Computably Enumerable Sets and Quasi-reducibility

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Abstract

We consider the computably enumerable sets under the relation of Q-reducibility. We first give several results comparing the upper semilattice of c.e. Q-degrees, $\langle \mathcal{R}_Q, \leq_Q \rangle$, under this reducibility with the more familiar structure of the c.e. Turing degrees. In our final section, we use coding methods to show that the elementary theory of $\langle \mathcal{R}_Q, \leq_Q \rangle$ is undecidable.

1 Introduction

Classical recursion theory first arose in order to study the inherent difficulty of mathematical problems. By far the most deeply studied notion relating the difficulty of one problem to another has been that given by Turing reducibility. The reason for this is that Turing reducibility seems to give the most general means of obtaining finite information about one object given finite information about another; hence, as the limiting case of using information, it is the most natural object of study for purely theoretical investigations of relative computability and definability. Nevertheless, for specific problems, particularly those arising in the study of algebraic structures, other reducibilities are actually the correct ones to consider. These reducibilities are usually less general, or "stronger", since they arise by putting limits of some kind on what sort of information can be used in a relative solution of one problem given another.

For example, weak truth table (wtt) reducibility imposes the additional condition that the amount of information used in a relative computation can be bounded in advance by a computable function. In the case of (computably presentable) infinite dimensional vector spaces, it turns out that the inherent difficulty of constructing bases for subspaces coincides exactly with the relation of wtt reducibility, rather than Turing reducibility. A similar situation arises in combinatorial group theory, where so-called quasi-reducibility, or Q-reducibility, turns out to be a more useful means of comparing word problems than ordinary T-reducibility.

Recall ([11]) that A is Q-reducible to B via f if f is a computable function such that for every $x \in \omega$,

$$x \in A \iff W_{f(x)} \subseteq B.$$

In this case we say that $A \leq_Q B$ via f. It is not hard to see that this relation is transitive and reflexive, so that it gives rise to a degree structure on 2^ω. Q-reducibility is a natural weakening of many-one reducibility, where singletons are replaced by c.e. sets. On the c.e. sets, it is not hard to show that $\leq_Q$ is a strictly stronger reducibility than $\leq_T$, since $A \leq_Q B$ implies $\omega - A$ is B-computably-enumerable.

The relationship of Q-reducibility to word problems is most easily seen by considering the theorem of Dobritsa (unpublished, see [1]) that for every set of natural numbers X there is a word problem with the same Turing degree as that of X. The proof of this fact depends, for the hard direction, on producing a group G in which every element has a normal form with an associated finite list of numbers, which are a subset of X if and only if the element is the identity. Thus, the proof actually shows that X has the same Q-degree as the word problem of G. In fact, since the sets produced are finite, the relationship is the slightly stronger one of *positive* equivalence, but for c.e. sets, and hence for computably presented groups, the relationship is the same (see Lemma 1 below). On the other hand, Ziegler in [14] showed that this fails for bounded truth table reducibility.

A further relationship between Q-degrees and groups lies in the relationship of a computably presented group to its algebraically closed extensions. A. Macintyre in [9] showed that whenever G and H are computably presented groups with word problems of Turing degrees $\mathbf{g}$ and $\mathbf{h}$, respectively, then if $\mathbf{g} < \mathbf{h}$, G must be a subgroup of every algebraically closed group of which H is a subgroup. O. Belegradek, however, showed in [1] that the converse fails, but becomes true if Turing reducibility is replaced by Q-reducibility. That is, for any computably presented groups G and H, if G is a subgroup of every algebraically closed group of which H is a subgroup, then G's word problem must be quasi-reducible to that of H. Thus any fact about the partial order of the c.e. Q-degrees has an immediate translation into one involving inclusion relations between the classes of finitely generated subgroups of algebraically closed groups. Since any countable algebraically closed group is determined up to isomorphism by the class of its finitely generated subgroups, this gives a natural relationship between a purely recursion-theoretic notion and a purely algebraic one.

In the case of groups without c.e. word problems, we note that the notion of Ziegler reducibility (see [15]) appears to be the natural one for studying relationships between general word problems. For this reducibility, one adds to Q-reducibility the requirement that A be enumeration reducible to B, and allows the function via which the reduction is given to give indices for sets enumeration reducible to B, rather than ordinary c.e. sets as in the case of a Q-reduction. In the case of c.e. sets, neither of these modifications makes any difference, so that Q-reducibility and Ziegler reducibility are the same, although clearly this fails in general. Both of these reducibilities arise as abstractions of the process of effectively using sets of equations true in one structure to check the truth of equations in another.

For non-c.e. sets, it is also true that there is no simple relationship between $\leq_Q$ and $\leq_T$. In fact, for any set A, the set $A^Q = \{ e : W_e \subseteq A \}$ is Q-equivalent to A, yet, by Rice's theorem, for any A we have $0' \leq_T A^Q$, since any such A^Q is a nontrivial index set. In particular, the Q-degree of the empty set contains a set Turing-above $0'$. It is partly for this reason that the notion of Ziegler reducibility was introduced in [15], since it is strictly stronger than Turing reducibility on arbitrary sets. Since we restrict ourselves in what follows to the c.e. sets this difference is unimportant for our results. We also assume in the statements of our theorems that none of the sets below is ω, since it is easy to see that ω has a Q-degree strictly below that of any other set.

We first establish some pathologies of the upper-semilattice of the Q-degrees relative to the more familiar structure of the c.e. Turing degrees. In particular, we show that the computable join has the following unusual property:

Corollary 4**:** Let C be an c.e. set. Then there exists an c.e. $D \equiv_T C$ such that for all A and B, if $D \leq_Q A \oplus B$, either $D \leq_Q A$ or $D \leq_Q B$.

We also show that infima in the c.e. Q-degrees can differ wildly from infima in the Turing degrees, constructing a Turing degree which contains a minimal pair of Q-degrees.

Theorem 5**:** There exists a noncomputable c.e. set A and a c.e. B with $A \equiv_T B$ such that A and B form a minimal pair in the Q-degrees.

We continue our discussion of infima in the Q-degrees by showing that, as in the case of the Turing degrees, non-branching c.e. Q-degrees exist outside of every (nontrivial) upper cone.

Theorem 6**:** For every c.e. $C \not\equiv_Q 0$, there exists an c.e. A, which is non-branching in the c.e. Q-degrees such that $C \not\leq_Q A$.

We then show that the c.e. Q-degrees form a dense partial order, just as the c.e. Turing degrees do, answering an open question of Omanadze..

Theorem 7**:** For every pair of c.e. sets $B <_Q A$, there exists an c.e. set C with $B <_Q B \oplus C <_Q A$.

The proofs of these last two results, particularly that of the density theorem, are more difficult than in the case of the Turing degrees. This shows up particularly in the more subtle methods that have to replace standard permitting techniques for ensuring a constructed set will be computable from some set given in advance. The added complexity serves to illustrate the ideas needed to modify standard finite injury and infinite injury constructions in order to deal with the c.e. Q-degrees. Some light seem to be shed on the (easier) proofs of the analogous Turing results, by way of contrast, as well.

We finish our analysis of the structure of the c.e. Q-degrees by adapting coding techniques developed for the Turing degrees to this context in order to obtain the following

Theorem 10**:** $\mathcal{R}_Q$ has an undecidable first order theory.

In what follows, our notation is standard, as in [12]. We adopt the practice of writing $[s]$ after any expression to indicate that every (dynamic) object in the expression is being taken to be its approximation at stage s.

2 Q-degrees and semirecursive sets

Before proceeding, we introduce a useful technical fact which will enable us to adapt some of the standard methods used in the study of Turing degrees to the more restrictive Q-degree context. Informally, the main point of the lemma below is that in the case of c.e. sets, if $X \leq_Q Y$ via f, then we may assume that $|(W_{f(x)} - Y)[s]|$ is always at most 1, and in fact the eventual "use" is finite.

Lemma 1. *Let X and Y be enumerable sets. For any partial computable function Φ_e, there exists a partial computable function Φ_{e_0} such that if $X \leq_Q Y$ via Φ_e, then*

$$X \leq_Q Y \text{ via } \Phi_{e_0};$$

$$\forall x (W_{\Phi_{e_0}(x)} \subseteq W_{\Phi_e(x)});$$

$$\forall x (W_{\Phi_{e_0}(x)} \text{ is finite});$$

$$\forall x \forall s |(W_{\Phi_{e_0}(x)} - Y)[s]| \leq 1;$$

$$\forall x \forall s \forall z (z \in (W_{\Phi_{e_0}(x)} - Y)[s] \bigcap Y[s+1]) \Longrightarrow (W_{\Phi_{e_0}(x)} - Y)[s+1] = \emptyset.$$

Furthermore, the index e_0 can be found effectively from e and indices for X and Y.

Proof. We simultaneously enumerate X, Y, and every $W_{\phi_e(x)}$, only allowing a new $y \in W_{\phi_{e_0}(x)}[s_{j+1}]$ when $x \notin X[s_{j+1}]$ and $(W_{\phi_{e_0}(x)} - Y)[s_j] = \emptyset$. This is clearly a computable procedure with the required properties. If $x \in X$, the enumeration of $W_{\phi_{e_0}(x)}$ stops at the first s with $x \in X[s]$; while if $x \notin X$, eventually some element $\phi_e(x) \in W_{\Phi_e(x)} - Y$ appears in $W_{\Phi_e(x)}$, and the enumeration into $W_{\Phi(e_0)}(x)$ stops after this point. Since we only allow a new element to enter $W_{\phi_{e_0}(x)}$ at a stage immediately after one at which $(W_{\phi_{e_0}(x)} - Y)[s] = \emptyset$, the last property is also clearly satisfied. The uniformity of the procedure for finding e_0 is obvious from the *s-m-n* Theorem. ☐

A set D of natural numbers is said to be *semirecursive* (Jockusch, [5]) if there exists a computable function f of two variables such that for every x and y,

$$f(x,y) = x \,\text{or}\, f(x,y) = y$$

$$(x \in D \,\text{or}\, y \in D) \Longrightarrow f(x,y) \in D.$$

The significance of this notion lies in the fact that f reduces membership questions for D about any finite number of elements to membership questions about one element, since whenever $f(x,y) = x$,

$$y \in D \Longrightarrow x \in D,$$

and similarly for $f(x,y)=y$. Because of this, if A and D are c.e. sets, with D semirecursive, and $A \leq_T D$, say $A = \Phi(D)$, then $A \leq_Q D$, since we can use f on $\big((\omega - D) \upharpoonright \phi(D;x)\big)[s]$, whenever $\Phi(D;x)[s] = 0$, to get uniformly an element which must eventually enter D if $x \in A$. This gives an c.e. set for the Q-reduction in the obvious way.

The following definition is from Downey, [2].

Definition 1. *Let g be a computable enumeration of an c.e. set A. The dump set of A relative to g, $D^g(A)$ is the c.e. set defined by the following procedure.*

Stage $0: D^g(A)_0 = \emptyset$.

Stage $s+1:$ *Let* $\omega - D^g(A)_s = \{d_{0,s} < d_{1,s} < \dots\}$. *Then*

$$D^g(A)_{s+1} = D^g(A)_s \cup \{d_{g(s),s}, \dots, d_{g(s)+s,s}\}, \text{ and}$$

$$D^g(A) = \bigcup \{\, D^g(A)_s \,:\, s \in \omega \,\}.$$

It is possible to show that for any such g, $D^g(A)$ has the same Turing degree as A and is semi-recursive. Since Q-reducibility and T-reducibility coincide for c.e. semirecursive sets, we suppress g in what follows and merely write $D(A)$ for the dump set, even when we have some fixed enumeration g under consideration.

Lemma 2. *The c.e. Q-degree of $D(A)$ is the greatest c.e. Q-degree contained in the Turing degree of A.*

Proof. This is clear, since $D(A)$ is semi-recursive, as witnessed by the function $f(x,y)$ which gives x if $x \in D(A)_{(y-x)+1}$ and y otherwise. □

Because of this property of the dump set, it is tempting to consider the possibility of an embedding of the Turing degrees into the Q degrees given by $A \mapsto D(A)$. This clearly embeds the partial ordering. It is also not hard to see that this mapping preserves infima. Let $\mathbf{a} = \mathbf{b} \cap \mathbf{c}$ in the c.e. Turing degrees. Suppose some c.e. $D \leq_Q D(B), D(C)$ where A, B, and C are c.e. representatives for these degrees. Then $D \leq_T D(B), D(C)$, since all of these sets are c.e. Hence $D \leq_T B, C$, hence $D \leq_T A \leq_T D(A)$, so $D \leq_Q D(A)$, as both are c.e. and $D(A)$ is semirecursive.

Unfortunately, this proposed embedding fails rather badly to preserve the *upper* semilattice structure of the Turing degrees, as we now show.

Theorem 3. *Let A, B and D be c.e. sets, with D semirecursive. Then if $D \leq_Q A \oplus B$, either $D \leq_Q A$ or $D \leq_Q B$. In particular, for any T-incomparable c.e. A and B, $D(A) \oplus D(B) <_Q D(A \oplus B)$.*

Proof. The proof is a non-uniform construction reminiscent of Lachlan's proof that the many-one degree of K cannot be the supremum of incomparable many-one degrees. Suppose $D \leq_Q A \oplus B$ via g, and let f be the function witnessing that D is semirecursive. We use Lemma 1 to guarantee that for every x, $|W_{g(x)} - (A \oplus B)| \leq 1$.

We first construct a recursive function h in an attempt to reduce D to B via h. If y is any number, we search for $x > y$ such that $x \notin D_s$, $(W_{g(x)} - (A \oplus B))[s] = \{2b(x,s)+1\}$ and $f(x,y) = x$. (In other words, the Q-reduction g counts x as out of D with an odd witness, $2b(x,s)+1$, at stage s.) For each y, we let $W_{h(y)}$ be the set of all such $b(x,s)$ where $s > x$. Since f witnesses the semirecursiveness of D, $y \in D$ implies $x \in D$. Hence, if $y \in D$, $W_{h(y)} \subseteq B$.

Suppose $D \not\leq_Q B$ via h, so that the procedure outlined above fails at some y. For this y, we must then have $y \notin D$, yet $W_{h(y)} \subset B$. Let $\bar{a}$ be some fixed number that is not an element of A. So, for every $x > y$, and $s > x$, if $x \notin D_s$, $(W_{g(x)} - (A \oplus B))[s] = \{2b(x,s)+1\}$ and $f(x,y) = x$, then $x \in D$. So, given an $x > y$, we first check whether $f(x,y) = y$. If so, then $x \notin D$, so we set $W_{h'(x)} = \{\bar{a}\}$. Otherwise, $f(x,y) = x$. In this case we set $W_{h'(x)} = \{\, a : 2a \in W_{g(x)} \,\}$. If $x \in D$, then $W_{g(x)} \subseteq A \oplus B$, so $W_{h'(x)} \subseteq A$. If $x \notin D$, then, clearly $x \notin D_s$ for any $s > x$. But if $W_{g(x)} - (A \oplus B) = \{2b+1\}$, then for almost every $s > x$, $(W_{g(x)} - (A \oplus B))[s] = \{2b+1\}$, implying $x \in D$. So $W_{g(x)} - (A \oplus B) = \{2a\}$, and $W_{h'(x)} \not\subseteq A$. So $D \leq_Q A$ via h'. $\square$

Corollary 4. *Let C be a c.e. set. Then there exists an c.e. $D \equiv_T C$ such that for all A and B, if $D \leq_Q A \oplus B$, either $D \leq_Q A$ or $D \leq_Q B$.*

Since the Q-degree of a dump set is the greatest c.e. Q-degree contained in a Turing degree, this extends the result of Ambos-Spies in Fischer, [4], that the Q-degree of K does not split. We note that every c.e. Turing degree, in particular that of K, is the join of incomparable low c.e. Turing degrees by a straightforward argument based on the Sacks Splitting Theorem (see [12], VII.3.4). Hence this result gives an elementary difference between the the structure of the c.e. Q-degrees and that of the c.e. Turing degrees.

3 More about infima

Since the mapping $C \mapsto D(C)$ preserves lower bounds, the fact that minimal pairs of c.e. Turing degrees exist implies that there are minimal pairs of c.e. Q-degrees. Hence 0 is a Q-degree which branches in the c.e. Q-degrees. In fact, however, something more surprising is true: there exists a single Turing degree containing a branching of 0 in the c.e. Q-degrees.

Theorem 5. *There exists a noncomputable c.e. set A and an c.e. B with $A \equiv_T B$ such that A and B form a minimal pair in the Q-degrees.*

Proof. We must satisfy two infinite sequences of requirements:

$N_{\langle a,b,c\rangle} : W_c \leq_Q A$ via ϕ_a and $W_c \leq_Q B$ via $\phi_b \Longrightarrow W_c \leq_Q \emptyset$,

and

$P_e : A \neq \phi_e$.

Additionally, we have the additional global requirement

$R : A \leq_T B \leq_T A$.

The positive requirements will be satisfied by the usual diagonalization strategy. The strategy for the negative requirements is also the usual one for constructing a minimal pair of c.e. degrees: whenever the length of agreement between the Q-reductions and W_c increases, we define a partial computable characteristic function χ for W_c. If we only allow one of A or B to change below the Q-uses of the reductions for elements in the domain of $\chi[s]$, this will satisfy the requirement.

The problems with making the two types of strategies cohere in this case, come from the imposition of the highest priority requirement R. Since we must guarantee $A \leq_T B \leq_T A$, each time A changes, B must also change to reflect this in B's Turing reduction for A, causing a further A-change to reflect this B-change. The basic idea for solving this problem comes from the embedding of the nondistributive lattice M_5 into the c.e. Turing degrees constructed in Lachlan [7]. If an element x is targeted at stage s to go into one of the sets A or B and x does not enter its target set by stage $s+1$, then a trace $T(x)$ is assigned to it at $s+1$ targeted for the opposite set. For each x this gives rise to a finite sequence of traces targeted alternately for A and B which must enter the appropriate sets (in reverse order to that in which they were chosen) before x is allowed to enter its target set.

This strategy appears to clash very badly with the minimal pair restraints, since many elements will be chosen as traces between successive expansionary stages for a particular negative requirement. Of course, this is just what makes it impossible to combine this kind of tracing procedure with the analogous minimal-pair-type requirements for infima in the Turing degrees. Here, since we are using Q-reductions, we can use the fact that the actual use of a Q-reduction at any given stage consists of at most one element. By picking many traces at each stage for each untraced element, we will guarantee that some sequence of traces produced by a follower for a positive requirement will eventually be allowed past the higher priority restraints for negative requirements.

The most convenient pattern for a priority argument combining a tracing procedure with infimum-preserving strategies is the pinball machine method introduced in Lerman [8]. Unlike the simpler construction of an ordinary minimal pair of Turing degrees, a whole sequence must in general be allowed to pass the restraint imposed by a given negative requirement at once, but then be forced to wait to pass the next higher-priority restraint one element at a time (with appropriate traces). The reason for this is that our trace-choosing procedure only avoids the uses on which the negative requirement depends for one negative requirement at a time. This will become clear in the verification below.

The machine we use has a sequence of holes, H_e, one for each positive requirement, and a sequence of gates, $G_{\langle a,b,c\rangle}$ each with a corral $C_{\langle a,b,c\rangle}$, one for each negative requirement. In order to control the operation of the gates, we define the length of agreement functions and the notion of expansionary stages as follows:

We let $l(\langle a,b,c\rangle, s)$ be the greatest y such that

$$\forall x < y((W_{\phi_a(x)} \subseteq A \Leftrightarrow W_{\phi_b(x)} \subseteq B, \text{ and } (W_{\phi_a(x)} \subseteq A \Leftrightarrow x \in W_c))[s].$$

As usual, we let $m(\langle a,b,c\rangle, s) = \max\{\, l(\langle a,b,c\rangle, t) : t < s \,\}$, and call s $\langle a,b,c\rangle$-*expansionary* if $l(\langle a,b,c\rangle, s) > m(\langle a,b,c\rangle, s)$. In general, even if $W_c \leq_Q A$ via ϕ_a and $W_c \leq_Q B$ via ϕ_b, this will not guarantee infinitely many $\langle a,b,c\rangle$-expansionary stages; however, by Lemma 1 above, there will exist some a', b', and c' such that $W_c = W_{c'}$, and there are infinitely many $\langle a',b',c'\rangle$-expansionary stages. In this case, the $N_{\langle a',b',c'\rangle}$-strategy will ensure W_c is computable.

Construction

Stage 0: $A_0 = B_0 = \emptyset$.

Stage $s+1$: A requirement P_e is said to *need attention at stage s* if one of the following conditions holds:

(1) P_e has no current follower waiting at hole H_e and there is no $x \in A[s]$ such that $\phi(x)[s] = 0$.

(2) P_e has a current follower x waiting at hole H_e, and $\phi(x)[s] = 0$.

(3) Some element y descended from a follower for P_e is waiting at a gate $G_{\langle a,b,c\rangle}$ and s is a $\langle a,b,c\rangle$-expansionary stage.

We take action for P_e where $e < s$ is the least requirement P_e that needs attention at stages s. We initialize all $P_{e'}$, where $e' > e$, by undefining all any followers and sets of traces associated with $P_{e'}$. If P_e needs attention under case (1) above, we appoint a new follower $x(e,n,s+1)$ greater than any number yet mentioned in the construction, where n is the least number such that $x(e,n,s)\uparrow$. If P_e needs attention under case (2), we allow x and its entire trace sequence to drop to the first gate which is unoccupied, we place the last member of the sequence, $T^k(x)$, where $k+1$ is the length of the sequence, at the gate, and put the rest of the sequence into the corral. If P_e needs attention under case (3), the situation is a little more complicated, since we must insure that the trace sequence descended from y causes a minimal amount of damage to the negative requirement associated with G. First, note that y is a member of a trace sequence associated with some $x(e,n,s)$. For all $n' > n$, we first undefine all followers $x(e,n',s)$ and their associated trace sequences. Next, let s^- be the stage at which y came to rest at gate G. Suppose $G = G_{\langle a,b,c\rangle}$. By the trace assignment procedure described below, y is actually the root of a (finite) tree of traces, which has at least an $s^- + 1$-fold branching at every level. If x is any number less than $l(\langle a,b,c\rangle, s)$, then either $x \in W_c[s]$, or there exist both a unique $a(x,s) \in (W_{\phi_a(x)} - A)[s]$ and a unique $b(x,s) \in (W_{\phi_b(x)} - B)[s]$. Clearly $|\{\, a(x,s) : x < s^- \,\}| \leq s^-$, and similarly for $\{\, b(x,s) : x < s^- \,\}$, so that there exists a path through y's tree of traces that avoids these sets in the sense that no trace targeted for A in the path is equal to any $a(x,s)$ for $x < s^-$, and similarly for traces targeted for B. We pick the lexicographically least such path, $y = y_0 < y_1 < \ldots < y_m$ and assign it as the trace sequence for y, defining $T(y_l, s+1) = y_{l+1}$ for each $l < m$. We then allow y and this entire trace sequence to drop to the first unoccupied gate, discarding all the other potential trace-descendents of y in the tree. As under case (2), we put the entire sequence associated with y except for the last element, into the corral, and we place the last element at the gate. If there is no unoccupied gate, then we put y and its associated trace sequence into the relevant target sets. If y

was actually a follower for requirement P_e, we are done. Otherwise, $y = T(y', s)$ for some y' associated with P_e. This y' is waiting in a corral by some gate G', a fact which is straightforward to check by induction. We remove y' from the corral and place it at gate G'.

Whether or not any action has been taken for any P_e at stage $s+1$, we need to assign potential traces to any elements waiting at a gate or hole at the end of stage s. We do this recursively in increasing order for all such y. If y is waiting at a hole and y has no trace assigned to it, then we assign the least number not yet mentioned in the construction to y as $T(y, s+1)$. If y is waiting at some gate and y has no set of potential traces assigned to it, we assign the least $s+1$ numbers not yet mentioned in the construction as potential traces for y, from which we will choose a unique trace if y is ever freed from the gate.

This completes the construction.

Verification:

Satisfaction of the positive requirements is straightforward to check, as is the fact that each positive requirement needs attention at most finitely often. To see that $A \leq_T B$, note that any y not eventually bypassed as a potential witness, either enters its target set on the stage immediately after it is chosen, or receives a trace $T(y, s)$ at this stage or a set of potential traces. If none of y's potential traces enter B, then y cannot enter A. If one does, then on the immediately following stage, y is again assigned a trace or set of traces if it has not entered A. This process eventually must come to an end, either with y and some final set of potential traces stuck at some gate or corral, or with y being cancelled by a higher priority requirement, or with y entering A by the stage immediately after the stage at which some trace for y enters A.

Satisfaction of the negative requirements is a little more delicate to verify. Suppose $W_c \leq_Q A$ via ϕ_a and $W_c \leq_Q B$ via ϕ_b. We can assume that the reductions have the properties described in Lemma 1 by merely replacing a, b, and c with some indices that do have these properties with respect to A and B. This guarantees infinitely many $\langle a, b, c\rangle$-expansionary stages, since W_c is an c.e. set, and hence Δ_2, so that it must settle down on each initial segment. Let $x \in \omega$. Call a stage s *$\langle a, b, c\rangle$-good for x* if

(1) $x < s$,

(2) All gates and corrals G_p and C_p with $p \leq \langle a, b, c\rangle$ have only permanent residents, or else are unoccupied,

(3) No P_e for $e < \langle a, b, c\rangle$ receives attention after stage s, and

(4) $l(\langle a, b, c\rangle, s) > x$.

First note that there are infinitely many such stages, since (1), (3) and (4) happen cofinitely often, and (2) must happen infinitely often, in particular at every stage at which some highest priority element already in the machine acts for the last time, thereby undefining all non-permanent residents. Also, any gate and corral which has permanent residents must in fact have some least permanent resident at its gate with which all but finitely many of its permanent residents are associated. Also, any resident associated with this element must be permanent, since it is stuck at the gate permanently. This implies that it is computable to find an infinite sequence of $\langle a, b, c\rangle$-good stages for x. Let s_0 be the first such stage, and s_1 be the second. We claim that $x \in W_c$ if and only if $(W_{\phi_a(x)} \subseteq A)[s_1]$. Since x is below the length of agreement at s_1, clearly, if $(W_{\phi_a(x)} \subseteq A)[s_1]$, then $x \in W_c$, so we need only worry about the case where $(W_{\phi_a(x)} \nsubseteq A)[s_1]$. We claim that at every stage $s > s_1$, either $(W_{\phi_a(x)} \nsubseteq A)[s]$ or $(W_{\phi_b(x)} \nsubseteq B)[s]$. Since $W_c \leq_Q A$ via ϕ_a, this implies $x \notin W_c$ by Lemma 1, since there are infinitely many stages at which the Q-reduction from B agrees with that from A on x. So let s be the least stage at which our claim fails, and let s^- be the greatest $\langle a, b, c\rangle$-good stage for x less than s. Since s^- was good, $G_{\langle a,b,c\rangle}$ must have have been unoccupied at the end of this stage. We need to show that either $a(x) = a(x, s^-) \notin A_s$, or $b(x) = b(x, s^-) \notin B_s$. So suppose this is false.

Without loss of generality, suppose $a(x)$ entered A first. Then $b(x)$ must be associated with a requirement of at least as great a priority as $a(x)$, since otherwise $a(x)$'s movement would cause $b(x)$ to become undefined. Let s_a be the stage at which a was chosen, and s_a^+ be the stage at which $a(x)$ entered A. Clearly, $s_a \leq s^- < s_a^+$. Let s_a^* be the stage at which $a(x)$ passed gate G. Note that $s_a^* > s^-$, since all the gates and corrals G_l and C_l with $l \leq \langle a, b, c\rangle$ have only permanent residents. If gate G had a higher-priority resident at s_a^*, then this resident would have destroyed $a(x)$ when it moved. Clearly, $a(x)$ would have destroyed any lower priority resident: so $a(x)$ itself was waiting at the gate. This also implies that $b(x)$ has at least as high a priority as $a(x)$. Clearly, $a(x)$ is the first element of its trace sequence at the gate, since $x < s^-$, and $b(x)$ cannot be a member of $a(x)$'s trace sequence, by choice of a safe path through $a(x)$'s potential witness tree. But now $b(x)$ must get stuck at G after $a(x)$ has already passed, since it must be the highest priority element that moves between s_a^* and s_b^*, else it would become undefined. Of course, if $a(x)$ is a descendent of $b(x)$, $b(x)$ cannot move until $a(x)$ enters A anyway. In short, $b(x)$ is above gate G at stage s_a^+, and it is the highest priority element that moves between $s_b < s_a$ and s_b^*. This implies that $b(x)$ must wait at gate G at stage s_a^+, and so it cannot enter B until after the next $\langle a, b, c\rangle$-expansionary stage, which is greater than s. This establishes the result, since it shows that the negative requirements are satisfied. □

Although 0 branches in the c.e. Q-degrees, it is not immediately obvious that non-branching c.e. Q-degrees exist, a fact which we now prove by constructing such a degree avoiding any nontrivial upper cone.

Theorem 6. *For every c.e. set $C \not\equiv_Q 0$, there exists an c.e. set A, $C \nleq_Q A$, such that A is non-branching in the c.e. Q-degrees.*

Proof. We must construct A to satisfy the following two infinite types of requirements:

$$N_a : \ C \nleq_Q A \text{ via } \phi_a \text{ and}$$

$$P_{\langle V,W\rangle} : (V \not\leq_Q A \wedge W \not\leq_Q A) \Longrightarrow \\ \exists B(B \leq_Q (V \oplus A) \wedge B \leq_Q (W \oplus A) \wedge B \not\leq_Q A).$$

Where $\langle V, W\rangle$ is some enumeration of the c.e. sets, and $\{ \phi_a : a \in \omega \}$ is the standard enumeration of partial computable functions. The requirements N_a are satisfied by the usual Sacks preservation strategy. We would ordinarily define the maximum length of agreement at stage s to be

$$\mu y(\phi_a(y)\uparrow \vee (y \notin C \wedge W_{\phi_a(y)} \subseteq A) \vee (y \in C \wedge W_{\phi_a(y)} \not\subseteq A))[s],$$

and attempt to restrain A from changing on the sets involved in the partial reduction below this length. For Q-reductions in general, it may be that at every stage s, some $y \in C[s]$, yet $(W_{\phi_a(y)} \not\subseteq A)[s]$, although $W_{\phi_a(y)} \subseteq A$. Because we cannot know in advance whether or not $C \leq_Q A$ via ϕ_a, we cannot use the procedure of Lemma 1 to keep elements out of $W_{\phi_a(y)}$ merely because y has entered C. After all, for us, the next element of $(W_{\phi_a(y)} - A)[s]$ may be the witness we are seeking that ϕ_a is not a Q-reduction. We can slow down the enumeration of $W_{\phi_a(y)}$ as in Lemma 1, by demanding that $(W_{\phi_a(y)} - A)[s]$ consists of at most one element at any stage, and that a stage at which it is empty must intervene before a new element is enumerated; such a modified function will have the same properties as the original ϕ_a whether or not ϕ_a really threatens $C \leq_Q A$. Unfortunately, even with this modified version of ϕ_a, we are only guaranteed that ϕ_a will appear correct infinitely often at every argument if $C \leq_Q A$ via ϕ_a. In the case of a Turing reduction, we are automatically guaranteed something much stronger, for if $C = \Phi(A)$, then, for every x, $(C(x) = \Phi(A; x))[s]$ at cofinitely many stages s. Because this guarantees that the lim sup of the length of agreement approaches ∞, we can use the notion of expansionary stages to control our constructions. Fortunately, we can also achieve this in the case of Q-reducibility, although it means we are forced to use a device like the Soare-Lachlan hat-trick even in what is essentially a finite injury construction. Let a_s be the least element enumerated into A at stage s (or s, if $A_s - A_{s-1} = \emptyset$). A stage s is A-true if $A_s \upharpoonright a_s = A \upharpoonright a_s$. Since A will be c.e., there will be infinitely many A-true stages. If $C \leq_Q A$ via ϕ_a, then at any A-true stage, if $\phi_a(y)\downarrow$ and $y \in C$, then $W_{\phi_a(y)} \upharpoonright a_s \subseteq A_s$. Defining the length of agreement function to be

$$l^C(a, s) = \mu y(\phi_a(y)\uparrow \vee (y \notin C \wedge W_{\phi_a(y)} \upharpoonright a_s \subseteq A) \vee \\ (y \in C \wedge W_{\phi_a(y)} \upharpoonright a_s \not\subseteq A))[s],$$

we, therefore guarantee that $\limsup_{s\to\infty} l^C(a, s) = \infty$, if $C \leq_Q A$ via ϕ_a.

For each requirement $P_{\langle V,W\rangle}$, we actually construct $B_{\langle V,W\rangle}$ to satisfy the following infinite list of subrequirements

$$P_{\langle V,W,e\rangle} \qquad B_{\langle V,W\rangle} \not\leq_Q A \,\text{via}\, \phi_e.$$

The strategy is derived from the familiar one for constructing a nonbranching c.e. Turing degree (see [12], IX.4.2), namely, the failure to satisfy requirement $P_{\langle V,W\rangle}$ implies that at least one of V and W is Q-reducible to A. We need to introduce some modifications in order to deal with the differences between Q-reducibility and T-reducibility. In what follows, we consider the subrequirement tied to a fixed V, W, and e and write B for $B_{\langle V,W\rangle}$. Suppose ϕ_e is a total computable function. To avoid the lower cone, we pick a marker $x \notin B$ and wait for a stage s at which the length of agreement of B and the set Q-reduced via ϕ_e to A has grown beyond x. Since $x \notin B$, this can only mean that $(W_{\phi_e(x)} - A)[s] = \{a(x,s)\}$, by our convention about the cardinality of this set derived from Lemma 1. Now we add x to B and restrain A on $\{a(x,s)\}$. Of course, we have to somehow force $V \oplus A$ and $W \oplus A$ to permit x to enter B after this stage. In the Turing case, this is relatively straightforward, since we can threaten to reduce V to A on longer and longer initial segments, and thus get an increasing infinite subsequence $x_{v(i)}$ of markers on which permission from V is given, then using this infinite subsequence, we can threaten to reduce more and more of W to A by considering the initial segment bounded by each $x_v(i)$ and thereby force W to give permission on at least one of these markers. The problem in adapting this to the Q-degree context lies in the fact that the apparent "use" of a Q-reduction at some fixed stage is too small to force changes in this simple way, since it is either the empty set (when $W_{f(x)} \subseteq A$), or a single element (when $W_{f(x)} \not\subseteq A$).

We get around this problem by doubly indexing our markers for B, to get whole blocks of potential uses, rather than just the isolated points we would otherwise get. We start out by defining $b(i,j) = \langle e,i,j\rangle$, where e is the index of f. This is just the $\langle i,j\rangle$th element of $\omega^{[e]}$. Because B is to be Q-reducible to $V \oplus A$ and $W \oplus A$, we must at least preliminarily define the Q-reductions g_V and g_W that are intended to achieve this. So, for each i and j we define $W_{g_V(b(i,j)),0} = \{2i\}$ and $W_{g_W(b(i,j)),0} = \{2j\}$. Again, we must use the true stage method to guarantee enough expansionary stages. As in the case of the cone-avoidance requirements, we therefore define

$$l(e,s) = \mu y(\phi_e(y)\uparrow \vee (y \notin B \wedge W_{\phi_e(y)} \restriction a_s \subseteq A) \vee (y \in B \wedge W_{\phi_e(y)} \restriction a_s \not\subseteq A))[s].$$

To construct our reductions, we also need to have available some element $\bar{a}$ which we agree will never enter A under any circumstances. In fact, we simply pick $\bar{a} = 0$, but continue to write $\bar{a}$, when this special fact about it is what is relevant.

Construction:

Stage 0: $A_0 = \emptyset$, $B_{\langle V,W\rangle,0} = \emptyset$. All markers $b(i,j)$ are said to be *waiting for a set-up.*

Stage $s+1$: For each $a \in \omega$, we define the restraint set for N_a at stage s by $R^a[s] = \{\, z \in (W_{\phi_a(y)} - A)[s] \,:\, \exists t \leq s(y \leq l^C(a,t)) \,\}$. These are the elements which the strategy for N_a would like to keep out of A. The maximum length of agreement below stage s has to be used in setting these restraints, since even if $C \leq_Q A$ via ϕ_a, $\liminf_{s\to\infty} l^C(a,s)$ can still be finite.

The strategy for positive requirements is a little more involved. To avoid the introduction of excessive subscripts, we write $b(i,j)$ for $b_{\langle V,W,e\rangle}(i,j)$; similarly for $B_{\langle V,W\rangle}$ and any auxiliary functions and parameters below. To threaten $V \leq_Q A$ and $W \leq_Q A$, we define auxiliary partial computable functions f_0 and f_1. For each $\langle V,W,e\rangle < s$, we treat each $b(i,j) < s$ assigned to $P_{\langle V,W,e\rangle}$ as follows, taking the first case that applies:

Case 1: $b(i,j)$ is waiting for a set-up.

(a) If $i \in V_s$ we define $W_{f_0(i)} = \emptyset$, $W_{g_V(b(i,j)),s+1} = W_{g_V(b(i,j)),s} \cup \{2\bar{a}+1\}$, and $W_{g_W(b(i,j)),s+1} = W_{g_W(b(i,j)),s} \cup \{2\bar{a}+1\}$. In this case, $b(i,j)$ will never be used as a potential candidate for $B \not\leq_Q A$, so we say $b(i,j)$ is *permanently cancelled at stage* $s+1$.

(b)If $j \in W_s$ we define $W_{f_1(j)} = \emptyset$, $W_{g_V(b(i,j)),s+1} = W_{g_V(b(i,j)),s} \cup \{2\bar{a}+1\}$, and $W_{g_W(b(i,j)),s+1} = W_{g_W(b(i,j)),s} \cup \{2\bar{a}+1\}$. Again, $b(i,j)$ is permanently cancelled at stage $s+1$.

(c) $l(e,s) > b(i,j)$ (and neither $i \in V_s$ nor $j \in W_s$), then we define $W_{f_0(i),s+1} = W_{f_0(i),s} \cup \{a(b(i,j),s)\}$, the unique element of $(W_{\phi_e(b(i,j))} - A)[s]$. In this case, we say $b(i,j)$ is *waiting for V-permission.*

Case 2: $b(i,j)$ is waiting for V-permission.

(a) If $a(b(i,j),s) \in A_{s+1}$, then $b(i,j)$ returns to waiting for a set-up.

(b) If $i \in V_s$, then we define $W_{f_1(j),s+1} = W_{f_1(j),s} \cup \{a(b(i,j),s)\}$, we choose some $a_{i,j}$ bigger than any number yet mentioned in the construction, and set $W_{g_V(b(i,j)),s+1} = W_{g_V(b(i,j)),s} \cup \{2a_{i,j}+1\}$. In this case, we say $b(i,j)$ is *waiting for W-permission.*

Case 3: $b(i,j)$ is waiting for W-permission.

(a) $a(b(i,j),s) \in A_{s+1}$, then we permanently cancel $b(i,j)$ at stage $s+1$, set

$$W_{g_V(b(i,j)),s+1} = W_{g_V(b(i,j)),s} \cup \{2\bar{a}+1\}, \text{ and}$$
$$W_{g_W(b(i,j)),s+1} = W_{g_W(b(i,j)),s} \cup \{2\bar{a}+1\}.$$

(b) $j \in W_{s+1}$, and $a_{i,j} \notin R[s]$ for any restraint R of higher priority, then we add $b(i,j)$ to B_{s+1} and $a_{i,j}$ to A. We put $a(b(i,j),s) \in R^{\langle U,V,e\rangle}_{s+1}$, the restraint set for $P_{\langle U,V,e\rangle}$. In this case, we say $P_{\langle U,V,e\rangle}$ *appears satisfied at stage* $s+1$, and we never again add elements to either A or B for the sake of this requirement unless $a(b(i,j)) \in A_t$ at some stage $t > s+1$. We may have to correct some functions g^V and g^W at later stages when elements enter V or W and permanently cancel potential witnesses in order to keep $B \leq_Q V \oplus A$ and $B \leq_Q W \oplus A$, but this action does not cause any enumeration into the sets A and B.

This ends the construction.

Verification:

We prove by induction that each restraint set is finite, that all requirements are satisfied, and that each requirement only adds finitely many elements to A. So, suppose this is the case for all N_a and $P_{\langle V,W,e\rangle}$ where a and $\langle V,W,e\rangle$ are less than b. The requirement N_b is satisfied as usual with the Sacks agreement strategy: we give the details only for completeness sake in this possibly unfamiliar context. Since only finitely many elements are added to A by any of these higher priority requirements, we must have some stage s_0 after which no more elements are added to A by any of these requirements. If $C \leq_Q A$ via ϕ_b, then because N_b has higher priority than any requirement seeking to add elements to A, a straightforward induction shows that $C \leq_Q \emptyset$ via any computable function g such that

$$W_{g(x)} = \{1\} \Longleftrightarrow \exists s > s_0 (W_{\phi_b(x)} \not\subseteq A)[s].$$

Hence ϕ_b fails to Q-reduce C to A. Let x be the least element at which ϕ_b fails. If $\phi_b(x)\uparrow$, then the restraint set for N_b is obviously finite, since it can have at most x elements at any stage after s_0. If $\phi_b(x)\downarrow$, then, if $x \in C$, we must have $W_{\phi_b(x)} \not\subseteq A$, and this can only be because some element $a(x)$ is permanently restrained from A by N_b at some stage $s > s_0$ such that $x \in C_s$. Again, this clearly leads to a finite restraint set. If $x \notin C$, then our strategy guarantees that once $l^C(b, s) \geq x$ permanently, $s \geq s_0$ implies $(W_{\phi_b(x)} \subset A)[s]$, since otherwise we would force ϕ_b to be correct at x by setting a restraint.

Assume the inductive hypothesis for all requirements below $P_{\langle V,W,e\rangle}$, and assume we are past some stage s_0 such that all the higher priority requirements have finished setting restraints and acting by stage s_0. The discussion is simplified if we assume $s_0 = 0$ in what follows. By merely ignoring the finite part of each set that is stable before s_0, the reader can see that this involves no loss of generality. If any element is added to A at some stage s for the sake of this requirement, it can only be some $a_{i,j}$ added under Case $3b$ to record that $b(i,j) \in B \leq_Q V \oplus A$. Since $P_{\langle V,W,e\rangle}$ is the highest priority requirement remaining, $a(b(i,j), s)$ never enters A, so that no further action is ever taken for the sake of this requirement. Thus $P_{\langle V,W,e\rangle}$ only adds finitely many elements to A (and B). Clearly, the restraint set is also finite.

Now, suppose $V \not\leq_Q A$ and $W \not\leq_Q A$. Suppose $B \leq_Q A$ via ϕ_e. Then, by what was just argued, no $b_{i,j}$ can enter B after stage s_0. Fix $j \in \omega$. Then there must be some infinite subsequence of the form $b(v(i), j)$ such that each $b(v(i), j)$ eventually is waiting for W-permission. If not, then we compute $V \leq_Q A$ via g defined as follows: Since there is some i_0 such that all $b(i,j)$ with $i > i_0$ never wait for W-permission, we can obviously ignore this finite piece of V. (In other words, the definition of g is automatic there.) For all $i > i_0$, let $W_{g(i)} = W_{f_0(i)}$. Notice that f_0 is total, since $B \leq_Q A$ via ϕ_e. If $i \notin V$, then, since $b(i,j) \notin B$, we must have some $a(b(i,j)) \in W_{\phi(e)} - A$. But then, $a(b(i,j)) \in W_{f_0(i)}$ as well. On the other hand, if $i \in V$, then $W_{f_0(i)} \subseteq A$, since otherwise, $b(i,j)$ waits for W-permission after any stage s where $a(b(i,j), s) = a(b(i,j)) \notin A_s$ and $i \in V$. This infinite sequence implies that f_1 is total. If $j \notin W$, then, since for almost all i, $b(i,j) \notin B$,we must have $a(b(i,j)) \notin A$ for some i such that $b(i,j)$ is waiting for W-permission. Then $a(b(i,j)) \in W_{f_1(j)} - A$ by construction. On the other hand, suppose $j \in W$. Then $j \in W_s$ at some least stage $s > s_0$, and, at any $t > s$, $W_{f_1(j),t} = W_{f_1(j),s}$. But any $a(b(i,j)) \in W_{f_1(j),s}$ must enter A before stage s, since otherwise $b(i,j)$ is added to B and the requirement is permanently satisfied by restraining $a(b(i,j))$ from A at every stage thereafter. But this contradicts $W \not\leq_Q A$, so the requirement is satisfied.

Notice that $B \leq_Q V \oplus A$ via g^V and $B \leq_Q W \oplus A$ via g^W, since we always correct these functions whenever they become incorrect, and this only happens finitely often for any $b(i,j) \notin B$, since if i or j enters V or W respectively while $b(i,j)$ is waiting for a set up, $b(i,j)$ is permanently cancelled, and $2\bar{a}+1$ is enumerated into the relevant reduction sets.

This establishes the result. □

4 Density

Now we turn to the more difficult task of establishing the density of the c.e. Q-degrees. The fact that Q-reductions are given via computable functions rather than relatively computable ones causes extra technical difficulties in carrying over the standard Sacks coding technique to this context.

Theorem 7. *For every pair of c.e. sets $B <_Q A$, there exists an c.e. set C with $B <_Q B \oplus C <_Q A$.*

Proof. As in the case of the Turing degrees, we must construct C to satisfy the following two infinite sequences of requirements:

$$N_a : \ A \not\leq_Q B \oplus C \,\text{via}\, \phi_a \,\text{and}$$

$$P_e : \ C \not\leq_Q B \,\text{via}\, \phi_e.$$

Each of these requirements gives rise to new problems in the context of Q-reducibility. N_a will be met by restraining C whenever possible, in the manner of the similar requirements N_a of the previous theorem. Again, because Q-reductions involve a single element, rather than an initial segment of ω, it is not immediately possible to give up all higher restraints whenever the length of agreement between A and the apparent Q-reduction from $B \oplus C$ given via ϕ_a drops back. The potential for infinite injury by B from below makes this situation worse than that in the non-branching construction above, since keeping all restraints forever will clearly make it impossible to satisfy the positive requirements below N_a. The solution is to give up restraints only when the length of agreement drops back because of the one truly infinitary outcome: that there is some least $z \notin A$, yet $W_{\phi_a(z)} \subseteq (B \oplus C)$, and every member of $W_{\phi_a(z)}$ enumerated after N_a has highest priority is even.

P_e will be met by a modification of the Sacks coding strategy. We appoint a sequence of *followers* and an auxiliary computable function h with the property that if ϕ_e is a correct Q-reduction for C on this sequence, then $A \leq_Q B$ via h. The feature that makes this situation different from the Turing degree case is that we cannot immediately give up followers which have already been appointed whenever action is taken for some previously-appointed follower. Since the function h is required to be computable in this case, rather than merely C-computable, as in the analogous argument for Turing reducibility, once we enumerate an element y into $W_{h(j)}$ for the sake of an attempt tied to a particular follower, we are forced to use some follower tied to y as our coding marker for the jth attempt until y enters B. (Since h continues to Q-compute $j \notin A$ until

such a stage is reached.) On the other hand, hanging on to markers permanently whenever they are appointed will make it impossible for lower priority negative requirements to make good on their threat to Q-reduce A to B. The solution is to add so-called tokens to any follower which we are currently prevented from giving up at some stage, even though we would like to do so then. This is intended to mark that we hope to give the follower up later (unless some further action changes this situation), thereby insuring for the sake of lower priority requirements that we will give up higher priority followers whenever we can.
Construction:

We use a tree of strategies to construct C. This is by now the standard approach; we refer the reader who is unfamiliar to Soare, [12], chapter XIV, for the notation and basic ideas involved in such constructions. Our tree of strategies T is isomorphic to ${}^{<\omega}\omega$. We assign N_a to each node of length $2a$ and P_e to each node of length $2e+1$. Even nodes have outcomes $n \in \omega$ ordered in the usual way, representing the least number at which ϕ_a fails to Q-reduce A to $B \oplus C$. We let U and D be new symbols, and allow odd-length nodes to have outcomes in $n \times \{U, D\}$, ordered lexicographically by the rule $U < D$. These represent the possible outcomes of each positive strategy: either P_e will be satisfied by some follower x appointed for n but never added to C, possibly because the use $W_{\phi_e(x)}$ is unbounded ($\langle n, U\rangle$), or the requirement will be satisfied by achieving diagonalization against ϕ_e on some follower for n ($\langle n, D\rangle$). It is essential for our Q-reduction that we distinguish these outcomes on the tree, since each gives different information about why elements are blocked from entering C.

The construction, as is usual in tree arguments, proceeds in stages at which the strategies assigned to a node α are only employed when α appears to have the correct information about how higher priority requirements are satisfied. To affect this, an approximation f_s to the true path f through the tree is defined at each stage s. Any stage s at which $\alpha \subset f_s$ is called an α-stage. The strategies employed by each $\alpha \in T$ to satisfy its requirement are controlled through length of agreement functions defined on these stages. As in the non-branching degree construction, we guarantee that the length grows enough by the true stage method. If s is an α-stage, we define d_s^α to be the least element enumerated into $B \oplus C$ between stage s and the last previous α-stage (or stage 0 if s is the first α-stage. We let $d_s^\alpha = s$ if no element was enumerated throughout this period. For each even-length α, say with $|\alpha| = 2a$, we define

$$\begin{aligned} l^\alpha(s) = \mu y(\phi_a(y)\uparrow \text{ or } \quad & (y \notin A \text{ and } W_{\phi_a(y)} \restriction d_s^\alpha \subseteq B \oplus C) \text{ or} \\ & (y \in A \text{ and } W_{\phi_a(y)} \restriction d_s^\alpha \not\subseteq B \oplus C))[s], \end{aligned}$$

just as in the manner of the negative requirements in the non-branching result above. Each even-length α also has restraint sets $R^\alpha(s)$ defined at each stage of the construction.

Each odd-length α has a potentially infinite sequence of followers, to be used as coding markers, which are given by a partial computable function $x^\alpha(j, s)$. Let b_s^α be the least element enumerated into B since the last previous α-stage, or s if no element has been enumerated in this period. For the length of agreement function at α, if $|\alpha| = 2e+1$, we would ordinarily use the definition

$$\begin{aligned} l^\alpha(s) = \mu y(\phi_e(y)\uparrow \text{ or } \quad & (y \notin C \text{ and } W_{\phi_a(y)} \restriction b^\alpha \subseteq B) \text{ or} \\ & (y \in C \text{ and } W_{\phi_a(y)} \restriction b^\alpha \not\subseteq B))[s], \end{aligned}$$

for our length of agreement function. Unfortunately, there are two problems with this. The first is easy to address, and arises from the need to use something like the hat-trick here, so actually α has a "slowed-down" version of ϕ_e, with the property that if if $s_0 < s$ are two immediately successive α-stages, and $W_{\phi_a(z),t} \subseteq B_s$ for any t with $s_0 < t \leq s$, then $W_{\phi_a(z),s} \subseteq B_s$. This is clearly possible by a slight modification of Lemma 1 above, and it ensures that $l^\alpha(s)$ will drop back correctly.

The second problem arises from the fact that Q-reductions have an "intermittent" character as compared with Turing reductions. For suppose $X \leq_T Y$, say $X = \Phi(Y)$. Then we can assume that the use is increasing in the argument and non-decreasing in the stage, so that if $\Phi(Y)$ appears correct through $y > x$ at stage s, yet fails to appear correct for x at some $s' > s$ because of a change in Y, new computations must appear for all the elements between y and x by the time $\Phi(Y;x)$ is restored to correctness at stage $t > s'$. The fact that this fails in the analogous situation for Q-reducibility leads to an extra problem, since we wish to ensure failure at some permanent coding marker for requirement P_e, in order for us to be able to achieve $C \leq_Q A$. We therefore restrict our length of agreement to merely checking the sequence of markers defined at a given stage, rather than every number, defining $l^\alpha(s)$ to be the least y such that for some j, $y = x^\alpha(j,s)$, y is not yet permanently stable (see below), and

$$\begin{aligned} & \phi_e(y)\uparrow \text{ or } (y \notin C \text{ and } W_{\phi_a(y)} \restriction b^\alpha \subseteq B) \text{ or} \\ & \qquad (y \in C \text{ and } W_{\phi_a(y)} \restriction b^\alpha \not\subseteq B))[s], \end{aligned}$$

if any such y exists, otherwise $l^\alpha(s)$ is $x^\alpha(j,s)+1$, where $x^\alpha(j,s)$ is the greatest marker yet defined. Now we are assured that the next marker $\geq l^\alpha(s)$ is the number on which we plan to achieve $C \not\leq_Q B$ via ϕ_a. As usual, we define a stage t to be α-expansionary if $\forall t' < t(l^\alpha(t') < l^\alpha(t))$.

As in the result above, we also choose some $\bar{b} \notin B$, in order to have an element which is permanently restrained from B available in defining our Q-reductions. As usual a node is *(re-)initialized* by undefining all parameters associated with it and setting its restraint set to be the empty set.

Stage 0: $C_0 = \emptyset$, and initialize all $\alpha \in T$. For every even-length $\alpha \in T$, let $R^\alpha(-1) = \emptyset$.

Stage $s+1$: We define the current approximation to the true path f_s recursively for each $n < s$, and take action for each node $\alpha = f_s \restriction n$. If $|\alpha| = 2a$, then we define the maximum length of agreement function $m^\alpha(s) = \max\{\, l^\alpha(t) : t < s \,\}$. If $l^\alpha(s) < m^\alpha(s)$ and there is some least $y_0 < m^\alpha(s)$ such that $y_0 \notin A$ yet $(W_{\phi_a(y_0)} \subseteq (B \oplus C))[s]$, then let let $R^\alpha(s) = \{\, z : 2z+1 \in (W_{\phi_a(y)} - (B \oplus C))[s] \text{ and } y \leq y_0 \,\}$ and set $f_s(n) = y_0$. This is the potentially infinite outcome which causes the restraint set to drop back. Otherwise, let $R^\alpha(s) = \{ z : 2z+1 \in (W_{\phi_a(y)} - (B \oplus C))[s] \text{ and } y \leq m^\alpha(s) \}$ and set $f_s(n) = m^\alpha(s)$.

If $|\alpha| = 2a+1$, then the situation is more complicated. If $x^\alpha(0,s)$ is undefined, then choose $x^\alpha(0,s)$ to be the least number greater than any yet mentioned in the construction.

First we check to see whether we already appear to have achieved a win through diagonalization. Let s^- be the greatest α-stage less than s, If α received outcome $\langle k, D\rangle$ at stage s^- and $b(x^\alpha(k,s)) \notin B_s$, then $\alpha^\frown\langle k, D\rangle \subseteq f_s$ and we take no action for α at stage s.

Otherwise, we must continue to try to satisfy P_e at α. If s is α-expansionary, then there exists some least j such that $x^\alpha(j+1,s)$ is undefined, choose $x^\alpha(j+1,s)$ to be the least number greater than any yet mentioned in the construction. If $j \notin A_s$, we declare $x = x^\alpha(j,s)$ to be *waiting for A-permission.* If $j \in A_s$, then we declare x to be *permanently stable.* This means that at no stage $t \geq s+1$ will $x(j,t)$ become undefined, so no new markers for j ever appear. We repeat this procedure for any $j' < j$ which is not already waiting for A-permission.

If some $x^\alpha(k,s)$ is waiting for A-permission, and $k \in A_s$, then we pick the least such k with $x^\alpha(k,s) \notin \bigcup\{ R^\beta[s] : \beta < \alpha \}$ and enumerate $x^\alpha(k,s) \in C_{s+1}$. We let α have outcome $\langle k, D\rangle$ in this case, and we say k is *waiting for a B-change.* For all $j > k$ such that $x^\alpha(j,s)\downarrow$ and j has not been permanently cancelled, we assign $x^\alpha(j,s)$ a token, and declare $x^\alpha(j,s)$ to be no longer waiting for A-permission. If no such k exists, we enumerate nothing into C. If $x^\alpha(j,s)$ began waiting for a B-change at some stage $t \leq s$ and $b(x^\alpha(j,t),t) \in B_s$, then we declare $x^\alpha(j,s)$ permanently stable, never again allowing $x^\alpha(j,s)$ to become undefined.

We may still have to take some actions to reduce the effect of α on lower priority nodes, and to ensure that all the Q-reductions later to be associated with α will be correct. We would like to immediately discard all markers above the first one below which ϕ_e appears to fail. If $l^\alpha(s) < x^\alpha(j-1,s)$, and $b^\alpha(j,s) \notin B_{s+1}$, then, as explained above, we cannot immediately discard $x^\alpha(j,s)$. Also, if we have been preserving α's outcome because of an apparent diagonalization, we wish have as outcome the least k with a marker for which ϕ_e appeared to fail throughout this whole period. Let s^* be the greatest α-stage before s at which α's outcome was not $\langle l, D\rangle$ for any l. We search for the least k such that there exists a t, $s^* < t \leq s$ such that $l^\alpha(t) \leq x^\alpha(k,s)$. For all $j > k$ we assign $x^\alpha(j,s)$ a token if $b^\alpha(j,s) \notin B_{s+1}$ to mark the fact that we will discard $x^\alpha(j,s)$ if $b(x^\alpha(j,s),s)$ enters B before j enters A. If $b^\alpha(j,s) \in B_{s+1}$, then we undefine $x^\alpha(j,s)$. ($x^\alpha(j,s)$ may already have a token assigned at a previous stage in this case.)

We also cancel markers below this lowest length of agreement, if they have been given tokens. For every j such that $x^\alpha(j,s)$ is not permanently stable, if $x^\alpha(j,s)$ has a token, and $b(x^\alpha(j,s),s) \in B_{s+1}$, then we undefine $x^\alpha(j,s)$. If $l^\alpha(t)$ is later above $x^\alpha(j-1,t)$, we can then redefine some new (bigger) number to be $x^\alpha(j,s)$. We let $\alpha^\frown\langle k, U\rangle \subseteq f_s$, for the k chosen above.

At the end of stage $s+1$, we re-initialize all β with $f_s < \beta$.

Let $C = \bigcup_{s\in\omega} C_s$. This completes the construction.

Verification:

Define the true path f to be $\liminf_{s\to\infty} f_s$. We show that the true path exists and witnesses that C has the required properties. So we show by induction that for every $\alpha \subseteq f$, the sequence of α's outcomes has a finite lim inf, α's requirement is satisfied, and, if α is even, R^α, the set of elements that are in $R^\alpha(s)$ at cofinitely many stages s, is finite.

First, suppose α has even length, say $2a$. Suppose $A \leq_Q B \oplus C$ via ϕ_a. We describe how to define a computable function h such that $A \leq_Q B$ via h, a contradiction. Since $\alpha \subset f$, there must exist some stage s_0 such that $\alpha \leq f_s$ for every $s > s_0$, and, for every odd-length node $\beta \subset \alpha$ with $\beta^\frown\langle k, U\rangle$ for some k, if there are only finitely many β-expansionary stages, then no action is taken for any β-strategy after s_0. This is possible, since if β's requirement only leads to finitely many expansionary stages, then only finitely many followers $x^\beta(j,s)$ are ever defined. For all $z \in \omega$, and $s \leq s_0$, let $W_{h(z),s} = \emptyset$. If $s \geq s_0$ is an α-expansionary stage and $l^\alpha(s) > z$, then we go through in order defining $h(z)$ for each $z < l^\alpha(s)$. If $z \in A_s$, $W_{h(z),s+1} = W_{h(z),s}$. Otherwise, $z \notin A_s$. If already $W_{h(z),s} \not\subset B_s$, then we are already correct at s, so we need take no action. Otherwise, we have two cases to consider, depending on whether B or C is being used by $\phi_{a,s}$ to Q-compute that $z \notin A$:

Case 1: $(W_{\phi_a(z)} - (B \oplus C))[s] = \{2b(z,s)\}$, for some $b(z,s) \notin B_s$. Then let $W_{h(z),s+1} = W_{h(z),s} \cup \{b(z,s)\}$.

Case 2: $(W_{\phi_a(z)} - (B \oplus C))[s] = \{2c(z,s)+1\}$, for some $c(z,s) \notin C_s$.

(a) If there does not exist any $\beta < \alpha^\frown\langle z\rangle$ and $j \in \omega$ with $c(z,s) = x^\beta(j,s)$, then let $W_{h(z),s+1} = W_{h(z),s} \cup \{\bar{b}\}$.

Otherwise, there exists some $\beta < \alpha^\frown\langle z\rangle$ and $j \in \omega$ such that $c(z,s) = x^\beta(j,s)$. β appears to threaten injury, because it has higher priority than α together with the outcome α believes correct at s. Recall that we cannot later remove elements from $W_{h(z)}$ when α gets some smaller outcome l at a stage after s. This involves several more cases, depending on whether $\beta \subset \alpha$ or merely $\beta \supset \alpha^\frown\langle l\rangle$ for some $l < z$. We apply the first case that we can below.

(b) If there is some k with either $\beta^\frown\langle k, D\rangle \subseteq \alpha$, or $\beta^\frown\langle k, U\rangle \subseteq \alpha$ with only finitely many β-expansionary stages, then let $W_{h(z),s+1} = W_{h(z),s} \cup \{\bar{b}\}$.

(c) If $\beta^\frown\langle k, U\rangle \subseteq \alpha$. and $j \leq k$, let $W_{h(z),s+1} = W_{h(z),s} \cup \{\bar{b}\}$.

(d) If $\beta^\frown\langle k, U\rangle \subseteq \alpha$. and $j > k$, then $x^\beta(j,s)$ must have a token, hence $b^\beta(x(j,s),s)\downarrow$, so we let $W_{h(z),s+1} = W_{h(z),s} \cup \{b^\beta(x(j,s),s)\}$.

(e) If there is some $l < z$ such that $\alpha^\frown\langle l\rangle \subset \beta$, and $l \in A$, then by construction, the path can never branch back through $\alpha^\frown\langle l\rangle$, so let $W_{h(z),s+1} = W_{h(z),s} \cup \{\bar{b}\}$.

(f) If there is some $l < z$ such that $\alpha^\frown\langle l\rangle \subset \beta$, and $l \notin A$, but $(W_{\phi_a(l)} - (B \oplus C))[s] = \{2c(l,s)+1\}$ for some $c(l,s) \notin C$, then note that $W_{h(l),s+1} - B_s = \{b(l,s)\}$ already since $l < z$, and so $h(l)$ has already been defined correctly at this stage. Let $W_{h(z),s+1} = W_{h(z),s} \cup \{b(l,s)\}$.

(g) Finally, if there is some $l < z$ such that $\alpha^\frown\langle l\rangle \subset \beta$, and $l \notin A$, and $(W_{\phi_a(l)} - (B \oplus C))[s] = \{2b\}$ for some $b \notin B$, then let $W_{h(z),s+1} = W_{h(z),s} \cup \{b\}$, since b will have to enter B before β is ever allowed to act.

We remark that the procedure to define h is not even uniform in α, since we have to know additionally which of the finitely many outcomes of the form $\langle k, U\rangle$ along α actually represent finitary outcomes without diagonalization.

Suppose z is some number for which h fails as a Q-reduction of A to B. If $z \in A$, then we must have $W_{h(z)} \not\subseteq B$. So, suppose $b \in (W_{h(z)} - B)[s+1]$, where s is the least such α-stage. Clearly, b was added to $W_{h(z),s+1}$ under case 2 above, since otherwise, $2b \in W_{\phi_a(z)} - (B \oplus C)$. Recall that s was an α-expansionary stage. This can only mean that $z \notin A_s$, while $(W_{\phi_a(z)} - (B \oplus C))[s] = \{2c+1\}$ for some $c \notin C_s$. Since $z \in A$, we must have $c \in C$, yet $b \notin B$.

This implies that there must exist at least one z' and $s' > s_0$ such that $z' \notin A_{s'}$, $(W_{\phi_a(z')} - (B \oplus C))[s'] = \{2c+1\}$ for some $c \notin C_{s'}$, yet $c \in C$, while $W_{h(z),s'+1} \not\subseteq B$. We will show that this is impossible, hence for every z, $z \in A$ implies $W_{h(z)} \subseteq B$. We may suppose z is the least number for which such an s' ever exists. So, suppose $z \notin A_s$ and $b \in (W_{h(z)} - B)[s+1]$, where s is the least such α-stage.

All $\beta > \alpha^\frown\langle z\rangle$ are re-initialized at s; all $\beta <_L \alpha$ never act again after stage s_0; and all $\beta \supset \alpha^\frown\langle z\rangle$ are restrained from ever adding c to C forever after this point. Since $c \in C$, subcase (a) therefore cannot have applied at $s+1$, hence $c = x^\beta(j, s)$ for some $\beta < \alpha^\frown\langle z\rangle$and $j \in \omega$. We analyze the remaining possible subcases. If subcase (b) applied at s, then by choice of s_0, β never acts again after stage s_0. This is also true for $\beta^\frown\langle k, D\rangle \subseteq \alpha$, for if β were to act again, once it did, $\langle k, D\rangle$ could never again be its outcome, as $x^\beta(k, s_0)$ will be declared permanently stable. This implies $c \notin C$. If case (c) applied, $\beta^\frown\langle k, U\rangle \subseteq \alpha$ for some k. If $j < k$, then $c = x^\beta(j, s)$ will never be added to C. If $j = k$, then $x^\beta(k, s)$ can never be added to C by construction, since $\langle k, U\rangle$ can only be the outcome when $x^\beta(k, t) \notin C_{t+1}$, and it is the outcome infinitely often. If case (d) applied, then $k < j$. In this case, $b = b(x^\beta(j, s), s)$. If $c \in C$, but $b \notin B$, then once c enters C at some stage t_0, $l^\beta(t) \leq x^\beta(j, s)$ for every $t \geq t_0$. Since this contradicts the infinitary outcome occurring infinitely often at β, this is impossible. If case (e) applied at s, then the path can never branch back through $\alpha^\frown\langle l\rangle$, since $m^\alpha(s) > l$. If case (f) applied, then by construction, the path can only branch back through $\alpha^\frown\langle l\rangle$ at $t > s$ if $(W_{\phi_a(l} \subseteq (B \oplus C))[t])$. This can only happen if the element $c(l, s)$ enters C, in which case $b(l, s)$ must enter B, since z was chosen least. But, $l \notin A_{s'}$, $(W_{\phi_a(l)} - (B \oplus C))[s] = \{2c(l, s)+1\}$, and $c(l, s) \in C$, while $W_{h(l),s+1} \not\subseteq B$. Since $l < z$, this contradicts z being chosen least with this property. Finally, suppose case (g) applied. As already pointed out, b has to enter B before β is ever allowed to act, so this too is impossible. So h must be correct at z.

Now suppose h fails for some $z \notin A$. Then $W_{h(z)} \subset B$. Clearly, if ϕ_a is correct, this can only be because $W_{\phi_a(z)} - (B \oplus C) = \{2c+1\}$ for some $c \notin C$. Furthermore, $W_{h(z)} \subset B$ can only happen if there is some $\beta^\frown\langle k, U\rangle \subseteq \alpha$ and $c = x^\beta(j, s)$, for some $j > k$, at some stage $s > s_0$. In this case, $b(x^\alpha(j, s), s) \in W_{h(z)} \subset B$. So there is some least β-stage $t > s$ with $b(x^\alpha(j, s), s) \in B_t$. Then $x^\beta(j, s)$ is permanently given up, since it has a token. But then $\bar{b}$ is enumerated into $W_{h(z)}$ at the next α-stage. So, again h must be correct at z.

This implies $A \leq_Q B$ via h, and is therefore impossible. So ϕ_a must be incorrect: there must be some least z such that either $\phi_a(z)\uparrow$; $W_{\phi_a(z)} \subseteq B \oplus C$, yet $z \notin A$; or $W_{\phi_a(z)} \not\subseteq B \oplus C$ for some $z \in A$. Only in the second case, can $m^\alpha(s)$ fail to have a fixed bound, and then only if some infinite sequence of numbers $2b$ are enumerated into $W_{\phi_a(z)}$ and then later each b is enumerated into B. But this is exactly the situation which causes the restraints to drop back, and so infinitely often the restraint set is finite and the same. In this case, $\alpha^\frown\langle z\rangle \subset f$, and every node extending this gets to act essentially without restraint. This takes care of the even-length nodes.

Next, suppose $|\alpha| = 2e+1$. Suppose further that $C \leq_Q B$ via ϕ_e. In fact, suppose merely that ϕ_e is correct on the set of all markers eventually defined for the sake of this requirement. As before, we now describe how to construct a function h such that $A \leq_Q B$ via h. Let s_0 be chosen so that for every $s \geq s_0$, $\alpha \leq f_s$. The total restraint imposed by all $\beta < \alpha$ is a fixed finite set at every α-stage $s \geq s_0$, so we can assume for convenience that all witnesses are chosen greater than any of these restraints, and so can always be added in an attempt to satisfy α's requirement at any α-stage. As before, for every $z \in \omega$, $W_{h(z),s_0} = \emptyset$. Note that since $C \leq_Q B$ via ϕ_e, there must be infinitely many α-expansionary stages $s \geq s_0$. At any such stage s, we define in sequence $W_{h(z),s+1}$ for each z such that $x^\alpha(z,s) < l^\alpha(s)$. If $z \in A_s$, we let $W_{h(z),s+1} = W_{h(z),s}$. If $z \notin A_s$, then clearly, $x^\alpha(z,s) \notin C_s$, by construction. Since $x^\alpha(z,s) < l^\alpha(s)$, there must exist some unique $b(x^\alpha(z,s),s) \in (W_{\phi_e(x^\alpha(z,s))} - B)[s]$. We define $W_{h(z),s+1} = W_{h(z),s} \cup \{b(x^\alpha(z,s),s)\}$.

We must show that $A \leq_Q B$ via h. If not, then the reduction must fail at some least number z. Since h is obviously total, we first suppose that $z \in A$, yet $W_{h(z)} \not\subset B$. Clearly, there must be some α-expansionary stage s such that $b(x^\alpha(z,s),s) \in W_{h(z),s+1} - B$, and $z \notin A_s$. Since $b(x^\alpha(z,s),s) \notin B$, $x^\alpha(z,s)$ can never be given up after s. But then, once $z \in A_t$ at some α-stage $t > s$, $x^\alpha(z,s) \in C_{t'}$, for some $t' > t$. Since $b(x^\alpha(z,s),s) \in W_{\phi_e(z)} - B$, this contradicts $C \leq_Q B$ via ϕ_e, and so is impossible. On the other hand, suppose $z \notin A$. Then no marker $x^\alpha(z,s)$ is ever enumerated into C, yet, since h fails at z, every $b(x^\alpha(z,s))$ is enumerated into B. Notice that, by a straightforward inductive argument, for every $z' < z$, $x^\alpha(z',s)$ has a finite limit, and eventually $l^\alpha(t) > x^\alpha((z',t)$ at every α-stage t. This follows because $z' \in A$ means that $x^\alpha(z',s)$ is eventually permanently stable, and z' can no longer count as the greatest element seeking to satisfy the requirement. But $z' \notin A$ means that eventually some $x^\alpha(z') = x^\alpha(z',s)$ appears which never can be undefined, since for some t, $b(x^\alpha(z'),t) \notin B$, else z' would be the least number at which h fails. But then $\langle z, U\rangle$ is the true outcome of α, and hence $x = x^\alpha(z,s)$ is eventually permanent, and $W_{\phi_e(x)} \subseteq B$, while $x \notin C$. This contradicts $C \leq_Q B$ via ϕ_e. Thus $A \leq_Q B$ via h, a further contradiction, so ϕ_e fails at some marker $x = x^\alpha(j,s)$. Then, infinitely often, $l^\alpha(s) = x^\alpha(j,s)$, so either $\langle j, U\rangle$ or $\langle j, D\rangle$ must be the true outcome of α.

This shows that all the requirements are satisfied, and the true path f exists. We now only have to show $C \leq_Q A$. In fact, since $B \oplus A \leq_Q A$, we show $C \leq_Q B \oplus A$. We define a Q-reduction h as follows. If x has not yet been chosen as a marker when the bound on possible markers grows beyond x at some stage t, or if x is chosen as a marker for some α at s and some stage $t > s$, $f_t <_L \alpha$, then enumerate $2\bar{b}$ into $W_{h(x)}$ at stage $t+1$ if $x \notin C_t$. Also, if $x = x^\alpha(j, s)$ we enumerate $2j+1$ into $W_{h(z)}$. This guarantees correctness on every x with $j \notin A$. Now we have to decide what to do when $x = x^\alpha(j, s)$ is defined, $j \in A_s$, but $\alpha \le f_s$. If $x \in C_{s+1}$, then, of course, we do nothing, letting $W_{h(x),s+1} = W_{h(x),s}$. If $x \notin C_{s+1}$, but $(W_{h(x)} \not\subseteq (B \oplus A))[s]$, then, again, there is no need to act, since $h(x)$ is still correct. If $\alpha \subseteq f_s$, then there are only two things that can prevent x from entering C_{s+1}. Either some $x^\alpha(j', s)$ enters C_{s+1}, and $\alpha^\frown\langle j', D\rangle \subseteq f_s$, in which case we enumerate $b(x^\alpha(j', s)) \in W_{h(x),s+1}$, or $x \in R^\beta[s]$ for some $\beta < \alpha$, in which case we enumerate $2\bar{b} \in W_{h(z),s+1}$, since $x \in R^\beta[s]$ at every α-stage, so α never gets to put $x \in C$.

The final possibility is $\alpha <_L f_s$. Let $\beta = \alpha \bigcap f_s$. If β is an even-length node, and $\beta^\frown\langle k\rangle \subset \alpha$, yet $\beta^\frown\langle k'\rangle \subset f_s$, we first note that for α to act at $t+1 > s+1$,, $\beta^\frown\langle k\rangle \subset f_t$. If $k \in A_s$, this can never happen again, so we enumerate $2\bar{b}$ into $W_{h(x),s+1}$. If $k \notin A_s$, since $\beta^\frown\langle k'\rangle \subset f_s$, $(W_{\phi_a(k)} - (B \oplus C))[s] = \{z\}$. If $z = 2b$, then β can only have outcome k at a stage when $b \in B$, so enumerate $2b$ in $W_{h(x),s+1}$. If $z = 2c+1$, for some $c \notin C_s$, then there are three possibilities. If c has not yet been chosen as a marker or c is a marker for some node $\gamma > \beta^\frown\langle k'\rangle$, then β never has outcome k' again, so we enumerate $2\bar{b}$ in $W_{h(x),s+1}$. Otherwise $c = x^\gamma(l, s)$ for some $\gamma < \beta^\frown\langle k'\rangle$. Notice that $\gamma < \alpha$, so $W_{h(x^\gamma(l,s)),s+1}$ is already defined. Then β cannot have outcome k' unless $x^\gamma(l, s)$ enters C, so we enumerate the unique element of $W_{h(x^\gamma(l,s)),s+1} - (B_s \oplus A_s)$ into ${}_{h(x),s+1}$. This takes care of the possibility that β is even.

So, assume β is odd, say $|\beta| = 2e+1$. First, suppose there is some j such that $\beta^\frown\langle j, D\rangle \subseteq f_s$. Then this will always be β's outcome unless $b(x^\beta(j, s), s)$ enters B. So we enumerate $2b(x^\alpha(j, s), s)$ into $W_{h(x),s+1}$. Otherwise, let $\beta^\frown\langle k, O\rangle \subseteq \alpha$. If $k \in A_s$, or $O = D$, then α can never act again, so we enumerate $2\bar{b}$ in $W_{h(x),s+1}$. Otherwise $k \notin A_s$, and $O = U$. Clearly, $(W_{\phi_e(x^\beta(k))} - B)[s] = \{b\}$, since otherwise $\beta^\frown\langle k', U\rangle \subseteq f_s$ for some $k' \le k$. Clearly, β can only get this outcome at a later stage if b enters B, so we can enumerate $2b$ into $W_{h(z),s+1}$.

This takes care of all the possibilities. It is clear from our description, that $x \in C$ implies $W_{h(x)} \subseteq B \oplus A$. It is also clear that if x is never chosen as a marker for any node $\alpha < f$, where f is the true path, then $x \notin C$ implies $2\bar{b} \in W_{h(x)} - (B \oplus A)$. Also, if $x = x^\alpha(j, s)$ for some $j \notin A$, $2j+1 \in W_{h(x)} - (B \oplus A)$. So we only have to show that if $x = x^\alpha(j, s)$ with $j \in A$ and $\alpha \le f$, then $x \notin C$ implies $W_{h(x)} - (B \oplus A) \neq \emptyset$. First, suppose $\alpha \subset f$. If $\alpha^\frown\langle k, D\rangle \subseteq f$, then $2b(x^\alpha(k)) \in W_{h(x)} - (B \oplus A)$. If $\alpha^\frown\langle k, U\rangle \subseteq f$, then either $2b(x^\alpha(j, s), s) \in W_{h(x)} - (B \oplus A)$, or $x^\alpha(j, s)$ is eventually either permanently stable, or permanently cancelled, so $2\bar{b} \in W_{h(x)} - (B \oplus A)$.

Next suppose $\alpha <_L f$. Let x be chosen least such that h fails and x is a marker for a node left of f. Let $\beta = \alpha \bigcap f$. First, suppose β has even-length, say $2a$.. Then $\beta^\frown\langle k\rangle \subset f$, while $\beta^\frown\langle k'\rangle \subseteq \alpha$. If $k' \in A$, then $2\bar{b} \in W_{h(x)} - (B \oplus A)$. If $k' \notin A$, then either there is some b with $2b \in W_{\phi_a(k')} - (B \oplus C)$, and also $2b \in W_{h(x)} - (B \oplus A)$, or $W_{\phi_a(k')} - (B \oplus C) = \{2c+1\}$. If $2\bar{b} \notin W_{h(x)} - (B \oplus C)$, then c is a marker for some $\gamma < \beta^\frown\langle k'\rangle$, and $W_{h(x)} - (B \oplus C) = W_{h(c)} - (B \oplus C)$. If $\gamma \subset \beta \subset f$, $c \notin C$ implies $W_{h(c)} \not\subseteq (B \oplus C)$, as already shown. If $\gamma <_L \beta^\frown\langle k'\rangle$, then α was initialized after c was chosen for γ, hence $c < x$, and, since x is least such that h fails, $W_{h(c)} \not\subseteq (B \oplus C)$, in this case as well. Clearly, $\beta \neq \gamma$, since β has even length.

Finally, we suppose β has odd-length, say $\beta = 2e + 1$. If $\beta^\frown\langle k, D\rangle \subset \alpha$, then $2\bar{b} \in W_{h(x)} - (B \oplus A)$. Otherwise, suppose a stage has been reached such that the true path never branches back through $\beta^\frown\langle k, U\rangle \subseteq \alpha$. If $k \in A$, then eventually $2\bar{b} \in W_{h(x)} - (B \oplus A)$. Also, if $\beta^\frown\langle k', D\rangle \subset f$, then $b(x^\beta(k')) \in W_{h(x)} - (B \oplus A)$. If $k \notin A$, and some $\beta^\frown\langle k', U\rangle \subset f$, for $k < k'$, then, if for infinitely many s, $b(x^\beta(k,s),s) \in B$, then $\beta^\frown\langle k, U\rangle \subset f$, a contradiction. So there is some stage s with $b = b(x^\beta(k,s),s) \notin B$. But then $2b \in W_{h(x)} - (B \oplus C)$. This takes care of the last remaining possibility, and establishes the result. $\square$

5 Undecidability in the c.e. Q-degrees

In this section, we show that any computable, countable partial order can be interpreted with parameters in the c.e. Q-degrees. This carries over enough elements of the coding construction used in characterizing the theory of the c.e. Turing degrees to show that theory of $\mathcal{R}_Q$ is undecidable.

Because the the reader may be unfamiliar with the technique of using codings of partial orders to show the undecidability of structures like $\langle \mathcal{R}_Q, \leq_Q \rangle$, we briefly sketch the overall plan. First, one shows that one can interpret a model of true arithmetic in a particular countable, computable partial order $\mathcal{P}$. Such an interpretation can be found in [10], so we do not repeat it here. Then, using this fact, one shows that one can embed this partial ordering into the degree structure in such a way that it is definable with four parameters by a formula $\phi(\mathbf{x}, \mathbf{y}; \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{l})$. This shows the undecidability of the structure in question, by the classical method of Tarski, [13].

We therefore merely show, using ideas due to Slaman and Woodin, that any computable partial order on ω can be embedded into $\mathcal{R}_Q$ via a countable antichain. The following result is the key lemma for the method.

Theorem 8 (Coding Lemma). *Let $\mathcal{P} = \langle \omega, \preceq \rangle$ be a computable partial order. There exist c.e. sets A, B, L and a sequence of c.e. sets $\mathcal{G} = \langle G_i | i \in \omega\rangle$ such that, letting $C = \oplus G_i$,*

(1) for all i, $A \leq_Q G_i \oplus B$,
(2) for all $i \neq j$, $G_i |_Q G_j$,
(3) for all c.e. $W((W \leq_Q C \wedge A \leq_Q W \oplus B) \Longrightarrow \exists i(G_i \leq_Q W))$,
(4) for all i and j, $i \preceq j \Longleftrightarrow G_i \leq_Q G_j \oplus L$, and
(5) $B \oplus C$ is low

In other words, the degrees of sets in $\mathcal{G}$ are the minimal c.e. Q-degrees below C which sup with B above A and L can be used to define the computable partial order P on $\mathcal{G}$ in $\mathcal{R}_Q$. We require $B \oplus C$ to be low for technical reasons which will become clear below.

Proof. Although our proof is self-contained, we only briefly give the intuitions behind the strategies. It would be helpful to the reader to have some familiarity with the main technical lemma used in [10], although the constructions there are more complicated than these, since they incorporate permitting to determine the degree of $Th(\mathcal{R}_T)$. Our proof involves the construction of many computable functions serving as Q-reductions. For convenience of notation, we switch to the use of uppercase Greek letters for these functions. In this way, for example, if we build, for some sets A and B, a Γ via which $A \leq_Q B$, then for each $a \notin A$, we can write $\gamma(a)$ for the (unique) element in $W_{\Gamma(a)} - B$.

Given $\mathcal{P}$, to prove the result, we must construct A, B, L, $\langle G_i | i \in \omega\rangle$ as well as indexed sequences of functions, Γ_i, $\Delta_i^{a,b,e}$, and $\Theta_{i,j}$ satisfying the following six kinds of requirements:

(T_i) $\quad A \leq_Q G_i \oplus B$ via Γ_i,
$(D_{i,j,e})$ $i \neq j \Longrightarrow G_i \not\leq_Q G_j$ via Φ_e,
$(M_{a,b,e})$$(W_e \leq_Q C$ via $\Phi_a \wedge A \leq_Q W_e \oplus B$ via $\Phi_b) \Longrightarrow$
$\qquad \exists i \exists \Delta(G_i \leq_Q W_e$ via $\Delta)$,
$(N_{i,j,e})$ $i \not\preceq j \Longrightarrow G_i \not\leq_Q G_j \oplus L$ via Φ_e,
$(P_{i,j})$ $\quad i \prec j \Longrightarrow G_i \leq_Q G_j \oplus L$ via $\Theta_{i,j}$, and
$(K_{e,x})$ $\quad \exists^\infty s\, \Phi_e(B \oplus C; x)\downarrow [s] \Longrightarrow \Phi_e(B \oplus C; x)\downarrow$.

Our construction is an infinite injury priority argument, using a tree of strategies. We only use the tree to control the actions of minimality requirements like $M_{a,b,e}$. These are the only requirements for which explicit restraints need to be set. To set these restraints, we must be able to nest 2^i different strategies for requirement M_i appropriately, using a modification of the method of expansionary stages. The priority ordering is somewhat unusual in that we demand that T_i and T_j must come before any requirements $D_{i,j,e}$ or $N_{i,j,e}$ which refer to the sets G_i and G_j; the intuition being that the requirement T_i introduces the set G_i into the construction. In the case of the minimality requirements, $M_{a,b,e}$ and the lowness requirements $K_{e,x}$, we of course have to use an approximation to the entire set $C = \oplus_{i\in\omega} G_i$, although the strategy for satisfying requirement $M_{a,b,e}$ will only involve the set of G_i such that $|T_i| < |M_{a,b,e}|$, where $|R|$ denotes the place of R in the priority ordering. The $P_{i,j}$ do not occur in the priority listing. They will be taken care of locally by simple permitting whenever a strategy for some $D_{i,j,e}$, $N_{i,j,e}$, or $M_{a,b,e}$ changes G_i to achieve some diagonalization.

The basic strategies for satisfying these requirements individually are simple. For requirements like $D_{i,j,e}$ or $N_{i,j,e}$, we use the standard Friedberg-Muchnik diagonalization strategy, picking a follower x, waiting for realization by Φ_e, then adding x to G_i, and restraining to achieve permanent disagreement. For requirements T_i and $P_{i,j}$, we build appropriate functionals and keep them correct at every stage of the construction. A requirement $M_{a,b,e}$ will be satisfied by building a sequence of functionals $\Delta_i^{a,b,e}$ at stages where the length of agreement in the condition expands such that one of these functionals succeeds by Q-reducing G_i to W_e.

As usual we are faced with the problem that the diagonalization strategies conflict directly with the strategies seeking to restrain the sets involved. For a potentially infinitary requirement $M_{a,b,e}$, with n requirements T_i above it, we intend to first attempt to define a sequence of reductions Δ_i, $i < n$, via one of which $G_i \leq_Q W_e$. If $W_e \leq_Q C = \oplus_{i \in \omega} G_i$, the failure of all of our reductions Δ_i infinitely often will eventually produce a sequence of G_i-changes enabling us to diagonalize against $A \leq_Q W_e \oplus B$ via Φ_b (or $W_e \leq_Q C$ via Φ_a) while still satisfying the higher priority T_i requirements.

Construction:

To specify the construction, we first define a priority listing $\langle R_n | n \in \omega \rangle$ as discussed above, ensuring that T_i and T_j always come before any $D_{i,j,e}$, or $N_{i,j,e}$. To control the strategies for satisfying minimality requirements like $M_{a,b,e}$, we use the tree of strategies $T = {}^{<\omega}2$. We assign to each $\alpha \in T$ the requirement $R_\alpha = M_{|\alpha|}$. For the sake of brevity in describing uses, we adopt the following general notational convention: if we have some approximation to $X \leq_Q Y$ via Φ and for some x less than the length of agreement at stage s, $x \notin X_s$, then we write $\phi(x)[s]$ for the least (usually unique) element of $(W_{\Phi(x)} - Y)[s]$.

Stage 0: All strategies are initialized; that is, all functionals and parameters associated to any strategy for any requirement are undefined. All c.e. sets we are constructing are empty

Stage $s+1$: We examine in order the requirements R_u for each $u < s$, and take whatever action is required to satisfy R_u if possible. As usual, any functional, parameter or set not mentioned explicitly does not change at stage $s+1$. There are several cases to consider. As usual we suppress any subscripts and superscripts that are clear from the context; for instance, in the first case below, we write G, Γ, and γ for G_i, Γ_i, and γ_i.

Case 1: $R_u = T_i$.

We define Γ by eventually defining $W_{\Gamma(y)}$ for every $y \in \omega$. A number y can only be added to A for the sake of some minimality requirement $M_{a,b,e}$. Our plan is to satisfy such an $M_{a,b,e}$ of lower priority than T_i by clearing an original G_i-use so that we can add y to A without disturbing the W_e-computation that $y \notin A$. Since Q-reductions depend on ordinary computable functions rather than $G_i \oplus B$-computable ones, we are faced with the problem that the use for $\Gamma(y)$ must be set, at least originally, without any reference to what takes place in the construction. Fortunately, we can at least determine in advance which requirement each y is associated with and exactly how many uses from members of the independent sequence need to be defined before M ever acts. We therefore set things up in the most natural way possible, assigning each $y \in \omega^{[<a,b,e>]}$ to requirement $M_{a,b,e}$, and treating each such y as coding a sequence of markers representing its uses for the finitely many members of the independent sequence which it must take into account. More precisely we do the following:

If $y < s$ and $y \in \omega^{[<a,b,e>]}$, then let $n = \mu j(|T_j| \not< |M_{a,b,e}|$. If $n \leq i$, then $\gamma(y)[s+1]\uparrow$, since $M_{a,b,e}$ has higher priority than T_i in this case anyway. Else

$i < n$. Suppose $y = \langle\langle a, b, e\rangle, \langle j_0, \ldots, j_{n-1}\rangle\rangle$. If $y \in A[s]$, then $\gamma(y)[s+1]\uparrow$. Else, $y \notin A[s]$. If $j_i \notin G_i[s]$, then $\gamma(y)[s+1] = 2j_i$. Else, if $\gamma(y)[s]\downarrow \notin G_i[s]$, then $\gamma(y)[s+1] = \gamma(y)[s]$. Finally, if $\gamma(y)[s] \in G_i[s]$, $\gamma(y)[s+1] = 2(s(y)) + 1$, where $s(y) = \langle y, s+1\rangle$, the $s+1$st element of $\omega^{[y]}$. We show below that these definitions ensure that $y \in A$ if and only if $W_{\Gamma(y)} \subseteq G_i \oplus B$, where

$$W_{\Gamma(y)} = \{\, \gamma(y)[s] \,:\, s \in \omega \,\}.$$

Case 2: $R_u = M_{a,b,e}$.

Let $v = \langle a, b, e\rangle$, so that there are exactly v requirements $M_{v'}$ of higher priority than $M_{a,b,e}$. We have 2^v different strategies, one for each $\alpha \in T$ with $|\alpha| = v$ for satisfying $M_{a,b,e}$, at most one of which is allowed to act at s. We define the notion of α-stage recursively on v. If s is an α-stage, then only α is allowed to act for $M_{a,b,e}$ at s.

We may assume that we have used the recursion theorem to fix indices for the sets A, B, and C in advance of the construction. In the manner of Lemma 1 above, using these indices, and e, the index of W_e, we may replace Φ_a and Φ_b by some $\Phi_{a'}$ and $\Phi_{b'}$ such that at every stage s, if $x < s$, then $(W_{\Phi_{b'}(x)} - (W_e \oplus B))[s]$ contains at most one element, and, similarly, for any $y < s$, $(W_{\Phi_{a'}(y)} - C)[s]$ contains at most one element. Below, we write a and b for a' and b', respectively, for notational convenience.

First we define the length of agreement function $l^\alpha(s)$. As in the preceding proofs, we have to be careful to define l^α in such a way that we are guaranteed infinitely many expansionary stages if $W_e \leq_Q C$ via Φ_a and $A \leq_Q W_e \oplus B$ via Φ_b. Recall that this problem arises whenever some element $x \in X \leq_Q Y$ via g has the property that $(W_{g(x)} \not\subseteq Y)[s]$ at any stage s because Y's enumeration is too slow relative to that of $W_{g(x)}$.

We modify the length of agreement function by using the method of W_e-true stages. Let $w^\alpha[s]$ be the least element to enter W_e since the last previous α-stage, or s, if no such element exists. Since we are in a position to control enumeration of elements into C and B, we can afford to ignore positive errors in the approximations.

We define $l(s) = l^\alpha(s)$ as follows:

$$\begin{aligned} l(s) = \mu x(\Phi_b(x)\uparrow \text{ or } &\exists y \in W_{\Phi_b(x)}(\Phi_a(y)\uparrow) \text{ or} \\ &x \notin A \text{ and } W_{\Phi_b(x)} \subseteq ((W_e \upharpoonright w^\alpha) \oplus B) \text{ or} \\ &\exists y \in W_{\Phi_b(x)} \exists y' < w^\alpha(y = 2y',\, y' \notin W_e, \text{ and } W_{\Phi_a(y')} \subseteq C))[s]. \end{aligned}$$

We say s is *α-expansionary* if $l(s') < l(s)$ for every $s' < s$. Notice that $l(s)$ only changes value on α-stages s, so is α-expansionary if and only if $l(s') < l(s)$ for every α-stage $s' < s$.

Let $n = |\{\, T_i \,:\, |T_i| < |R_u| \,\}|$.

If some strategy for R_u has initialized all lower priority strategies since R_u was itself last initialized, then R_u appears permanently satisfied at $s+1$. In this case, we say s is an $\alpha^\frown\langle 1\rangle$-stage, allowing $\alpha^\frown\langle 1\rangle$ to act for M_{v+1} at stage $s+1$, and go on immediately to the next requirement, R_{u+1}.

Otherwise, we first look to see whether or not we have the ability to immediately satisfy R_u permanently by a finite action. Recall that if $\Phi(y)\downarrow[s]$, and some putative reduction $X \leq_Q Y$ via Φ is under consideration, we write $\phi(y)[s]$ for the least (usually unique) element of $(W_{\Phi(y)} - Y)[s]$, if such exists; otherwise $\phi(y)[s]\uparrow$.

If there is some $y \in W_e[s]$ such that $\phi_a(y)[s]\downarrow$, hence $(W_{\Phi(y)} - C)[s] \neq \emptyset$, immediately initialize all strategies for requirements $R_{u'}$ with $u < u'$, and go on to stage $s+2$.

If there is some $y \in A[s]$ such that $\phi_b(y)[s]\downarrow$ and either $\phi_b(y)$ is odd, or $\phi_b(y)$ is even and $\phi_a(\frac{\phi_b(y)}{2})[s]\downarrow$, then immediately initialize all strategies for requirements $R_{u'}$ with $u < u'$, and go on to stage $s+2$.

If there is some $y \notin A[s]$, with $\phi_b(y)[s]\downarrow$, and $\phi_b(y) = 2y' + 1$ is odd, then we are prevented from using y below in our attempts to define the functionals Δ_j. Our plan for defining this functional depends on linking G_j to W_e on $\gamma_j(y)$ using the fact that $y \notin A$, and hence both $\gamma_j(y) \notin G_j \oplus B$ and $\phi_b(y) \notin W_e \oplus B$. But in this case there is no relationship between W_e and the fact that $y \notin A$, since the potential Q-reduction Φ_b is focused entirely on B. Fortunately, if every $\gamma_j(y)[s]$ is still set to its original (even) value, we can immediately satisfy the requirement if we are not restrained from adding the necessary elements to C to preserve the requirements $P_{i,j}$ and T_i when we add y to A, since then we can be certain not to be forced to add $\lfloor\frac{\phi_b(y)}{2}\rfloor$ to B.

Suppose $y = \langle\langle a, b, e\rangle, \langle x_0, \ldots, x_{n-1}\rangle\rangle$, and for every $j < n$, $\gamma_j(y)[s] = x_j$. If for every $j < n$ and $\beta < \alpha$, $\langle j, x_j\rangle \notin r_C^\beta[s+1]$, then we can add y to A and preserve the requirements T_i. For the requirements $P_{i,j}$, we also have to check that if $j \preceq i$ and x_j is assigned to some requirement $N_{i',j',e'}$, then we can also add x_j to G_i. So, if additionally, for every $j < n$, $\beta < \alpha$, if $x_j \in \omega^{[u(j)]}$ and $R_{u(j)} = N_{i'j'e'}$, then $\langle i, x_j\rangle \notin r_C^\beta[s+1]$, we enumerate $y \in A[s+1]$, for each $j < n$, $x_j \in G_j[s+1]$, and, for each i such that $j \preceq i$, with $x_j \in \omega^{[u(j)]}$, then we enumerate $x_j \in G_i$ if for some i', j', e', $R_{u(j)} = N_{i'j'e'}$, and $x_j \in L[s+1]$, if $R_{u(j)}$ is any other kind of requirement. Immediately initialize all strategies for requirements $R_{u'}$ with $u < u'$, and go on to stage $s+2$.

For the final finitary strategy, suppose there is some $y \notin A[s]$, $\phi_b(y)\downarrow[s]$ and is even, say $\phi_b(y) = 2y'$. If $\phi_a(y')[s]\downarrow$ (so $W_{\Phi_a(y')} \not\subseteq C$), and for every $j < n$, $\gamma_j(y)[s]$ is odd and $\lfloor\frac{\gamma_j(y)[s]}{2}\rfloor \notin r_B^\beta[s+1]$, then enumerate $y \in A[s+1]$ and for each $j < n$, $\lfloor\frac{\gamma_j(y)[s]}{2}\rfloor \in B[s+1]$. Immediately initialize all strategies for requirements $R_{u'}$ with $u < u'$, and go on to stage $s+2$.

Otherwise, if s is not α-expansionary, then we do not act for α at stage $s+1$. In this case, s is an $\alpha^\frown\langle 1\rangle$-stage, allowing $\alpha^\frown\langle 1\rangle$ to act for M_{v+1} at $s+1$. Let s^* be the last α-expansionary stage before s, or s^- (the stage at which α was last initialized), otherwise. Set

$$r_B^\alpha[s+1] = \bigcup_{\beta<\alpha} r_B^\beta[s+1] \cup \quad \{\, x \,:\, x \leq s^* \,\} \text{ and}$$

$$r_C^\alpha[s+1] = \bigcup_{\beta<\alpha} r_C^\beta[s+1] \cup \quad \{\, x \,:\, x \leq s^* \,\}.$$

Otherwise, suppose s is α-expansionary.

In this case we attempt to define some Δ_i for $i < n$ such that $G_i \leq_Q W_e$ via Δ_i. To define $\Delta_i(x_i)$, for some $x_i \notin G_i$, we intend to wait for some $y < l^\alpha[s]$ to appear such that $y = \langle\langle a,b,e\rangle, \langle x_0,\ldots,x_i,\ldots,x_{n-1}\rangle\rangle$ and define $\delta_i(x_i)$ to be $\frac{\phi_b(y)[s]}{2}$. This will enable us to get G_i permission to diagonalize against R_α if x_i enters G_i before $\delta_i(x_i)$ enters W_e. For our strategy to work, we must ensure that some y appears that is suitable for this purpose. We could be prevented from even getting started, however, if, for instance, for every $y \in \omega^{[<a,b,e>]}$,

$$y < l^\alpha[s] \Longrightarrow \gamma_i(y)[s] \text{ is odd.}$$

For in that case, there is nothing suitable to set δ_i to be on y's i-coordinate. To avoid this, we define an α-target, $y^\alpha[s]$, in order to slow the enumeration of α-expansionary stages even further until a suitable y appears.

There are three cases.

(1) $y^\alpha[s]\!\uparrow$.

The total C-restraint on α at stage s is $\mathcal{R}(C,\alpha)[s] = \{\, x \in r_C^\beta[s] \,:\, \beta < \alpha \,\}$. Since we expect this to be a permanent, though finite, obstruction to our strategy, we ignore any $x \notin G_j[s]$ such that for some j, $\langle j,x\rangle \in \mathcal{R}(C,\alpha)[s]$. For all $i < n$, let x_i be least such that $x_i \notin G_i$, for all j, $\langle j,x_i\rangle \notin \mathcal{R}(C,\alpha)[s]$, and $\delta_i(x_i)[s]\!\uparrow$. Define $y^\alpha[s+1] = \langle\langle a,b,e\rangle, \langle x_0,\ldots,x_n\rangle\rangle$, and for each $i < n$, $y(x_i)[s+1] = y$. Immediately end stage $s+1$ and go on to stage $s+2$.

(2) $y^\alpha[s]\!\downarrow$ and $(l^\alpha \leq y^\alpha)[s]$ or there is some y' such that $\phi_b(y^\alpha)\!\downarrow\![s] = 2y'$, but $w^\alpha[s] \leq y'$.

Then we take no action at stage s for α, essentially treating s as if it were non-α-expansionary. Let $s*$ be the last α-expansionary stage before s. We define

$$r_B^\alpha[s+1] = \bigcup_{\beta<\alpha} r_B^\beta[s+1] \cup \quad \{\, x \,:\, x \leq s^* \,\} \text{ and}$$

$$r_C^\alpha[s+1] = \bigcup_{\beta<\alpha} r_C^\beta[s+1] \cup \quad \{\, x \,:\, x \leq s^* \,\}.$$

Notice that each $x_i(y^\alpha[s]) \in r^\alpha[s+1]$. We set α's outcome to be 1, allowing $\alpha^\frown\langle 1\rangle$ to act for M_{v+1} at $s+1$.

(3) $y = y^\alpha[s]\!\downarrow$ and $(y^\alpha < l^\alpha)[s]$, and, if $\phi_b(y^\alpha)\!\downarrow = 2y'$, $y' < w^\alpha[s]$.

Then $y^\alpha[s+1]\!\uparrow$, since the current α-target has been reached. Note that $\gamma_i(y)$ is even for every $i < n$, since each $(x_i(y^\alpha) \notin G_i)[s]$ because of the C-restraint that obtained since $y^\alpha[s]$ was originally chosen as α-target. Also, $(y^\alpha \notin A)[s]$ and $(y^\alpha < l^\alpha)[s]$, hence $\phi_b(y^\alpha)[s]$ is even, say $\phi_b(y^\alpha)[s] = 2y'$, $\phi_a(y')[s]\!\downarrow$. (In fact, for some $j_0 < n$, $\phi_a(y')[s] = \langle j_0, \gamma_{j_0}(y^\alpha)[s]\rangle$.) Set $\delta_{n-1}(x_{n-1})[s+1] = y'$.

Next, for each $j < n$ say Δ_j *fails on x at s* if $x \in G_j$, $\delta_j(x)[s]\downarrow$, and $\delta_j(x)[s] \notin W_e$. This failure is active until there exists some stage t such that $\delta_j(x)[s] \in W_e[t]$ (at which point $\delta_j(x)[t+1]\uparrow$). For each $j < n$, and each x with $\delta_j(x)[s]\downarrow$, if Δ_j fails on x at s, then note that, for some $j < n$,

$$y(x_j)[s] = \langle\langle a, b, e\rangle, \langle x_0, \ldots, x_j, \ldots, x_{n-1}\rangle\rangle,$$

where $x_j = x$. If $j \neq 0$ and $x_{j-1} \notin G_{j-1}$, then define $\delta_{j-1}(x_{j-1})[s+1] = \delta_j(x)[s]$. Otherwise, do nothing. (Notice that if $j = 0$, then $\phi_b(y) = 2y'$, $\phi_a(y')[s]\downarrow$ and for all $i < n$,

$$\phi_a(y')[s] \neq \langle i, \gamma_i(y)[s]\rangle = \langle i, x_i\rangle.$$

Hence, unless restrained by $\mathcal{R}(C, \alpha)[s]$, we would already have enumerated $y \in A[s+1]$ and for each $j < n$, $\gamma_j(y)[s] \in (G_j \oplus B)[s+1]$, by the action specified above.)

We then set α's outcome to be 0, allowing $\alpha^\frown\langle 0\rangle$ to act for M_{v+1} at $s+1$. Let $r^\alpha_B[s+1] = \bigcup_{\beta<\alpha} r^\beta_B[s+1]$, and $r^\alpha_C[s+1] = \bigcup_{\beta<\alpha} r^\beta_C[s+1]$.

If α is the unique node on T that is allowed to act for requirement R_u at $s+1$, let $r^u_B[s+1] = r^\alpha_B[s+1]$ and $r^u_C[s+1] = r^\alpha_C[s+1]$.

Case 3: $R_u = D_{i,j,e}$.

Recall $D_{i,j,e}$ is $G_i \not\leq_Q G_j$ via Φ_e. We intend to satisfy this requirement in the usual way by diagonalization, picking a new follower $x^u[s]$ after each initialization of R_u. We also have to add this element to L whenever we add it to G_i in order to guarantee $G_i \leq_Q G_k \oplus L$ when $i \preceq k$. (Obviously, we cannot in general add the marker to G_k, since we may have $j = k$.) All followers for R_u will be picked from $\omega^{[u]}$, to insure that there is no interference between strategies for different requirements. Because of higher priority restraints, we define a sequence of potential witnesses, x^u_k, at least one of which is certain to be large enough to achieve diagonalization.

If, for some k, $x^u_k[s]\downarrow$ and $x^u_k[s] \in G_i[s]$, then R_u is currently satisfied, so we do nothing.

If $x^u_0[s]\uparrow$, then let $x^u_0[s+1] = \langle u, s+1\rangle$, and continue with stage $s+1$.

If $x^u_{k+1}[s]\uparrow$, $x^u_k[s]\downarrow$, $(W_{\Phi_e(x^u_k)} \not\subseteq G_j)[s]$, $(x^u_k \notin G_i)[s]$ (*i.e.*, R_u is currently unsatisfied), and $\langle i, x^u_k\rangle \in r^{u'}_C[s]$ for some $u' < u$, then we set $x^u[s+1]$ to be the least element of $\omega^{[u]}$ greater than $x^u_k[s]$.

If $x^u_k[s]\downarrow$, then if $(W_{\Phi_e(x)} \not\subseteq G_j)[s]$, and $\langle i, x^u_k\rangle \notin r^{u'}_C[s]$ for any $u' < u$ we enumerate $x^u_k[s] \in G_i[s+1]$, and $x^u_k[s] \in L[s+1]$. In this case, we initialize all requirements $R_{u'}$ such that $u < u'$ and go on to stage $s+2$.

Case 4: $R_u = N_{i,j,e}$.

We intend again to satisfy this requirement by diagonalization. We adopt essentially the strategies of $D_{i,j,e}$, but now, because we are diagonalizing against L, we must add the realized follower to G_l as well as G_i in order to satisfy $G_i \leq_Q G_l \oplus L$, when $i \preceq l$.

If, for some k, $x^u_k[s]\downarrow$ and $x^u_k[s] \in G_i[s]$, then R_u is currently satisfied, so we do nothing.

If $x_0^u[s]\uparrow$, then let $x_0^u[s+1] = \langle u, s+1\rangle$, and continue with stage $s+1$.

If $x_{k+1}^u[s]\uparrow$, $x_k^u[s]\downarrow$, $(W_{\Phi_e(x_k^u)} \not\subseteq G_j \oplus L)[s]$, $(x_k^u \notin G_i)[s]$ (*i.e.*, R_u is currently unsatisfied), and $\langle l, x_k^u\rangle \in r_C^{u'}[s]$ for some $u' < u$ and l with $i \preceq l$ (including the case $i = l$), then we set $x^u[s+1]$ to be the least element of $\omega^{[u]}$ greater than $x_k^u[s]$. If $x^u[s]\downarrow$, then if $(W_{\Phi_e(x)} \not\subseteq G_j \oplus L)[s]$, and $\langle i, x_k^u\rangle \notin r_C^{u'}[s]$ for any $u' < u$ we enumerate $x_k^u[s] \in G_i[s+1]$, and, for all l with $i \preceq l$, $x_k^u[s] \in G_k[s+1]$. In this case, we initialize all requirements $R_{u'}$ such that $u < u'$ and go on to stage $s+2$.

Case 5: $R_u = K_{e,x}$.

If $\Phi_e(C;x)\downarrow[s]$, then we initialize all $R_{u'}$ such that $u < u'$. We then proceed immediately to stage $s+2$.

Verification:

We show that each requirement R_u is satisfied, and imposes only finite restraint on the construction. After this, we define Q reductions to show that each $P_{i,j}$ is satisfied.

First, note that it is obvious from the construction that once a requirement has been initialized for the last time, we only act once for it to initialize other requirements or put numbers into A, B, or C, thus each requirement R_u is initialized only finitely often, and

This immediately implies that every lowness requirement $K_{e,x}$ is satisfied, since once a stage s is reached after which $K_{e,x}$ will never again be initialized, and $\Phi_e(B\oplus C;x)\downarrow[s]$, all lower priority requirements are initialized, forcing them to pick new witnesses greater than $\phi_e(B \oplus C;x)[s]$, so that $\Phi(B \oplus C;x)\downarrow[t]$ for every $t \geq s$.

We now go through the other requirements in order, proving by induction that each requirement R_u is satisfied and imposes only finite restraint. Therefore, suppose that the hypotheses are satisfied for all $R_{u'}$ with $u' < u$, and that we are beyond some stage at which every such $R_{u'}$ has enumerated an element into one of the sets A, B, or C if it ever will.

Case 1: $R_u = T_i$.

We only have to ensure that R_u is satisfied, since these requirements never act to initialize any others and they set no restraints. In fact, we merely show that Γ is correct on followers picked by requirements of lower priority than R_u. This is a computable set, and its complement's intersection with A is obviously Q-reducible to $G_i \oplus B$, since it is finite. Let $W_{\Gamma(y)} = \{\, \gamma(y)[s] \,:\, s \in \omega \,\}$.

If $x \notin A$, note that at every stage greater than x we correct Γ_i. The second time we do this, say at stage s, we have $\gamma_i(x)[s] = 2s(x)+1$, where $s(x) > s$ is the sth element of $\omega^{[x]}$. But then $\gamma_i(x)[s]$ never enters B unless x is enumerated into A, since it is only such actions which cause any enumeration into B. If $x \in A$, it is obvious from the description of the action of the minimality strategies of the lower priority requirements that x cannot be enumerated into $A[s+1]$ unless the use $\gamma_i(x)[s+1]$ is also enumerated into $(G_i \oplus B)[s+1]$, after which no new use $\gamma_i(x)[s']$ is ever set.

Case 2: $R_u = M_{a,b,e}$.

Let the true path f be defined recursively by

$$f(n) = \liminf_{s\to\infty} \{\, j \,:\, (f \upharpoonright n)^\frown\langle j\rangle \text{ is allowed to act at } s \,\}.$$

We show by induction that for each $\alpha \subset f$, R_α is satisfied, and $r^\alpha_B[s]$ and $r^\alpha_C[s]$ are both finite and eventually constant. First, we verify that if $W_e \leq_Q C$ via Φ_a and $A \leq_Q W_e \oplus B$ via Φ_b, then there are infinitely many α-expansionary stages. In fact, we show that $\lim_{s\to\infty} l^\alpha(s) = \infty$. Suppose not; then $\liminf_{s\to\infty} l^\alpha(s) = x \in \omega$. Clearly $\Phi_b(x)\downarrow$ and for every y in $W_{\Phi_b(x)}$, $(\Phi_a(\lfloor \frac{y}{2} \rfloor)\downarrow$.

If $x \notin A$, then $W_{\Phi_b(x),t} \not\subseteq (W_e \oplus B)$ at every sufficiently large t, since, in fact, there is some unique $\phi_b(x) \in W_{\Phi_b(x)} - (W_e \oplus B)$. If $\phi_b(x)$ is odd, then $x < l^\alpha[t]$, for every sufficiently large t. If $\phi_b(x) = 2y'$ for some $y' \notin W_e$, then since $W_e \leq_Q C$ via Φ_a, we must have some $\phi_a(y') \notin C$. But then, even after $w^\alpha[t]$ is greater than y', $x < l^\alpha[t]$.

If $x \in A$, then suppose we are at a W_e-true α-stage s. Such a stage obviously exists since there are infinitely many α-stages. If $W_{\Phi_b(x)} \not\subseteq (W_e \oplus B)[s]$, then either $\phi_b(x)[s]$ is odd, in which case $x < l^\alpha[s]$, or there is some y' such that $\phi_b(x)[s] = 2y'$. If $w^\alpha[s] \leq y'$, then $x < l^\alpha[s]$. Otherwise, suppose $y' < w^\alpha[s]$. Then, note that $y' \notin W_e$, since s is W_e-true. This is a contradiction, since then $W_{\Phi_b(y)} \not\subseteq (W_e \oplus B)$.

We next show that r^u_B and r^u_C are finite, and eventually constant. By inductive hypothesis, since α is on the true path, we can get only into trouble if there are only finitely many α-expansionary stages. But in this case, once the greatest one is reached after the approximation to the true path last initializes α, both restraint functions remain constant on every α-stage. (This follows straightforwardly by induction on the α-stages.)

There is nothing left to prove if there are only finitely many α-expansionary stages, since the requirement is satisfied in that case.

Otherwise, suppose there are infinitely many α-expansionary stages. Then $f(|\alpha|) = 0$, since infinitely many α-targets are defined and eventually reached. Let $n = |\{\, T_i \,:\, |T_i| < u \,\}|$. Recall that we have assumed we are beyond any stage at which any strategy for any $R_{u'}$ with $u' < u$ acts, so α is never again initialized.

If there is some $y \in W_e[s]$ such that $\phi_a(y)[s]\downarrow$, then $(W_{\Phi_a(y)} - C) \neq \emptyset$, since all $R_{u'}$ such that $u < u'$ are immediately initialized at $s+1$ and we go on to stage $s+2$. Hence $W_e \not\leq_Q C$ via Φ_a, and there are no more α-expansionary stages after s, a contradiction.

If there is some $y \in A[s]$ such that $\phi_b(y)[s]\downarrow$ and either $\phi_b(y)$ is odd, or $\phi_b(y)$ is even and $\phi_a(\frac{\phi_b(y)}{2})[s]\downarrow$, a similar argument shows that either $y \in A$ and $W_{\Phi_b(y)} \not\subseteq W_e \oplus B$, or $\frac{\phi_b(y)}{2} \in W_e$ and $W_{\Phi_a(\frac{\phi_b(y)}{2})} \not\subseteq C$, and again there are no more α-expansionary stages after s.

If there is some $y \notin A[s]$, with $\phi_b(y)[s]\downarrow$, $y = \langle\langle a,b,e\rangle, \langle x_0,\ldots,x_{n-1}\rangle\rangle$, and y is enumerated in $A[s+1]$, then all requirements $R_{u'}$ such that $u < u'$ are initialized at $s+1$. This can happen for two reasons. If $\phi_b(y) = 2y'+1$ is odd, then as above, $y \in A$ and $W_{\Phi_b(y)} \not\subseteq W_e \oplus B$. On the other hand, it may be that $\phi_b(y)$ is even, say $\phi_b(y) = 2y'$, $\phi_a(y')[s]\downarrow$ and for all $j < n$ $x_j \in G_j[s]$. In this case, every $\gamma_j(y)[s]$ is odd, so that adding each $\gamma_j(y)[s]$ to $(G_j \oplus B)[s+1]$, preserves $(W_{\Phi_a(y')} \not\subseteq C)[s+1]$. Hence, as above, either $y \in A$ and $W_{\Phi_b(y)} \not\subseteq W_e \oplus \ B$, or $\frac{\phi_b(y)}{2} \in W_e$ and $W_{\Phi_a(\frac{\phi_b(y)}{2})} \not\subseteq C$, and there are no more α-expansionary stages after s.

In all these cases, this shows that R_u is satisfied. It only remains to show that R_u is satisfied if there are infinitely many α-expansionary stages. For all $x \in \omega$, let $W_{\Delta_j(x)} = \{\, \delta_j(x)[s] \,:\, s \in \omega \,\}$. If there is any $j < n$ such that for almost all x,

$$x \in G_j \iff W_{\Delta_j(x)} \subseteq W_e,$$

then R_α is satisfied. Hence each Δ_j must be incorrect on infinitely many $x \in \omega$.

The total B-restraint on α at stage s is

$$\mathcal{R}(B,\alpha)[s] = \{\, x \in r_B^\beta[s] \,:\, \beta < \alpha \,\}.$$

By inductive hypothesis, there exists some stage t_0 such that for every α-stage $s > t_0$, $\mathcal{R}(B,\alpha)[s] = \mathcal{R}(B,\alpha)[t_0]$. Similarly, if

$$\mathcal{R}(C,\alpha)[s] = \{\, x \in r_C^\beta[s] \,:\, \beta < \alpha \,\}.$$

then we can assume $\mathcal{R}(C,\alpha)[s] = \mathcal{R}(C,\alpha)[t_0]$, at every α-stage $s > t_0$.

We first show that for almost all x, Δ_{n-1} can only be incorrect on x if $x \in G_{n-1}$ and $W_{\Delta_{n-1}(x)} \not\subseteq W_e$. To see this, suppose x is least such that $x \notin G_{n-1}$ yet $W_{\Delta_{n-1}(x)} \subseteq W_e$, and for all j, $\langle j, x\rangle \notin \mathcal{R}(C,\alpha)[s]$. Then, at some stage $s > s_0$ where $\delta_j(x)\uparrow$, some $y = \langle\langle a,b,e\rangle, \langle x_0, \dots, x_{n-1} = x\rangle\rangle$ is chosen as α-target. By the construction, once the α-target y is reached, at, say, stage s', we have some $y' \notin W_e[s']$, which is set to be $\delta_{n-1}(x)[s+1]$. Since $W_e \leq_Q C$, $W_e \leq_T C$, and hence there is some computable functional Φ_d such that $\Phi_d(B \oplus C; x)\downarrow$ if and only if $\exists s' \exists y'(y' \notin W_e$ and $\delta_{n-1}(x)[s'] = y')$. We have just shown that there exist infinitely many s' such that $\Phi_d(B \oplus C; x)\downarrow[s']$. Hence, since requirement $K_{d,x}$ is satisfied, $\Phi_d(B \oplus C; x)\downarrow$, so that Δ_{n-1} does not fail on x.

If $y \notin A$ and $\phi_b(y)$ is even, we write y' for $\frac{\phi_b(y)}{2}$. Since Δ_{n-1} does not reduce G_{n-1} to W_e, there must therefore exist an infinite sequence $x_{n-1}^0, x_{n-1}^1, \dots$ such that, for all $i \in \omega$,

$$y(x_{n-1}^i)' = \delta_{n-1}(x_{n-1}^i) \notin W_e, \text{ but } x_{n-1}^i \in G_{n-1}.$$

Clearly $y(x_{n-1}^i) = \langle\langle a,b,e\rangle, \langle x_0, \dots, x_{n-2}, x_{n-1}^i\rangle\rangle$, for some $x_0, \dots, x_{n-2}$.

Suppose that for some $j < n-1$, there exists an infinite sequence of numbers $x_{j+1}^0, x_{j+1}^1, \dots$ such that, for all $i \in \omega$,

$$y(x_{j+1}^i)' = \delta_{j+1}(x_{j+1}^i) \notin W_e, \text{ but } x_{j+1}^i \in G_{j+1}.$$

Then we must have $y(x_{j+1}^i) = \langle\langle a,b,e\rangle, \langle x_0, \dots, x_j, x_{j+1}^i, \dots, x_{n-1}\rangle\rangle$, for some $x_0, \dots, x_j, x_{j+2}, \dots, x_{n-1}$, such that $x_{j+2} \in G_{j+2}, \dots x_{n-1} \in G_{n-1}$, by the definition of δ_{j+1}.

We show that in this case, for almost all x, Δ_j can only be incorrect on x if $x \in G_j$ and $W_{\Delta_j(x)} \not\subseteq W_e$. To see this, suppose x is least such that $x \notin G_j$ yet $W_{\Delta_j(x)} \subseteq W_e$, where for all j, $\langle j, x\rangle \notin \mathcal{R}(C,\alpha)[s]$. Then, at any stage $s > s_0$ where $\delta_j(x)\uparrow$, some $y = \langle\langle a,b,e\rangle, \langle x_0, \ldots, x_j = x, x_{j+1}, \ldots, x_{n-1}\rangle\rangle$ is chosen as α-target. Recall, there are infinitely many x^i_{j+1} such that

$$y(x^i_{j+1})' = \delta_{j+1}(x^i_{j+1}) \notin W_e, \text{ but } x^i_{j+1} \in G_{j+1}.$$

Hence, for every s, since x is continually picked as the jth coordinate of the α-target at every stage s' where $\delta_j(x)[s']\uparrow$, eventually some α-target y is chosen at some s' such that

$$y = \langle\langle a,b,e\rangle, \langle x_0, \ldots, x_j = x, x_{j+1}, \ldots, x_{n-1}\rangle\rangle,$$

for some x_{j+1} such that at $t' > s'$, Δ_{j+1} fails on x_{j+1}. (This must eventually happen, since some $x^i_{j+1} > s'$) But then, at stage t', $\delta_j(x)[t'+1] = y'$. As in the case of Δ_{n-1} above, then, $\delta_j(x)\downarrow$ infinitely often, and δ_j is essentially a $(B \oplus C)$-computable partial function. Thus, the lowness requirements K guarantee that $\delta_j(x)\downarrow = y' \notin G_j$. This is a contradiction, since it implies Δ_j is correct on x.

Since Δ_j does not reduce G_j to W_e, by hypothesis, there must therefore exist an infinite sequence $x^0_j, x^1_j, \ldots$ such that, for all $i \in \omega$,

$$y(x^i_j)' = \delta_j(x^i_j) \notin W_e, \text{ but } x^i_j \in G_j,$$

where $y(x^i_j) = \langle\langle a,b,e\rangle, \langle x_0, \ldots, x^i_j, xj+1, \ldots, x_{n-1}\rangle\rangle$, for some $x_0, \ldots, x_{n-2}$.

This shows, by induction, that in fact, since Δ_0 is incorrect infinitely often, that there must in fact exist $x^0_0 < x^1_0 < \ldots$ such that, for all $i \in \omega$,

$$y(x^i_0)' = \delta_0(x^i_j) \notin W_e, \text{ but } x^i_0 \in G_0,$$

where $y(x^i_0) = \langle\langle a,b,e\rangle, \langle x^i_0, x_1 \ldots, x_{n-1}\rangle\rangle$, for some $x_1, \ldots, x_{n-1}$. Note that, by definition of $\delta_j(x_j)$, we must have each $x_{j+1} \in G_{j+1}$,for every $j < n-1$ in every such sequence attached to some x^i_0.

Let $b = \max(\mathcal{R}(B,\alpha))$, where $\mathcal{R}(B,\alpha)[s] = \mathcal{R}(B,\alpha)[t_0]$, the permanent B-restraint on α. To achieve a contradiction, we need only consider $x_0 = x^b_0$. Let $y = \langle\langle a,b,e\rangle, \langle x_0, x_1 \ldots, x_{n-1}\rangle\rangle$, and let s_j be the unique stage such that $x_j \in G_j[s_j+1] - G_j[s_j]$. Then $s_0 > s_1 > \ldots > s_{n-1}$. Clearly $s_{n-1} > b$. Let t be the next α-expansionary stage after s_0 such that $y' < w^\alpha[t]$. Note that $y \notin A[t]$ and $y' \notin W_e$, so such a stage is guaranteed to exist. Since t is α-expansionary, $y < l^\alpha[t]$, and thus, since $y' < w^\alpha[t]$, we must have $\phi_a(y)\downarrow[t]$. Note that, since each $x_j \in G_j[s_j+1]$, $\gamma_j(y) = 2t_j + 1$ for some $t_j > s_j + 1 > b$. Hence at stage $t+1$, the α strategy enumerates $y \in A[t+1]$ and for every $j < n$, $t_j \in B[t+1]$, after which all lower priority requirements are initialized. But now $2y' \in W_{\Phi_b(y)} \not\subseteq (W_e \oplus B)$, so $A \not\leq_Q W_e \oplus B$ via Φ_b, a contradiction. This shows that the requirement is satisfied.

Case 3: $R_u = D_{i,j,e}$.

Let $v = |\{ v' : \exists u' < u\, R_{u'} = M_{v'} \}|$. Let α be the initial segment of the true path of length v. There exists some stage t such that at every α-stage t' after t, $r^{\alpha}_{C}[t'] = r^{\alpha}_{C}[t]$, and this is the restraint on R_u. Clearly, at some stage $s > s_0$, the stage at which R_u is last initialized, some witness $x = x^u_k[s]\downarrow$ and $\langle i, x\rangle \notin r^{\alpha}_{C}[t]$. If there exists some α-stage $s' > s$ such that $(W_{\Phi_e(x)} \nsubseteq G_j)[s']$, then $x \in G_i[s'+1]$, and all lower priority requirements are initialized at $s'+1$. We also enumerate x into L at this stage, but this clearly has no effect on G_j. Hence $W_{\Phi_e(x)} \nsubseteq G_j$, and R_u is satisfied. This, of course, is also the case if x enters G_i at a non-α-stage. Otherwise, at every α-stage $s' > s$, $(W_{\Phi_e(x)} \subseteq G_j)[s']$, and $x \notin G_i$. Since there are infinitely many α-stages, this can only mean that $W_{\Phi_e(x)} \subseteq G_j$, again yielding that $G_i \nleq_Q G_j$ via Φ_e.

Case 4: $R_u = N_{i,j,e}$.

The argument here is analogous to that for the requirements $D_{i,j,e}$ above. Eventually some witness x is chosen which can bypass the eventually-constant restraint which holds at every α-stage, for an appropriate initial segment of the true path α. Once x is chosen, Φ_e is then forced to fail on x, just as in Case 3 above.

Finally, we show that the requirements $P_{i,j}$ are satisfied. In fact, the reduction is not merely Q, but many-one. We merely examine which requirement each $x \in \omega$ could possibly be assigned to, and choose $\theta(x) = 2x$ or $2x + 1$, depending on which sort of diagonalization we intend to achieve on x. More precisely, suppose $x \in \omega^{[u]}$. If $R_u = N_{i',j',e}$, then let $\theta(x) = 2x$. If $R_u = D_{i',j',e}$, then let $\theta(x) = 2x+1$. If R_u is any other type of requirement, let $\theta(x) = 2x+1$. It is clear from a straightforward examination of cases of the construction that if $x \in \omega^{[u]}$, then, if R_u is not a requirement of type $N_{i',j',e}$, R_u enumerates x into some G_i if and only if it enumerates x into L; and, if R_u is some $N_{i',j',e}$,then R_u enumerates x into G_i if and only if it enumerates x into G_k for every k with $i \preceq k$. It therefore follows that $\theta : G_i \leq_m G_j \oplus L$, and, hence, letting $W_{\Theta(x)} = \{\theta(x)\}$, that $G_i \leq_Q G_j \oplus L$ via Θ. This completes the proof.

□

By the Coding Lemma, we immediately have the following:

Corollary 9. *For any countable, infinite computable partial order $\mathcal{P}$ there exist Q-degrees* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{l}$, *such that an interpretation of $\mathcal{P}$ is first-order definable with parameters* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{l}$ *in $\mathcal{R}_Q$*

Proof. Let A, B, C, L, and $\langle G_i | i \in \omega\rangle$ be as in the Coding Lemma, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{l}$, and $\{ \mathbf{g_i} : \mathbf{i} \in \omega \}$ be their respective degrees. Note that $\mathbf{x} \in \{ \mathbf{g_i} : \mathbf{i} \in \omega \}$ if and only if

$$(\mathbf{a} \leq \mathbf{x} \vee \mathbf{b}) \text{ and } \forall \mathbf{y} \leq \mathbf{c}(\neg(\mathbf{y} < \mathbf{x} \text{ and } \mathbf{a} \leq \mathbf{y} \vee \mathbf{b})).$$

Also, for all i and j, $i \preceq j$ if and only if

$$\mathbf{g_i} \leq \mathbf{g_j} \vee \mathbf{l}.$$

This immediately gives the required interpretation of $\mathcal{P}$. □

Undecidability now follows as discussed above.

Theorem 10. *$\mathcal{R}_Q$ has an undecidable first order theory.* $\square$

We remark that this leaves open the question of the Turing degree of the theory of $\mathcal{R}_Q$. It seems natural to conjecture that, as in the analogous case of the c.e. Turing degrees, the degree is that of true arithmetic. In this case, however, we have not been able to achieve the permitting needed to embed partial orders below a given sequence of low degrees, which is needed to show that prime (and therefore standard) models of arithmetic can be picked out without parameters.

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