^{s-1}_{e}$ is a pruned clump, P meets the e-majority vote criterion at stage s.

The idea of this definition is that if we believe that Q forces e-divergence then there is no reason to remove any of its extensions from T_e, whereas if we have met the e-majority vote criterion we will attempt to preserve that at later stages. In either case, it appears that from the perspective of ρ, σ can safely be expected to remain on T_e. Of course, we should only believe that σ will stay on T_e if we already believe that it will stay on T_{e-1}, and thus adjust our beliefs accordingly.

At each stage s of the construction we will want to focus on a single pruned clump on which some requirement R_e is working, and which requires attention at stage s. If we identify such a pruned clump, we refer to it as our *target for action* at stage s.

We will say that a pair $\langle e, P\rangle$ consisting of a number $e < s$ and clump $P \subseteq T^{s-1}_{e-1}$ is a *candidate for action* at stage s if R_e is working on P at stage s, P requires attention at stage s, and furthermore g permits changes at the root of P at stage s.

A candidate for action $\langle e, P\rangle$ is the *target for action* at stage s if it meets each of the following conditions:

1. there is no pruned clump $Q \prec P$ such that for some i, $\langle i, Q\rangle$ is a candidate for action at stage s,

(2) there is no requirement $i < e$ which requires attention on a pruned clump $Q \sim P$,
(3) there is no pruned clump $Q \succsim P$ and number $i < e$ such that Q is the first witness to i-divergence at stage s.

There is at most one target for action at each stage. At some stages there may be no candidates for action, or candidates but no target.

In the next section, we will outline the construction proper.

We will build the trees T_e^s by attempting to find strings which force divergence of Φ_e, and, if that is not possible, will attempt to meet the e-majority vote criterion on the pruned clumps in T_{e-1}^s. If we meet the e-majority vote criterion on a pruned clump $Q \subseteq T_{e-1}^s$, we will want to preserve this at all future stages. However, it may be the case that at a later stage $t > s$ we have a target for action of form $\langle i, Q\rangle$, where $P \precsim Q$. At such a stage, if R_i requires attention at Q due to halting, then we will be forced to abandon our progress on Q. However, if R_i requires attention at Q due to complexity, we will ensure that Q remains a pruned clump on T_e^t. This will assist us in meeting the enumerability criterion given by Lemma 2.4.

4. The Construction

Initialization

At stage 0, we set ξ^0 to be the string consisting of a single 0, and let g_0 have empty domain. We adopt the convention that 0 is not working on the single pruned clump of T_{-1}^0 at this stage.

We will now describe how to use the situation at the end of stage $s-1$ of the construction to carry out stage s.

Defining the trees T_e^s and approximation ξ^s

How we proceed at stage s depends on whether there is a target $\langle e, P\rangle$ for action, and, if so, the reason that R_e requires attention at P.

Case 1: No candidates for action.

If there are no candidates for action, then for each $i < s$, define T_i^s as follows. If P is the $\prec$-least pruned clump on T_{i-1}^{s-1} on which ξ^{s-1} is working, but which does not meet the e-majority vote criterion at stage s, then let μ be the leaf of P which is an initial segment of ξ^{s-1}, and let

$$T_i^s = T_i^{s-1} \cup \{\tau \in T_{i-1}^s \mid \mu \prec \tau\}.$$

If every pruned clump P on T_{e-1}^s on which ξ^{s-1} is working meets the e-majority vote criterion at stage s, let

$$T_i^s = T_i^{s-1} \cup \{\tau \in T_{i-1}^s \mid \xi^{s-1} \prec \tau\}.$$

Define ξ^s to be some leaf λ of T_{s-1}^s such that $\xi^{s-1} \preceq \lambda$.

Case 2a: Target for action due to halting, and an apparently divergent computation is found.

Let $\langle e, P\rangle$ be the target for action. Suppose that R_e requires attention due to halting at P, and that the root of P is ρ. Suppose that there is a leaf λ of P which is not e-extendible at stage s.

Then we choose $\xi^s = \lambda$ (if there are several possible choices, choose the leftmost). For $i < e$, let $T_i^s = T_i^{s-1}$. For $e \leq i < s$, let

$$T_i^s = \{\sigma \in T_i^{s-1} \mid \neg(\rho \prec \sigma)\} \cup \{\sigma \in 2^{<\omega} \mid (\exists \mu \in P)[\sigma \preceq \mu]\}.$$

Case 2b: Target for action due to halting, but every leaf is e-extendible.

Let $\langle e, P\rangle$ be the target for action. Suppose that R_e requires attention due to halting at P, and that the root of P is ρ. Suppose that each leaf λ of P is e-extendible at stage s.

For each $\tau \in 2^{<\omega}$ of length $|\rho| \cdot 2^{-2e-4}$, define $E(\tau)$ to be the set of leaves λ of P which are e-τ-extendible at stage s. From amongst these strings, effectively find a string τ for which $|E(\tau)|$ is maximal. Let $D(\tau)$ be a subset of $E(\tau)$ with exactly $2^{|\rho|\left(1-\sum_{j=0}^{e} 2^{-2j-4}\right)}$ leaves[2].

Let $\widehat{D}(\tau)$ be an effectively chosen set of strings on T_{e-1}^{s-1} consisting of one e-τ-extension of each $\lambda \in D(\tau)$. Define ξ^s to be the leftmost member of $\widehat{D}(\tau)$.

Define $T_i^s = T_i^{s-1}$ for $i < e$.

There is some $\prec$-least pruned clump $Q \preceq P$ on T_{e-1}^{s-1} on which ξ^{s-1} is working, and such that Q does not meet the e-majority vote criterion at stage s. Define

$$T_e^s = \{\sigma \in T_{e-1}^{s-1} \mid \neg(\rho_0 \prec \sigma)\} \cup \{\sigma \in 2^{<\omega} \mid \exists \widehat{\lambda} \in \widehat{D}(\tau)[\sigma \preceq \widehat{\lambda}]\}.$$

For $e < i < s$, define

$$T_i^s = \{\sigma \in T_i^{s-1} \mid \neg(\rho_0 \prec \sigma)\} \cup \{\sigma \in 2^{<\omega} \mid \sigma \preceq \xi^s\}.$$

Case 3: Target for action due to complexity.

Finally, suppose that $\langle e, P\rangle$ is the target for action, that R_e requires attention due to complexity at P, and that the root of P is ρ.

For $0 \leq i < s$, let $T_i^s = T_i^{s-1}$.

In this case, P meets the e-majority vote criterion. For $e \leq i < s$ let $P_i = T_{i-1}^{s-1} \cap P$. Let D consist of the numbers i for which P_i is a pruned clump on T_{i-1}^{s-1} which meets the i-majority vote criterion. For each $i \in D$ let $\tau_i = \Phi_i^{\xi^{s-1}}[s] \upharpoonright 2^{-2i-4}|\rho|$. Let i_0 be the largest member of D. Let λ be an effectively chosen leaf of P_{i_0} with the property that $K_s(\lambda)$ is maximal amongst all such leaves.

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[2]We will later see that $E(\tau)$ has at least this many leaves.

Choose strings $\xi^s_e \succeq \xi^s_{e+1} \succeq \cdots \succeq \xi^s_{s-1} \succeq \lambda$ such that for each i, ξ^s_i is a leaf of T^{s-1}_i. Let $\xi^s = \xi^s_{s-1}$.

Finally, in all of the cases 1-3, let T^s_s consist of ξ^s together with all of its initial segments.

Defining the permission function

We will now define g_s. When deciding whether g assigns n to work on ρ, we will want to ensure that any numbers assigned at stage s are still assigned a stage $s+1$, and in addition to this, that a number assigned to work on ρ at stage s should be assigned to work on some initial segment of ρ at stage $s+1$.

Begin by checking if there is a target $\langle e, P\rangle$ for action at stage s. Suppose that there is one. Then let ρ_0 be the root of P.

For each number n which g assigns to work on a string $\rho \succeq \rho_0$ at stage s, declare that g assigns n to work on ρ_0 at stage $s+1$. For each $\rho \prec \rho_0$, if g assigns n to work on ρ at stage s, say g assigns n to work on ρ at stage $s+1$.

If there was no target for action at stage s, then for each ρ such that g assigned n to work on ρ at stage s, say that g assigns n to work on ρ at stage $s+1$.

In each of the above cases, if g assigns n to work on ρ at stage $s+1$, then define $g_s(n) = h_s(n)$ if some requirement R_e requires attention on a pruned clump with root ρ at stage $s+1$, but there was no such requirement at stage s. Otherwise, define $g_s(n) = g_{s-1}(n)$.

Finally, let $\rho_1 \prec \rho_2 \prec \cdots \prec \rho_k$ be the roots of the clumps on T^s_{-1} on which R_0 is working at stage s and on which g assigned no number to work at stage s. Let $n_1 < n_2 < \cdots < n_k$ be the least k numbers that g did not assign to work on any string at stage s. For $1 \le i \le k$, say that g assigns n_i to work on ρ_i at stage $s+1$. For $1 \le i \le k$, define $g_s(n_i) = h_s(n_i)$ if some requirement R_e requires attention on a pruned clump Q with root ρ_i on T^s_{e-1} at stage $s+1$, and $g_s(n_i) = 0$ if not.

5. Verification of construction

For each e, let $T_e = \{\sigma \in 2^{<\omega} \mid \sigma \in T^s_e$ at cofinitely many stages $s\}$, and $X = \lim_s \xi^s$.

We will begin our analysis of the construction by establishing that some of its basic features function as intended. We will check that the strings ξ^s come to a limit X, and that the permission process given by the functions g and h behaves as intended.

Remark 5.1. Let $0 \le i \le s$. Then $T^s_i \subseteq T^s_{i-1}$, and for each pruned clump P on T^s_i, there is a pruned clump Q on T^s_{i-1} such that $Q \sim P$.

In addition, $\xi^s \in T^s_i$ for each s and $i \le s$, so X is a path through T_i.

Each of these facts is easily verified by checking that they are preserved from one stage of the construction to the next.

Lemma 5.2. *For each s and each $i \leq s$, if P is a pruned clump on T_i^s with root ρ, then P has at least $2^{|\rho|\left(1-\sum_{j=0}^{i} 2^{-2j-4}\right)}$ leaves.*

Proof. If $i = -1$, then P has exactly $2^{|\rho|}$ leaves, since in that case $P = \rho 2^{\leq |\rho|}$.

Now, work by induction on i. Suppose that the result is true of every pruned clump Q on T_{i-1}^s for every s. Fix some s, and let P be some pruned clump on T_i^s. Consider the largest $t \leq s$ such that P is on T_i^t but not on T_i^{t-1}.

If the construction proceeds via case *1* at stage t, then there is some string $\mu \in T_{i-1}^{t-1}$ such that $T_i^t = T_i^{t-1} \cup \{\tau \in T_{i-1}^{t-1} \mid \mu \prec \tau\}$. Let $P \sim Q$, where Q is a pruned clump on T_{i-1}^{t-1}. The string μ must be an initial segment of the common root of P and Q, and therefore that every leaf of Q is also a leaf of P. But that implies that P has at least $2^{|\rho|\left(1-\sum_{j=0}^{i-1} 2^{-2j-4}\right)}$ leaves, by induction. This is more than the minimum required.

If the construction proceeds via case *2a* or *3* at stage t, then there are no pruned clumps on T_i^t that were not already on T_i^{t-1}, and there is nothing to prove.

If the construction proceeds via case *2b* at stage t, then it must be the case that $\langle i, Q\rangle$ is the candidate for action at stage t, where Q is the pruned clump on T_{i-1}^{t-1} with $P \sim Q$. In this case, there are at least $2^{|\rho|\left(1-\sum_{j=0}^{i-1} 2^{-2j-4}\right)}$ leaves on Q. But each such leaf λ is e-τ-extendible at stage t for some $\tau \in 2^{<\omega}$ with $|\tau| = 2^{-2i-4}|\rho|$, where the ρ is the root of P. Since there are $2^{2^{-2i-4}|\rho|}$ many such τ, it follows that there is some particular τ such that at least $\dfrac{2^{|\rho|\left(1-\sum_{j=0}^{i-1} 2^{-2j-4}\right)}}{2^{2^{-2i-4}|\rho|}} = 2^{|\rho|\left(1-\sum_{j=0}^{i} 2^{-2j-4}\right)}$ of the leaves of Q are e-τ-extendible. So the construction builds a pruned clump with exactly this many leaves. Hence P has at least $2^{|\rho|\left(1-\sum_{j=0}^{i} 2^{-2j-4}\right)}$ leaves, as desired. □

Corollary 5.3. *For each s and each $i \leq s$, if P is a pruned clump on T_i^s , then some leaf λ of P has $K(\lambda) \geq {}^{|\lambda|}\!/_4$.*

Proof. Any prefix-free set of binary strings of length at most $|\lambda|/4$ can have at most $2^{|\lambda|/4}$ members. However,

$$\left(1-\sum_{j=0}^{i}2^{-2j-4}\right)=\left(1-\frac{1}{12}(1-4^{-i-1})\right)$$
$$\geq\frac{11}{12}$$

so that $2^{|\rho|\left(1-\sum_{j=0}^{i}2^{-2j-4}\right)}\geq 2^{11|\rho|/12}>2^{|\rho|/2}=2^{|\lambda|/4}$, and therefore P has too many leaves for them to all have such short descriptions. $\square$

Lemma 5.4. *For each e and string ρ, there are only finitely many stages t at which there is a target for action of the form $\langle e,P\rangle$, where ρ is the root of a pruned clump P on T^s_{e-1}.*

In addition, the strings ξ^s approach a limiting real X. That is, for each k, there is some s such that $|\xi^s|\geq k$ and for each $t\geq s$, $\xi^s\restriction k=\xi^t\restriction k$.

Proof. We will prove the first result by induction on the length of ρ and (within that) by induction on e.

Fix a number e and string ρ which is the root of a pruned clump on T_{-1}. Applying the inductive hypothesis, choose t_0 such that for $s\geq t_0$, $\langle i,P\rangle$ is not the target for action at stage s for any P with root $\rho_0\prec\rho$, nor for any $i<e$ and clump P with root ρ.

Suppose that for some $s_0\geq t_0$, ξ^{s_0} has an initial segment λ which is a leaf of some pruned clump P on $T^{s_0}_{e-1}$ with root ρ.

Then P is also on T^s_{e-1} for each $s\geq s_0$ because after that stage there will never be a target for action which can cause P to be removed.

Now we check that amongst stages $t\geq s_0$, $\langle e,P\rangle$ can be the target for action at most finitely many times.

For each leaf λ of P there can be at most one stage t at which $\langle e,P\rangle$ is the target for action and at which R_e requires attention due to halting at P, since at such a stage, if λ is the leaf of P for which $\lambda\prec\xi^{t-1}$, we know that λ is e-extendible. But then we either are in case *2a* and define ξ^t in a way which guarantees that it extends a leaf λ_1 of P which is not e-extendible at stage t, or are in case *2b* and have verified that every leaf of P is e-extendible. In the latter case P will meet the majority vote criterion at the next stage, and R_e will never again require attention due to halting at P.

Likewise, $\langle e,P\rangle$ can be the target for action at a stage t where R_e requires attention due to complexity at P only finitely many times. At such a stage t we will note that the leaf λ of P such that $\lambda\preceq\xi^{t-1}$ has $K_t(\lambda)<|\lambda|/4$. We will then will define ξ^t to be an extension of a leaf

$\widetilde{\lambda}$ of a pruned clump $Q \sim P$ which is on a tree T_i^{t-1} for some $i < s$, and such that $K_t(\widetilde{\lambda})$ is maximal amongst such leaves. It follows that $K_t(\widetilde{\lambda}) \geq |\tilde{\lambda}|/4$, by Corollary 5.3. Once again, $\langle e, P\rangle$ can only be the target at a stage where R_e requires attention due to complexity once for each leaf of P.

Only finitely many requirements ever require attention on the pruned clump P (namely those R_e for which $e \leq |\rho|$). As has been seen, each $\langle e, P\rangle$ is a target for action at finitely many stages. So it follows that eventually $\xi^t \upharpoonright |\rho|$ will remain constant.

We will now check that $\lim_s \xi^s$ exists as a member of 2^ω. Note that if ξ^s has the root of P as an initial segment and $\langle e, P\rangle$ is never a target for action after stage s, then ξ^t will still have that root as an initial segment at any stage $t \geq s$. Thus it suffices to show that for any given k, ξ^s eventually remains at least k in length.

Our proof will be by contradiction. Assume there is some longest string ρ which is the root of a pruned clump on T_{-1} and which is an initial segment of ξ^s at all stages $s \geq t$ of the construction. In addition, choose t large enough that for $s \geq t$, the target $\langle e, P\rangle$ for action will never have the property that P has a root $\rho_0 \preceq \rho$. Thus if $s \geq t$, a target $\langle e, P\rangle$ for action must have the property that the root ρ_1 of P satisfies $\rho \prec \rho_1 \preceq \xi^s$.

If such a target exists at a later stage t_0, then ρ_1 is an initial segment of ξ^{t_0}. Suppose ρ_1 is $\prec$-minimal amongst strings which are roots of pruned clumps P for which there is some stage $t_0 \geq t$ at which $\langle e, P\rangle$ is the target for action. Then ρ_1 will be an initial segment of ξ^{t_0} for all sufficiently large t_0. This contradicts that ρ is the longest such string.

Thus we may assume that there are no stages $s \geq t$ at which there is a target for action. So at each stage $s > t$, and for each $e < s$,

$$T_e^s \supseteq T_e^{s-1} \cup \{\tau \in T_{-1}^s \mid \xi^{s-1} \prec \tau\},$$

and ξ^s is always chosen to be a leaf of T_{s-1}^s which extends ξ^{s-1}. But then ξ^s an initial segment $\rho_1 \succ \rho$ which is the root of a pruned clump on T_{-1}, and ρ_1 is an initial segment of ξ^s at cofinitely many stages s. This gives the desired contradiction.

So $\lim_s \xi^s$ does exist as a member of 2^ω. □

Note that from the above proof it follows that for each n, if s is sufficiently large, then $g_s(n)$ is defined.

Lemma 5.5. *If g assigns m to work on ρ_1 and n to work on ρ_2 at some stage s, and $m < n$, then $\rho_1 \preceq \rho_2$.*

Proof. If s is the first stage at which g ever assigns n to work on some string ρ_2, then for each $m < n$, g assigns m to work on a proper initial segment of ρ_2.

If g assigns n to work on ρ_3 at stage $s-1$ and on ρ_2 at stage s, there is some $i < n$ such that for $i \leq m < n$, g also assigns m to work on ρ_2 at stage s, and for $m \leq i$, g assigns m to work on the same string $\rho_1 \prec \rho_2$ at stages $s-1$ and s. This is sufficient to verify that the condition of the lemma is preserved from one stage to the next.

□

Lemma 5.6. *For each n, there are finitely many numbers s such that $g_s(n) \neq g_{s+1}(n)$. Indeed there is a computable bound $f(n)$ on the number of such stages s, and that bound can be given independently of h.*

Proof. Observe that g assigns n to work on a string ρ at some stage s, then at stage $s+1$, g must assign n to work on a string $\rho_0 \preceq \rho$.

Now we show that if s is the first stage at which g assigns n to work on the root of some pruned clump, that root has length at most 4^n. Suppose the result for each $m < n$. Thus at the first stage s at which g assigns n to work on some string ρ, any $m < n$ which g assigns to work at this stage will be working on a string of length at most 4^m. So if ρ is the $\prec$-least on T^s_{-1} which is the root of a pruned clump on which R_0 is working and such that g assigns no number $m < n$ to work on ρ at stage s, we must have $|\rho| \leq 4^n$.

Next, we note that if $g_s(n) \neq g_{s-1}(n)$, then g assigns n to work on the root of a pruned clump P on which a requirement R_e requires attention at stage s, and that furthermore either no requirement R_i required attention on a pruned clump $Q \sim P$ at stage $s-1$, or n was not assigned to work on ρ at stage $s-1$.

We must find a computable bound on how many times this can happen.

Definition 5.7. Suppose that at some stage s, g assigns n to work on some string ρ. We will say that the interval $[t_0, t_1)$ is *dedicated to e on ρ* if for $t_0 \leq t < t_1$,

i.) g assigns n to work on ρ at stage t, and

ii.) for $i < e$, if Q_i is a pruned clump on T^t_{i-1} with root ρ, then $\langle i, Q_i \rangle$ is not the target for action at stage t.

Note that if ρ is the root of a pruned clump Q on $T^{t_0}_e$ and $[t_0, t_1)$ is dedicated to e on ρ then Q is on T^t_e for $t_0 \leq t \leq t_1$.

Fix some number k, and suppose $|\rho| = k$, and that $[t_0, t_1)$ is dedicated to k on ρ. Recall that R_i can only require attention on a pruned clump with root ρ if $i \leq k$.

Then if $t_0 \leq t < t_1$ and $g_t(n) \neq g_{t+1}(n)$, R_k requires attention on a pruned clump Q with root ρ at stage $t+1$. In that case, Q is on T^t_{k-1} at each stage in $[t_0, t_1)$ because no requirement can affect this clump. In addition, we may count the number of stages $t \in [t_0, t_1)$ at which $\langle k, Q\rangle$ can be the target for action. For each leaf λ of Q, there is at most one such stage at which $\xi^t \succeq \lambda$ and the construction proceeds via case *2a*, and at most one such stage at which the construction proceeds via case *3* — as seen in Lemma 5.4. The target may also be $\langle e, Q\rangle$ for action at one stage at which the construction proceeds via case *2b*. Thus $g_t(n) \neq g_{t+1}(n)$ for at most $2^{k+1} + 1$ values of t such that $t_0 \leq t < t_1$ (since this is one more than double the maximum possible number of leaves on Q).

We now show that for each e, if $[t_0, t_1)$ is dedicated to e on ρ, there are at most $(k-e+1)(2^{k+1}+1)$ stages $t \in [t_0, t_1)$ at which $g_t(n) \neq g_{t+1}(n)$, proceeding by backward induction. The base case ($e = k$) is given above.

Fix $e \leq k - 1$. Suppose that whenever $[t_0, t_1)$ is dedicated to $e + 1$ on ρ, there are at most $(k - e)(2^{k+1} + 1)$ many stages $t \in [t_0 < t < t_1)$ at which $g_t(n) \neq g_{t+1}(n)$.

Suppose that $[t_0, t_1)$ is dedicated to e on ρ. Let t_2 be the largest number in $[t_0, t_1)$ such that $[t_0, t_2)$ is dedicated to e on ρ. There are at most $(k - e)(2^{k+1} + 1)$ many stages t such that $t_0 < t < t_2$ and $g_t(n) \neq g_{t+1}(n)$.

If $t_2 < t_1$, then at stage t_2, the target for action is of form $\langle e, Q\rangle$, where Q has root ρ. Thus for $t_2 \leq \tilde{t} < t_1$, only $\langle e, Q\rangle$ can be the target for action at stage t. Applying the reasoning given above in the case $e = k$, we see that there are at most $2^{k+1}+1$ stages $t \in [t_2, t_1)$ at which $g_t(n) \neq g_{t+1}(n)$. So the total number of stages $t \in [t_0, t_1)$ at which $g_t(n) \neq g_t(n)$ is at most $(k-e)(2^{k+1}+1)+2^{k+1}+1 = (k-e+1)(2^{k+1}+1)$, completing the induction.

Now consider the stages at which g assigns n to work on the root of P. There are at most $(k + 1)(2^{k+1} + 1)$ many such stages t for which $g_t(n) \neq g_t(n+1)$. Because g first assigns n to work a string ρ for which $|\rho| \leq 4^n$, and at later stages assigns n to work on initial segments of ρ, there are at most $\sum_{i=0}^{4^n}(i + 1)(2^{i+1} + 1)$ stages throughout the entire construction at which $g_s(n) \neq g_{s+1}(n)$. □

Lemma 5.8. *For each n there is some string ρ such that g assigns n to work on ρ at all but finitely many stages.*

Proof. For some least s, $g_s(n)$ is defined. At stage s, g assigns n to work on some string ρ. If $t > s$, g assigns n to work on some string ρ_t. Furthermore, ρ_t is $\prec$-decreasing as a function of t, and therefore is eventually constant. □

Definition 5.9. If g assigns n to work on ρ at all stages $t \geq s$, we will say that n *settles* on ρ by stage s. If n settles on ρ by some stage, then we will simply say that n settles on ρ.

We are now ready to check that for each e the requirement R_e is met. To do so we must check that X is a path through each T_e, and that that either $\mathrm{Dim}(\Phi_e^X) < 1$ and there is some string $\xi \prec X$ with $|\xi| \geq e$ and $K(\xi) \geq |\xi|/4$, or that Φ_e^X is a nontotal function. In the former case, the required inequality on the effective packing dimension of Φ_e^X will be verified indirectly using Lemma 2.4.

Lemma 5.10. *Suppose that n_0 is a number such that $h(n_0) > g(n_0)$, that n_0 settles on some string ρ by stage s with $|\rho| \geq e$ and that ρ is the root of a pruned clump P which is on T^s_{e-1} at every stage $t > s$. Suppose also that for each $i < e$ and pruned clump $Q \precsim P$, Q is not the first witness to i-divergence at any stage $t > s$.*

Then one of the following conditions holds:

(a) There is a leaf λ of P and stage t_1 such that for $t > t_1$, ξ^t has λ as an initial segment, and λ is not e-extendible at stage t.
(b) There is a leaf λ of P and stage t_1 such that for $t > t_1$, ξ^t has λ as an initial segment, P meets the e-majority vote criterion at stage $t + 1$, and $K(\lambda) \geq \dfrac{|\lambda|}{4}$.

Proof. We proceed by induction on n_0. Fix n_0 such that $g(n_0) < h(n_0)$, and assume the result for $n < n_0$.

Suppose n_0 settles on some string ρ by stage s. Note that at stages $t \geq s$, if Q has a root which is a proper initial segment of ρ, then $\langle i, Q\rangle$ cannot be the target for action, since that would cause g to assign n_0 to a different string.

Fix some number e, and let $P \subset T^s_{e-1}$ be a pruned clump with root ρ. Suppose that for $t \geq s$ and $i < e$, R_i does not require attention on any clump $Q \sim P$ at stage t. Then P is a pruned clump on T^t_{e-1} at each stage $t \geq s$, since we have just ruled out all of the possible targets for action which could prevent that. If $t_0 \geq s$ is a stage at which R_e requires attention at P, then at a later stage $t \geq t_0$, $g_{t-1}(n_0) < h_t(n_0)$. At the first such stage, either R_e no longer requires attention on P, or $\langle e, P\rangle$ is a target for action.

Suppose that for some $t_1 > s$, ξ^{t_1} has an initial segment which is a leaf λ of P which is not e-extendible at any stage $t \geq t_1$. If so, we may choose t_1 so that if P is the first witness to e-divergence at any stage $t > t_1$, then P is the first witness to e-divergence at stage t_1.

If so, P is the first witness to e-divergence at every stage $t \geq t_1$. In that case, if $t \geq t_1$ and Q has a root $\rho_0 \preceq \rho$, $\langle i, Q\rangle$ cannot be the target for action at stage t. Thus $\xi^t \succeq \lambda$ for all $t \geq t_1$.

If P is not the first witness to e-divergence at stage t_1, then no leaf of P is e-extendible at any stage $t > t_1$, and there is some leaf λ_0 of P and $t_2 > t_1$ such that for $t > t_2$, $\xi^t \preceq \lambda_0$.

Thus in this case the first of the two conditions is satisfied.

Otherwise there is some stage t_0 at which every leaf λ of P is e-extendible. Because $\lim_t \xi^t$ exists there is some $t_1 > t_0$ such that for $t \geq t_1$, ξ^t has some fixed leaf λ of P as an initial segment. But $h(n_0) > g(n_0)$, so if t_1 is large enough, R_e does not require attention at P at any stage $t \geq t_1$. This implies that at each stage $t \geq t_1$, P meets the e-majority vote criterion and that $K_t(\lambda) \geq {}^{|\lambda|}/_4$. □

Definition 5.11. If P is a pruned clump on T_{e-1} such that there is a leaf λ of P and stage t_1 such that for $t > t_1$, ξ^t has λ as an initial segment, and λ is not e-extendible at stage t, then we will say that P *forces e-divergence of X*.

Note that in the preceding definition and lemma, P forcing e-divergence merely guarantees that we never find any $(e-1)$-ρ-verified extensions of λ which threaten to make Φ_e^X total. We will later see that if P forces e-divergence, then Φ_e^X really is nontotal.

Lemma 5.12. *At infinitely many stages there is no target for action.*

Proof. If ρ is the root of a pruned clump P on T_{i-1}^s but is not the root of a pruned clump on T_{i-1}^{s-1} for some $i \geq 0$, then there is no target for action at stage s.

If there are only finitely many stages without targets for action, there are only finitely many strings ρ which are roots of pruned clumps on any tree T_{i-1}^s at some stage s. For each such ρ, there are finitely many stages at which the target for action is of the form $\langle i, P\rangle$, where ρ is the root of P. If t is the last such stage for any ρ, every stage $s > t$ must be one at which there is no target for action, contrary to hypothesis. □

Lemma 5.13. *For each $n \in \omega$ let ρ_n be the string on which n settles. For each e, there are finitely many numbers n such that $h(n) > g(n)$ and ρ_n is not the root of a pruned clump on T_e.*

The finitely many exceptions to this assertion are numbers amongst those for which either $|\rho_n| < e$ or when there is some $i \leq e$ such that

ρ_n is an initial segment of the root of a pruned clump on T^{s-1}_{i-1} which is the first witness to i-divergence at stage s for all sufficiently large s.

Proof. First, fix some number e. There are finitely many numbers n for which $|\rho_n| < e$. Likewise, for each $i \leq e$, there is at most one string ρ which is the root of a pruned clump on T^{s-1}_{i-1} that is the first witness to i-divergence at stage s for all sufficiently large s, and hence only finitely many n for which $\rho \succeq \rho_n$. So the list of purported potential problems is indeed finite.

Now, fix some n such that $h(n) > g(n)$. Fix some e and assume the result of the lemma for each $i < e$.

Assume that ρ_n does not satisfy either exceptional condition. Note that if either of the exceptional conditions discussed above holds of ρ_n and e, the same condition also applies to ρ_n and i, for each $i < e$.

There are two possible scenarios.

The first is as follows: n settles on ρ_n by some stage t, and ρ_n has a proper initial segment which is the root of a pruned clump P on T^{s-1}_{e-1} such that for $s \geq t$, P is the first witness to e-divergence at stage s. Assume t is large enough that for $s \geq t$, there is no target for action of the form $\langle j, Q\rangle$, where the root of Q is an initial segment of ρ_n. At stages $s > t$ at which there is no target for action, if μ is the leaf of P which is an initial segment of ξ^{s-1}, we have $T^s_e = T^{s-1}_e \cup \{\tau \in T^s_{e-1} \mid \mu \prec \tau\}$. By our inductive hypothesis, ρ_n is the root of a pruned clump on T^s_{e-1} for all sufficiently large s. Because $\mu \prec \rho_n$, it follows that ρ_n is also the root of a pruned clump on T^s_e. But then ρ_n is the root of a pruned clump on T^{s-1}_e at all stages $s_1 \geq s$.

The second scenario is that ρ_n is the root of a pruned clump P on T_{e-1} with the property that there is a stage s at which R_e requires attention at P due to halting, and at which the target for action is $\langle e, P\rangle$. This stage may be assumed to be the last stage at which the target for action is of the form $\langle j, Q\rangle$, where $j \leq e$ and $\rho \preceq \rho_n$. In this case we add a pruned clump $Q \sim P$ to T^s_e, and never remove it again.

This concludes the induction. □

Lemma 5.14. *Fix e. Let t_1 be a stage in the construction and let $\rho \in T^{t_1-1}_e$ be a root of a pruned clump P on $T^{t_1-1}_{e-1}$ such that:*

1. *For each $i < e$ such that some pruned clump P_i on T_{i-1} forces i-divergence, ρ is an extension of the root of the $\prec$-least such P_i,*
2. *Some number n settles on ρ by stage t_1, and furthermore for each $i < e$, there is no stage $t > t_1$ at which there is a target for action of the form $\langle i, Q\rangle$, where Q is a pruned clump on T^{t-1}_{i-1}*

such that $Q \sim P$, except in case R_i requires attention due to complexity at Q,

(3) P meets the e-majority vote condition at stage t_1.

Then P is on T_e^t and meets the e-majority vote criterion at each stage $t > t_1$.

Proof. The only targets for action which might cause P to not be on T_{e-1}^t for some first stage $t > t_1$ are those of the form $\langle i, Q\rangle$ where $Q \prec P$ or $\langle i, Q\rangle$ where $i \leq e$ and $Q \sim P$. In the case where R_i requires attention due to complexity no pruned clump will be removed. But our assumption rules out any other target for action. □

We now introduce a noneffective version of verification which is called e-permanence.

Definition 5.15. Let $e \geq 0$. Say that $\sigma \in T_e^{s-1}$ is *e-permanent at stage s* if for $0 \leq i < e$, σ is i-permanent at stage s and either:

(1) there is a pruned clump Q with root ρ and a leaf λ such that for each $t \geq s$, Q is on T_{e-1}^t, appears to force e-divergence at stage t, and $\lambda \preceq \xi^t$, and such that $\sigma \preceq \lambda$ or $\lambda \preceq \sigma$, or
(2) each pruned clump P on T_{e-1}^s with root $\rho \prec \sigma$ for which $P \cap T_e^{s-1}$ is a pruned clump meets the e-majority vote criterion at stage s.

Lemma 5.16. *For each e, there is some stage t_1 such that for each $t > t_1$, any $\sigma \in T_e^t$ which is e-permanent at stage t is also in T_e^{t+1} and is e-permanent at stage $t + 1$.*

Proof. Fix some e, and assume the result for all $i < e$.

Let $\rho \prec X$ be the root of some pruned clump on T_{e-1} such that for each $i \leq e$ such that some pruned clump P forces i-divergence, such a clump has a root which is a proper initial segment of ρ.

Let t_0 be large enough that

(1) t_0 meets the condition given by the lemma for each $i < e$
(2) some number n settles on ρ by stage t_1

Now suppose that σ is an e-permanent string on T_e^t at some stage $t \geq t_1$.

At stages $s > t$, if $i < e$ and Q is a pruned clump on T_{i-1}^{s-1} with a root which is a proper initial segment of σ, $\langle i, Q\rangle$ can only be the target if R_i requires attention at Q due to complexity (otherwise Q could not have met the i-majority vote criterion, and hence σ was not i-permanent at stage $s - 1$).

If some pruned clump P on T_{e-1} forces e-divergence, then for $s \geq t_1$, ξ^s has some fixed leaf λ of P as an initial segment and P appears to force e-divergence at stage t. So the strings which are e-permanent at a stage $s \geq t_1$ are precisely the $(e-1)$-permanent strings σ on T_e^s such that $\sigma \preceq \lambda$ or $\lambda \preceq \sigma$. No such string can be removed from T_e^s at a stage s at which there is no target for action, nor at a stage at which there is a target for action because some requirement R_i requires attention due to complexity. Thus σ remains e-permanent at all stages $s \geq t$.

Otherwise, there is no pruned clump on T_e which forces e-divergence. Then every pruned clump P on T_{e-1}^{t-1} for which $P \cap T_e^{t-1}$ is a pruned clump with root ρ meets the e-majority vote criterion at stage t. At a stage s at which there is no target for action or at which there is a target for action chosen because some requirement requires attention due to complexity, a pruned clump on T_{e-1}^s cannot cease to meet the e-majority vote criterion. Once again, σ will remain e-permanent at all stages $s \geq t$.

□

Lemma 5.17. *Suppose that some pruned clump P on T_{e-1} forces e-divergence of X. Then Φ_e^X is nontotal.*

Proof. Let P force e-divergence of X. Assume that P has a root ρ such that $|\rho| \geq e$, and that for each $i < e$ such that there is some $\prec$-least pruned clump Q on T_{i-1} which forces i-divergence of X, $Q \prec P$ (choose P to be a clump with a longer root, if necessary). Suppose that Φ_e^X is total. Choose some stage t_0 such that for $t > t_0$, P is on T_{e-1}^t, such that some n_0 settles on ρ by stage t_1, and such that there is a leaf λ of P such that $\xi^t \succeq \lambda$ for all $t > t_0$.

Let $\sigma \succ \lambda$ be an initial segment of X such that $\sigma \in T_{e-1}^t$ for all $t > t_0$. Choose $t_1 > t_0$ such that if $\rho_0 \preceq \sigma$ is the root of any pruned clump P_0, the target for action cannot be $\langle i, P_0 \rangle$ at any stage $t > t_1$. For each $i < e$ such that there is some t_2 such that for each $t \geq t_2$, σ is i-verified at stage t, assume that $t_1 \geq t_2$.

If σ is not $(e-1)$-ρ-verified at every stage $t > t_1$, then there is some least $i < e$ such that σ is not i-ρ-verified at every stage $t > t_1$. We will show that this is impossible, by showing that σ is i-ρ-verified at all sufficiently large stages t. Thus it follows that σ is eventually $(e-1)$-ρ-verified, and since σ was an arbitrary extension of λ, P cannot force e-divergence of X.

By our assumption on P, the pruned clump $Q \sim P$ on T_{i-1} does not appear to force i-divergence at any stage $t \geq t_1$.

Find the $\prec$-least initial segment ρ_1 of X such that some number n settles on ρ_1 by a stage $t_2 > t_1$, and that t_2 is the largest stage at which there is a target for action of form $\langle i, P_1 \rangle$, where ρ_1 is the root of a pruned clump P_1 on $T^{t_2-1}_{i-1}$, R_i requires attention due to halting at P_1, and the construction proceeds via case *2b*.

Note that for $j < i$ and $P_2 \sim P_1$ there is never a target for action of the form $\langle j, P_2\rangle$ at any stage $t > t_2$ except if R_j requires attention due to complexity (or t_2 would not be the last stage at which there is a target of form $\langle i, P_1\rangle$ as specified above).

At stage t_2, consider the $\prec$-least pruned clump $Q \preceq P_1$ on $T^{t_2-1}_{i-1}$ on which ξ^{t_2-1} is working, and such that Q does not meet the i-majority vote criterion at stage t_2.

If $Q = P_1$ then σ is i-ρ-verified at stage t_2, by definition.

Otherwise $Q \prec P_1$ and $\rho_0 \prec \sigma$. But in that case the definition of $T^{t_2}_i$ ensures that every pruned clump Q such that $P \precsim Q \precsim P$ meets the i-majority vote criterion at stage $t_2 + 1$, and so σ is i-ρ-verified at stage $t_2 + 1$.

By our choice of t_2 and Lemma 5.14, there is no stage $t > t_2$ at which any target for action could cause σ to cease being i-ρ-verified.

This contradicts the minimality of i, as promised. $\square$

Remark 5.18. For each e and s, the e-permanent strings on T^s_e are downward closed, and therefore form a tree.

Lemma 5.19. *Suppose that e is a number such that Φ^X_e is total. For each s, let $\widehat{T^s_e}$ consist of the strings on T^s_e which are e-permanent at stage s, and t_1 be a number satisfying the condition of Lemma 5.16.*

Then $\bigcup_{s\geq t_1} \widehat{T}_e^{\,s}$ satisfies the conditions of Lemma 2.4.

Proof. Every pruned clump P on T^s_{e-1} such that $P \cap T^s_e$ is a pruned clump on $\widehat{T^s_e}$ meets the e-majority vote criterion at stage $s+1$, because its leaves are e-permanent. This shows that $\widehat{T^s_e}$ meets conditions *(2)* and *(3)* of Lemma 2.4.

By Lemma 5.16, we have $\widehat{T^s_e} \subseteq \widehat{T^{s+1}_e}$ for each $s \geq t_1$. Determining which of the leaves of T^s_e is e-permanent is a computable procedure, and so $\bigcup_{s\geq t_1} \widehat{T^s_e}$ is a c.e. tree.

Finally, suppose ρ is the root of a pruned clump on $\widehat{T^{s+1}_e}$ for some $s \geq t_1$. Then each string in that pruned clump must be $(e-1)$-verified at stage s, because otherwise there is no way that ρ can be the root of a pruned clump on T^{s+1}_e which meets the e-majority vote criterion. Suppose that $\rho \in \widehat{T^s_e}$. If there is some $\tau \succ \rho$ in $\widehat{T^s_e}$, then we may deduce that ρ is the root of a pruned clump P on T^s_{e-1} which meets the e-majority vote criterion. Hence $P \cap T^s_e$ is a pruned clump, and furthermore every string in $P \cap T^s_e$ is e-permanent. Thus ρ is the root of a pruned clump on $\widehat{T^s_e}$. If no such τ exists, then ρ is a leaf of $\widehat{T^s_e}$.

Thus $\bigcup_{s\geq t_1} \widehat{T_e^s}$ satisfies the conditions of Lemma 2.4, as desired. □

Lemma 5.20. $\mathrm{Dim}(X) \geq \frac{1}{4}$.

Proof. If Φ_e is total, then for infinitely many pruned clumps P on T_e with root $\rho \prec X$, condition *(b)* of Lemma 5.10 must be met. Thus there is a leaf λ of P such that $\lambda \prec X$ and $K(\lambda) \geq {}^{|\lambda|}/_4$. Therefore $\mathrm{Dim}(X) \geq {}^1/_4$. □

Lemma 5.21. *$X \leq_T A$.*

Proof. We will define a Turing functional Ψ which witnesses the fact. We will let ψ_s denote the use of the reduction Ψ^{A_s}. Recall that $\Theta^{A_s}(n) = h_s(n)$ for each s, and that for each n, $h_s(n)$ is an increasing function of s. Let θ_s denote the use of Θ^{A_s}. We may assume that for each s, and each $m < n$, $\theta_s(m) < \theta_s(n)$ if both are defined.

At stage 0 of the construction, specify none of the graph of Ψ.

At each stage $s > 0$ at which g assigns n to work on the root ρ of a pruned clump P, but such that g did not assign n to work on ρ at stage $s-1$, set $\Psi^{A_s}(n) = \rho$ with use $\psi_s(n) = \theta_s(n)$.

We must check that there are no strings $\sigma \prec \tau$ such that for some n, $\Psi^\sigma(n)\downarrow \neq \Psi^\tau(n)\downarrow$.

Suppose that g assigns n to work on ρ at stage s, and that at stage s we define $\Psi^{A_s}(n)$. Suppose also that g assigns n to work on some $\rho_0 \succ \rho$ at stage $s-1$. Then the target for action at stage s is of the form $\langle e, P\rangle$, where P has root ρ. So for some $m < n$ which g assigns to work on ρ at both stages $s-1$ and s, $g_{s-1}(m) < h_s(m)$. Furthermore, at some stage $t_0 < s$ at which g assigns m to work on ρ, some requirement began to require attention on a pruned clump with root ρ, and $g_{t_0}(m) = h_{t_0}(m)$. If t_0 is the greatest such stage, then $h_{t_0}(m) = h_{s-1}(m)$, because h is an increasing approximation. Thus $h_t(m) < h_s(m)$ for all $t < s$.

Now suppose $t < s$ and $A_t \upharpoonright \theta_t(n) = A_s \upharpoonright \theta_t(n)$. We will show $\Psi^{A_t\upharpoonright\theta_t(n)}(n)$ is not defined. If it were defined, note that $\theta_t(n) \geq \theta_t(m)$, from which it follows that $A_t \upharpoonright \theta_t(m) = A_s \upharpoonright \theta_t(m)$, and therefore that $\Theta^{A_t\upharpoonright\theta_t(m)}(m) = \Theta^{A_s\upharpoonright\theta_s(m)}(m)$, i.e. that $h_t(m) < h_s(m)$, which is absurd.

At each stage s of the construction, if $m < n$ then $\Psi^{A_s}(m) \prec \Psi^{A_s}(n) \preceq \xi^s$, by Lemma 5.5. Furthermore, Ψ^A is total, because g assigns each n to work on a string at all sufficiently large stages t.

To compute a desired initial segment of X it therefore suffices to consult the output of $\Psi^A(n)$ for sufficiently large n. □

Combining the results of Lemmas 5.19, 5.20, and 5.21, we see that our real X satisfies the requirements of the main result given by Theorem 1.3, and thus suffices to prove both that result and Corollary 1.4, our characterization.

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