Notre Dame Journal of Formal Logic
Volume 32, Number 4, Fall 1991

# On $\Pi_1^0$ Classes and their Ranked Points

ROD DOWNEY

**Abstract** We answer a question of Cenzer and Smith by constructing a nonzero degree each of whose members is a rank one point of a $\Pi_1^0$ class. The technique of proof is a rather unusual full approximation argument. This method would seem to have other applications.

***1 Introduction*** In this paper, we solve a problem of Cenzer and Smith [3] concerning ranked points for $\Pi_1^0$ classes in $2^\omega$. Here the reader should recall that a member (point) of a $\Pi_1^0$ class $P$ is called *ranked* if, for some ordinal $\alpha$, $x \notin D^\alpha(P)$ where $D^\alpha$ denotes the $\alpha$-th Cantor-Bendixon derivative. This is defined via

$$D(P) = \{x \in P : x \in \mathrm{Cl}(P - \{x\})\}$$
$$D^0(P) = P,\ D^{\alpha+1}(P) = D(D^\alpha(P)), \text{ and}$$
$$D^\alpha(P) = \bigcap_{\beta<\alpha} D^\beta(P) \text{ for } \alpha \text{ a limit ordinal.}$$

The *rank* of a ranked point $y$ *in* $P$ is the least $\alpha$ with $y \notin D^{\alpha+1}(P)$, and the rank of $y$ is the minimum rank in all $P$ such that $y$ is a ranked point in $P$. Ranked points in $\Pi_1^0$ classes have been extensively investigated in, for example, Jockusch and Soare [6,7], Clote [4], Cenzer et al. [1,2], and Cenzer and Smith [3]. It is known that all $\Delta_2$ degrees contain ranked points [3], but not all points can be ranked. For example, all the nontrivial iterated jumps of $\varnothing$ cannot be ranked and all hyperimmune degrees (see Section 2) contain unranked points.

These results left the following question open: do all degree $\neq 0$ contain unranked points? In Section 2 we answer this question by showing that:

**Theorem** *There is a completely ranked degree below* $0''$, *that is, there exists* $a$ *with* $0 < a < 0''$ *such that if* $A \in a$ *then* $A$ *is ranked. Indeed for all* $B \leq_T A$, $B$ *has rank* $\leq 1$.*

*The author thanks Richard Shore for pointing out that this stronger conclusion follows from the proof in §2.

*Received June 1, 1990; revised April 10, 1991*

The proof of the above theorem will consist of the construction of a set $A$ of hyperimmune-free degree (see Section 2) with $A$ of rank 1. As we will see, the result will then follow by some work of Miller and Martin [8] and an application of a lemma of Cenzer and Smith [3]. Notation is standard and follows, e.g. Odifreddi [9] or Soare [10].

*2 The proof* We achieve our goal by constructing a nonrecursive set $A$ of hyperimmune-free degree such that $A$ has rank 1. Here the reader should recall that $A$ is hyperimmune-free iff each function $f \leq_T A$ is majorized by a recursive function. The crucial property of such degrees is:

**Lemma 2.1** (Jockusch [5], Miller and Martin [8]) *If $A$ has hyperimmune-free degree and $B \leq_T A$, then $B \leq_{tt} A$.*

Our result will then follow due to the following result of Cenzer and Smith.

**Lemma 2.2** (Cenzer and Smith [3]) *If $B \leq_{tt} A$ and $A$ has rank $n$, then $B$ has rank $m$ for some $m \leq n$.*

We now turn to the construction of $A$. To do this we will use a '$\Pi_2$-full approximation' construction of a hyperimmune-free degree. We must be very careful not to have too many splittings. For example, the standard construction of Miller and Martin [8] (cf. Odifreddi [9, Ch. V.5]) uses perfect trees. To "fully approximate" such a construction would apparently result in unranked points. (See also the comments at the end of Section 3.) We must keep $A$ of rank 1, and therefore topologically we must essentially build "something like" the tree of Figure 1.

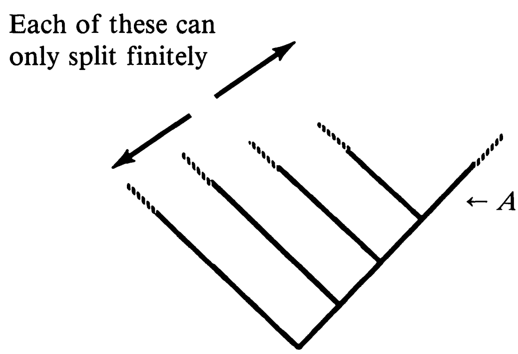


Figure 1.

This necessitates a great deal of delicacy in the construction. In particular, quite distinct from other full approximation constructions, we do not construct at each stage $s$ a finite set of trees and intersect them. Rather we construct $A$ as the unique rank 1 path through a $\Pi^0_1$ class $C$ with all other points of $C$ having rank 0 (as in Figure 1).

The reader should keep the following idea in mind: At any stage $s$, we will have a finite tree $C_s$ that will consist of a number of strings we have (irrevocably) declared to be *terminal*, together with a number of strings we currently believe form initial segments of paths through $[C] = [\bigcup_s C_s]$.

At any stage $s+1$ to create $C_{s+1}$ we will either extend a string $\sigma$ on $C_s$ to a longer string (of length $s+1$) or declare such a string to be terminal. Hence, the only possible transition of a string $\sigma$ on $C_s$ is to go from one we believe will have an infinite extension to one that is terminal. Note that if at all stages $s$ we believe $\sigma$ on $C_s$ has an infinite extension, then $\sigma$ has an infinite extension on $[C]$.

Another picture the reader should have is the following: At any stage $s$ we will have a unique string $\sigma$ on $C_s$ of length $s$ that we believe will be an initial segment of the unique rank one point $\beta$ of $[C]$. At stage $s$, this will be the only string from which we will allow (new) splittings to form. Thus, although other strings on $C_s$ can be extended at stage $s$, they will not form split extensions. This is, by itself, not enough to cause $[C]$ to be isomorphic to the tree of Figure 1, since we may still form many splittings at stages $t \neq s$ elsewhere; nevertheless, it is an underlying idea that the reader should keep in mind. The other main device we will use to keep $[C]$ isomorphic to the tree of Figure 1 is that if we change our minds at some stage $t > s$ and believe that $\sigma$ is really not an initial segment of $\beta$ then, roughly, we will cancel most of the false splittings we made which extended $\sigma$. (See Figure 4 wherein we once thought that $\gamma_1 (= \sigma)$ was an initial segment of $\beta$. As we will see when we discuss the $R_e$, we no longer believe this and believe that, e.g. $\gamma_2$ is an initial segment of $\beta$. So here, all the "false splittings" based on the "wrong guess" have died.)

We now turn to the exact details of the mechanism by which we achieve our goals. We must meet the requirements

$R_e : \Phi_e(A)$ total implies $\Phi_e(A)$ is majorized by some recursive function

$N_e : A \neq W_e$.

To meet $R_e$ the basic idea is to construct a recursive tree ($\Pi_1^0$ class) $T_e$ so that $A$ is a path on $T_e$. This tree will have the property that either there is an $n$ such that for all paths $P$ on $T_e$, $\Phi_e(P; n)\uparrow$ or $\Phi_e$ is total on all paths.

Before we consider the $R_e$ in detail, it is best to first review the Friedberg strategy we use for $N_j$. For the sake of $N_j$ there will be a follower $n$ (in fact many "guessed" versions as we will see) and ensure that in the $\Pi_1^0$ class $C$ we construct, there are nodes $\sigma, \sigma_1, \sigma_2$, each (initially) having extensions in $C_s$ (the part of $C$ we've built by stage $s$) with the following property: We have $\sigma_1(n) = 0$, $\sigma_2(n) = 1$ and $\sigma_1 = \sigma^\wedge 0$, $\sigma_2 = \sigma^\wedge 1$. Initially, we route the approximation to $A$ through $\sigma_2$ (this means no splittings for $\tau \supset \sigma_1$ in $C$). But we keep the option of routing through $\sigma_1$ by making sure that there is one extension of $\sigma_1$ in $C_t$ for $t \geq s$. If we ever see a stage $u$ with $n \in W_{e,u}$ we ask that (for the basic module) $A \supset \sigma_1$ by making all extensions of $\sigma_2$ die in $C_{u+1}$ and, with the appropriate priority, ask that henceforth we route all possible $A$ through $\sigma_1$. This wins since $n \in W_e - A$. See Figure 2 for a typical situation.

The reader should note that in the full construction since $A$ will be only $\Delta_3$ it will not always be possible to keep $A_q \supset \sigma_1$ for $q \geq u+1$, but if $n$ was the follower with the $\pi_2$-correct guess then, infinitely often, $A_q \supset \sigma_1$ and the only path

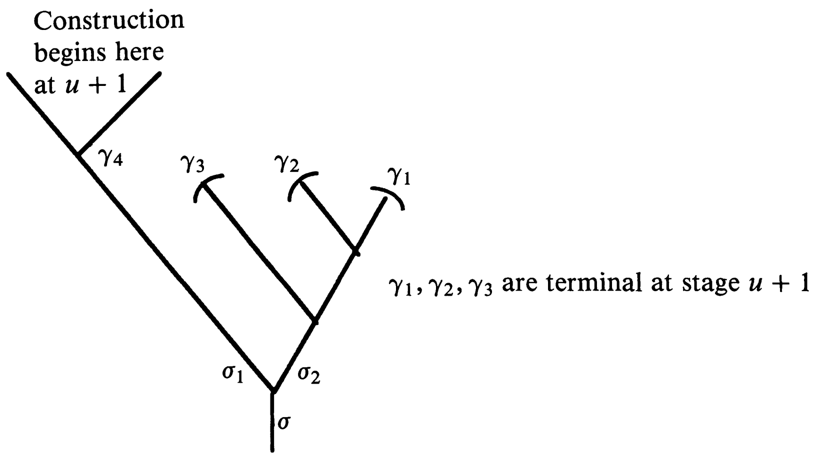


Figure 2.

of rank $\neq 0$ will extend $\sigma_1$. (This path can be figured out via a $\pi_2$ complete oracle. More on this later.)

But it is important for the reader to note that once we are dealing with the version of $N_j$ of the 'correct guess' and all the higher priority versions of $N_k$ have finished acting then the construction will ensure that $A$ either extends $\sigma_2$ or $\sigma_1$; and furthermore since $N_j$ has the correct guess, there will either be exactly one path in $[C]$ extending $\sigma_1$ (so $A \supset \sigma_2$) or there will be no path extending $\sigma_2$ (and so $A \supset \sigma_1$). This will make the verification that $A$ has rank one easy, provided we can argue that "guesses" settle down as we now see.

We need $R_e$ to operate in the arena given by the above. It will be the case that $R_e$ will not *cause splittings*, but only "shift around" where we think the "construction" should happen. (That is, where we *allow* splittings to be caused by $N_k$ for $k > e$.)

***The basic module for $R_e$*** The real underlying idea behind satisfying $R_e$ is to try to find some string $\gamma$ so that if $A \supset \gamma$ and $A \in [C]$ then $\Phi_e(A)$ is not total. This will entail finding some number $m$ and some string $\gamma$ so that

(i) for all $P \in [C]$ if $P \supset \gamma$ then $\Phi_e(P; m)\uparrow$, and
(ii) $\gamma$ has infinitely many extensions in $[C]$.

If we *fail* to find some $m$ and $\gamma$ we will ensure that $\Phi_e(P)$ is total for (almost) all $P \in [C]$. (This is the $\pi_2$ outcome.)

As we will see, we will verify the $\pi_2$ outcome as follows. First at some stage $s_0$ we will ensure that for all nonterminal $\sigma$ of length $s_0$ on $C_{s_0}$ we have $\Phi_{e,s_0}(\sigma; 0)\downarrow$. Having done this we will only then believe that $\Phi_e(A)$ is total (for one stage). We will then ensure that for all $\sigma$ of length $s_1$ on $C_{s_1}$ we have $\Phi_e(\sigma; 1)\downarrow$, etc. We will do this by testing strings one at a time in this level-by-level method. It is really best to think of our procedure as trying to find a cone where we can force divergence on some fixed argument. We will now look in more detail at the mechanism by which we achieve our goals. A typical situation we need to consider is given in Figure 3 below.

In Figure 3 we are waiting for $N_j$ to act. Suppose we "knew" that for some $n, i$ with $1 \leq i \leq 5$ and for all $\mu \in 2^{<\omega}$,

$$\text{if } \mu \supset \gamma_i \text{ then } \Phi_e(\mu; n)\uparrow.$$

In that case, our strategy would be easy. Without loss of generality, let $i = 3$. We would first either terminate $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ or at least make sure there were at most finitely many splittings in $C$ extending $\gamma_1, \gamma_2, \gamma_4, \gamma_5$. Second we would then ensure that we would work in the "cone" above $\gamma_3$. That is, ensure that if $\beta$ is a path on $[C]$ then, with the possible exceptions above, $\beta \supset \gamma_3$.

Unfortunately, we cannot know if an $n$ and $i$ exist (and furthermore we will not be working with all $\mu \supset \gamma_i$, rather only those $\mu$ on $C_t$ some $t \geq s$). The nonexistence of $(n, i)$ is a $\pi_2$ question. However, the idea is to approximate the $\pi_2$ question by an infinite collection of $\Sigma_1$ questions. We test the $n$'s one at a time, and the $\gamma_i$'s one at a time for each $n$, as described above. Now, inductively suppose we have considered $m - 1$. That is, we know for all $i$ with $1 \leq i \leq 5$ that $\Phi_e(\gamma_i; k)\downarrow$ for all $k \leq m - 1$. We wish to deal with $m$. Thus, we will test $(\gamma_1, m)$ then $(\gamma_2, m), \ldots, (\gamma_5, m)$ and then return to $\gamma_1$, if they all "test positive". So, to begin with, we try to *verify* $(\gamma_1, m)$. That is, we try to find a nonterminal extension $\tau \supset \gamma_1$ in $C_s$ such that $\Phi_{e,s}(\tau; m)\downarrow$.

While we do this, we will *directly extend* $\gamma_2, \ldots, \gamma_5$ by a single extension at each stage $u \geq s$ (e.g., if $\mu \supset \gamma_j$ in $C_u$ then $\mu\hat{\ }0 \supset \gamma_j$ in $C_{u+1}$ for $j \in \{2,3,4,5\}$). Meanwhile, we continue our construction based on the belief that

(2.1) For all $\tau \supset \gamma_1$ on $C$ if $\Phi_{e,t}(\tau; m)\downarrow$ then at some stage $q < t$ all extensions of $\tau$ in $C_q$ are terminal.

Note the *timing* element in (2.1). We do not claim that $\Phi_e(\tau; m)\uparrow$ for all $\tau \supset \gamma_1$, we claim only that such $\tau$'s are not initial segments of the $A$ we construct.

How should we perform such a construction based on (2.1)? Clearly the desirable thing to do (for lower priority $R_j$) is to begin anew with a version that

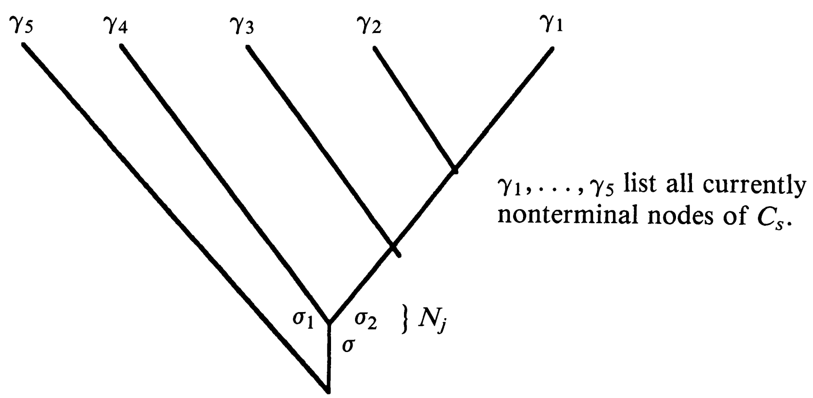


Figure 3.

only seeks verification in the cone above $\gamma_1$. That is, based on (2.1) we will only allow $N_k$ for $k \geq e$ to form splittings in the cone above $\gamma_1$ and $R_j$ for $j > e$ will only need to do their verifications for strings in $C_s$ in this cone. This is acceptable, since we get a *global win* on $R_e$, since the threat is that we are putting the rank one point above $\gamma_1$ where possibly (2.1) holds.

Note that another coherent strategy is to make all nodes extending $\gamma_1$ terminal (to verify (2.1)). We could do this if $N_j$ acts via $(\sigma, \sigma_1, \sigma_2)$, but must ask (for priority reasons) that the follower $n$ of $\sigma$ has $m > n$ so that this is an '$e$-correct' follower in the usual tree of strategies sense.

Assuming $N_j$ does not so act, if we see in the course of the construction $\Phi_{e,r}(\tau; m)\downarrow$ on some $\tau \supset \gamma_1$, not satisfying (2.1), then from $R_e$'s point of view $(\gamma_1, m)$ has been verified and we can (for instance, for a single $R_e$ alone) cancel (by making terminal) all *save one* extensions of $\gamma_1$ and keep only one extension of $\tau$. Figure 4 shows this for a *single $R_e$ alone.*

The reader should note that, if we desired (from $R_e$'s point of view), we could also split the extension $\mu_1$ of $\tau$ as $(\mu_1, m)$ is verified. We will need to do this sometimes, but need real care since we must keep the $\Pi_1^0$ class having a unique rank 1 point and all the others of rank 0. The time we will need to do this is when $N_k$ for some $k$ requests a follower. If it turns out that for almost all paths $P$ on $[C]$, "$\neg$(2.1)" holds (that is, for almost all $P$ on $[C]$ we force, for almost all $m$, $\Phi_e(P; m)\downarrow$) then we will need to ensure that $N_k$ has an arena in which to act. In particular, it will need a follower equipped with the correct guess as to $R_e$'s behavior. The idea is to designate essentially the rightmost nonterminal path on $C$ as the one we will use for splittings. Thus, for $\gamma_1$ we ask that we see two extensions $(\mu_1, \mu_2)$ with $\Phi_e(\mu_1, m)$, $\Phi_e(\mu_2, m)\downarrow$. As there are two extensions for some $g \geq m$ (inductively) we will have $\mu_1(g) \equiv \mu_2(g)$. We will give $\mu_1$ and $\mu_2$ the high $e$-state and appoint $g$ as a follower of $N_k$ with the correct (high) $e$-state. We can even wait until we see $\tau_1 \supset \mu_1$, $\tau_2 \supset \mu_2$ with $\Phi_e(\tau_1; j)\downarrow$, and $\Phi_e(\tau_2; j)\downarrow$ for all $j \leq g$; and then use $\tau_1$ for $\mu_1$ and $\tau_2$ for $\mu_2$. Note that unless some higher priority $N_q$ for $q < k$ terminates the extensions of $\gamma_1$, a consequence of this will be that we will terminate all save one extension of $\gamma_2, \gamma_3, \gamma_4, \gamma_5$ since they know that if we are in the height $e$-state, then the rank 1 point should extend $\gamma_1$. We discuss this further when we describe the inductive strategies later. After we verify $(\gamma_1, m)$ and get the situation in Figure 4, we will then verify $(\gamma_2, m)$ similarly. That is, we directly extend $\gamma_5, \gamma_4, \gamma_3$, and $\mu$ (or its two extensions, as the case may be) and begin the construction in the cone above $\gamma_2$ based on (2.1) with $\gamma_2$ in place of $\gamma_1$. We then move on from $\gamma_2$ to $\gamma_3$ to $\gamma_4$ to $\gamma_5$ similarly to get $\mu_5, \mu_4, \ldots, \mu_1$ all of whom are $m$-verified. We then begin again (at stage $s$, say) on, $\mu_1$, and for simplicity, demand that $\mu_1$ be verified simultaneously for all $j$ with $m < j \leq s$. (This makes the combinatorics easier.)

***The $\alpha$-module and the inductive strategies*** The above describes a procedure which, if it acts confinally with the construction, forces $\Phi_e(P; m)\downarrow$ for all possible paths on $[C]$. To compute the relevant majorizing function, simply wait until all currently nonterminal paths $P$ in $C_s$ are $m$-verified. König's lemma say that such a stage must exist. Then all are majorized by $f(m) = 1 + \max\{\Phi_e(P; m) : P \text{ as above}\}$.

In a standard $\Pi_2$ way, we will desire that we only meet $N_j$ via $j$-correct ver-

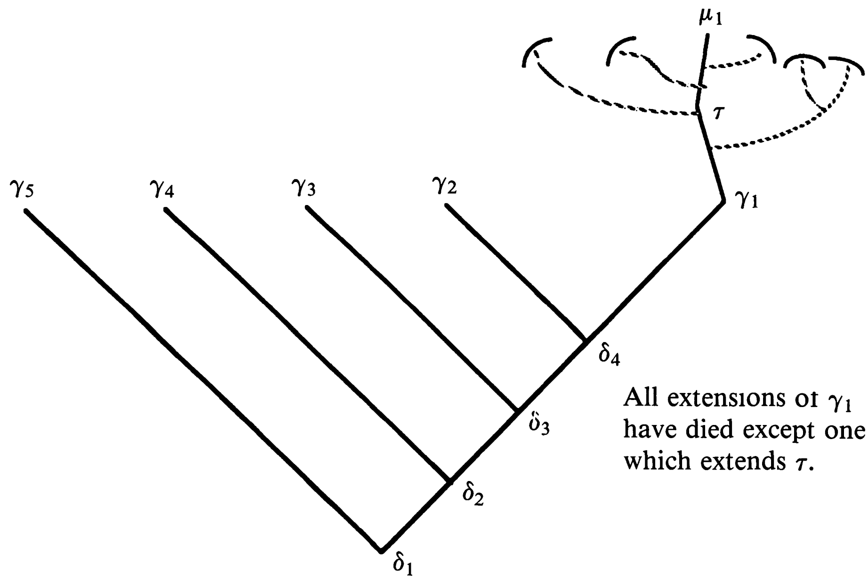


Figure 4.

sions (i.e., $e$-correct for $e \leq j$). Note that the stronger outcome (that $R_e$ acts infinitely often) will "appear correct" whenever we shift from $\gamma_1$ to $\gamma_2$, etc. At such stages we will terminate all work based on the false assumption that $R_e$ acts only finitely often. In particular, a node $\sigma$ devoted to $N_j\,(j \geq e)$ will be given a guess for the outcomes of $R_j$ for $j \leq e$. For a *single* $R_e$, this guess is either $\infty$ or $f$ (for infinitely or finitely active). The outcome $\infty\,(f)$ will be reflected in $e$-state $1\,(0)$. If the guess is $f$ then when $\infty$ looks correct the node $\sigma$ is terminated (That is, all extensions in $C$ (except perhaps 1) are terminated and in particular $\sigma$ will no longer give rise to splittings.)

In particular, for the situation in Figure 4, assuming none of the $\delta_1, \ldots, \delta_4$ act via their $N_k$-requirements, when $\gamma_1$ was $m$-verified we could define two extensions $\mu_1{}^\wedge 0$, $\mu_1{}^\wedge 1$ of $\mu_1$ and declare $g = \mathrm{lh}(\mu_1) + 1$ as the follower of the highest priority $N_k$ without a follower with guess $\infty$ (with $k \geq e$). We would declare $\mu_1$ to have $e$-guess $\infty$. When this guess is verified (i.e., for all $j \leq g$ that are extendible $\lambda_i \supset \mu_1{}^\wedge i$ with $\Phi_{e,s}(\lambda_i, j)\!\downarrow$) we would declare $g$ as *active* and then let it act as in the basic $N_k$ module.

Assuming $\delta_1, \ldots, \delta_4$ have not yet acted, we would not, however, split the extensions of $\gamma_5, \ldots, \gamma_2$, for we know that either one day the $\delta_i$ might act or infinitely often we will return to $\mu_1$ to build the class. Arguing by priorities, if ever $\delta_1, \ldots, \delta_4$ act, they must have been assigned to $N_j$ of higher priority than that to which $\mu_1$ is assigned. Thus, as usual, they can terminate $\gamma_1$ and so $\mu_1$ if they want. But also in the usual way, the $N_j$ assigned to $\mu_1$ will be met via some version. However, it is also important that the reader realize that no node with guess $f$ can cancel $\mu_1$.

The other interactions of $R_e$ with $N_j$ occur if $N_j$ has higher priority than $R_e$ (i.e., $j < e$). To make the combinatorics easier—particularly the verification that $[C]$ has only one rank one path—we will deal with this as follows. If $R_e$ has lower priority than $N_j$ it should cooperate with it. In particular, if a version of $N_j$ asks that the construction of $A$ only should be through $\sigma_1$ or $\sigma_2$ (as in Figure 3), and indeed should route through $\sigma_2$ unless $W_{j,s}(\mathrm{lh}(\sigma_2)) = 1$ at some stage $s$ (in which case we terminate all extensions of $\sigma_2$), then a version of $R_e$ guessing that this version of $N_j$ is correct should act only in the cone above $\sigma_2$. It will do so until such an $s$ occurs (in which case is will act only in the cone above $\sigma_1$) or this version of $N_j$ proves false.

Thus this version of $R_e$ will only verify strings in the cone above $\sigma_2$ (or $\sigma_1$) while $N_j$ looks good. In particular, in the situation of Figure 3, assuming all the $N_k$ of higher priority than $N_j$ have ceased activity, it will be the case that the only requirements that can force $A$ to extend $\gamma_5$ or $\gamma_4$ will be $N_j$ or requirements $R_q$ of higher priority then $N_j$.

To complete the proof, we need to describe the inductive strategies for $R_j$ for $j > e$. As usual, it suffices to consider two $R_j$, say $R_j$ and $R_e$ as above (with $j > e$).

The whole point is that $R_j$ must operate in the universe handed to it by $R_e$. Going back to the initial situation of Figure 3, we wish $R_e$ to verify $(\gamma_1, m)$. We would also desire $R_j$ to verify $(\gamma_1, m)$. If $R_j$ verifies $(\gamma_1, m)$ we would not abandon $\gamma_1$ for $\gamma_2$, since it still appears that (2.1) holds for $R_e$ (and hence we wish the construction to occur above $\gamma_1$). If this is the final outcome it could be reflected in a $j$-state with 0 for $R_e$ and 1 for $R_j$. Note that if $R_e$ later verifies $(\gamma_1, m)$ too, then in this situation, both $R_e$ and $R_j$ would begin anew on $\gamma_2$.

On the other hand, if $R_j$ has not (yet) verified $(\gamma_1, m)$ *but $R_e$ has* then when $R_e$ abandons $\gamma_1$ for $\gamma_2$ we leave $(\gamma_1, m)$ "pending" for $R_j$. The crucial step is not *not allow any node to get the high j-state* (*with* 1 *in the e-position and* 1 *in the j-position*) *until* $(\gamma_1, m)$ *is verified or permanently abandoned via some $N_k$-action.*

Thus, when we move to $\gamma_2$, $R_j$ seeks to verify $(\gamma_2, m)$ on the assumption $R_e$ has the $\Sigma_2$ outcome at $\gamma_1$. If $R_e$ does verify $(\gamma_2, m)$ and $(\gamma_1, m)$ is still active we now believe that $(\gamma_2, m)$ is not verified regardless.

The guiding principle is that if we visit $\gamma_1$ infinitely often and $R_j$ is not $m_j$-verified there, then it is here that we will have built $A$. On the other hand, if we visit $\gamma_1$ infinitely often then if (2.1) does not pertain to $R_j$ *in the set of stages we visit* (*extensions of*) $\gamma_1$, then we will get to $\hat{m}$-verify $\gamma_1$ for $j$ at some stage we $\hat{m}$-verify $\gamma_1$ for some $\hat{m} > m$ (for $e$).

That is, for each $j$-state $\tau$ we will have a "preferred place" to build $A$, denoted by $p(\tau, s)$. The reader should remember that the most desirable state of all is an "$f$" outcome since then $R_j$'s action is finite. Roughly we will have $p(\tau, s)$ as the rightmost appropriate place for the state $\tau$.

For a single $R_e$ above, suppose we had the situation in Figure 4, having just $e$-verified $\gamma_1$. Then unless some $N_k$ kills $\gamma$, it is clear that we'd desire $A$ to extend $\mu_1$, if the high $e$-state $(\infty)$ pertains, (i.e., $p(\infty, s) = \mu_1$) and until $\gamma_2$ is $m_e$-verified, $p(f, s) = \gamma_2$. Subsequently $p(f, s)$ moves to $\gamma_2, \gamma_3, \gamma_4, \gamma_5$ as they become $m_e$-verified, and then back to $\gamma_1$.

For the situation with $R_e$ and $R_j$ above things are a little more complex. Again looking at Figure 4, suppose we have $m_j$-verified and $m_e$-verified $\gamma_1$ and have moved to $\gamma_2$. Let $\delta$ be the high $j$-state, and $\rho$ the state with $\rho(e) = 1$ verified and $\rho(j) = 0$.

Suppose that at some stage $s_1 > s$ we get to $m_e$-verify $\gamma_2$ yet we do not $m_j$-verify. The observation is that should $\rho$ be the correct outcome, then infinitely often we will visit $\gamma_2$ unless it is cancelled by $N_k$-action (as $\rho(e) = 1$). If, during the stages we visit $\gamma_2$ we never get to $m$-verify it for $j$, then we would like to build $A$ in the zone above $\gamma_2$. Thus it is $\gamma_2$ that we set $p(\rho, s_j)$ equal to. Note that it is consistent with guess $\rho$ to not allow any new splittings of $e$-state $\rho$ or $\delta$ anywhere else until such time as $\rho$ proves wrong. Note the slight priority inversion: although $\mu_1$ has guess $\delta$ we won't allow new splittings extending $\mu_1$ to occur until such time as *all* of $\gamma_2, \ldots, \gamma_5$ have been $m$-verified. Hence we do not always build in the place where we can force the high $e$-state, as the lower one is *more desirable*.

The effect of the above is that if $\rho$ is the correct outcome then as in the example of Figure 4 we might build $A$ in the zone above $\gamma_2$ although $\gamma_2$ is not the rightmost path visited infinitely often (for instance $\gamma_1$ might be visited infinitely often, but no splittings might come from such visits).

On the other hand, if $\delta$ is the correct outcome, then all of $\gamma_1, \ldots, \gamma_5$ will be $m$-verified for both $e$- and $j$-. At such a stage $t$ we would be allowed to generate a splitting with the high $j$-state $\delta$ above $\mu_1 = (\delta, s)$ and we would then generate a longer $p(\delta, s)$.

In this way it can be seen that the strategies all cohere.

The above describes the strategies for, and the interactions of, the requirements. The remaining details are to implement the full construction using a $\Pi_2$ tree of strategies to coordinate the depth-$n$ strategies. We now give some formal details although we expect that the readers would prefer to supply them for themselves as the procedure is now well understood. We only sketch the verification as the key ideas are encapsulated in the preceding discussion.

As usual, we use the phrase "initialize". In our context, this means cancel all assignments of followers (by $N_j$'s), define them as unsatisfied, etc. (more on this later). We will build $C$, a $\Pi_1^0$ class in $2^{<\omega}$ with $0 < 1$, ordered lexicographically. The priority tree for the strategies is $T = \{f, \infty\}^{<\omega}$ with $\infty <_L f$. We will use $\delta, \tau, \rho$ for *guesses* (i.e., members of $T$) and reserve $\sigma, \mu, \gamma, \lambda, \pi$ for members of $2^{<\omega}$ (i.e., potential members of $C$). If $\mathrm{lh}(\tau) = 2j$ we will assign $N_j$ to $\tau$ for each such $\tau$. In this case we write $N_\tau$ for the version of $N_j$ assigned to guess $\tau$. Similarly, if $\mathrm{lh}(\tau) = 2j + 1$ we assign $R_j$ to $\tau$ and write $R_\tau$ for such a version of $R_j$. We will need an $m$(-verification) parameter, as described above, that we denote by $m(\gamma, \tau, s)$. Other parameters are:

- $\gamma(\tau, s)$: The string awaiting $\tau$-verification of $s$.
- $x(t, s)$: A follower of $N_\tau$: Note $x(\tau, s) \in \omega$.
- $\gamma\langle x(\tau, s)\rangle$: The string in $C_s$ associated with $x(\tau, s)$. It will be the case that $\gamma(x(\tau, s)) = 1$, and for some $\mu$, $\gamma\langle x(\tau, s)\rangle = \mu^\frown 1$. Also, both $\mu^\frown 1$ and $\mu^\frown 0$ are initial segments of nonterminal members of $C_s$.
- $\delta(s, t)$: A string in $T$ that appears correct at *substage t* of *stage s* (we write this as *stage* $(s, t)$).
- $\sigma(s, t)$: A string in $C_s$ that appears to be an initial segment of the rank one point of $[C]$ at stage $(s, t)$.
- $P_t$: The value of a parameter $P$ at the end of substage $t$.

**Important remark** When we initialize an $N_\tau$ then we will always do the following: If $N_\tau$ has a follower $x = x(\tau, s)$ with associated string $\gamma = \gamma\langle x(\tau, s)\rangle = \mu^\wedge 1$, we will declare as terminal all strings extending $\gamma$. When we initialize $R_\tau$ at stage $s$ we merely reset $m$ to be $s$ and reset $\gamma(\tau, s)$ to be undefined.

**The construction**

*Stage 0* Let $C_0 = \lambda$.
*Stage 1* Assign 0 to follow $N_0$. Declare both 0 and 1 to be (currently) nonterminal and set $C_1 = \{0,1\}$. Let $\sigma(1,1) = 1$, and $\delta(1,1) = f$.
*Stage $s + 1$* $(s \geq 1)$.

*Step 1.* Perform the following substages $t \leq 2s$.

*Substage* $t = 0$ (i.e., *stage* $(s + 1,0)$). (*Attend* $N_0$). If $0 \in W_{0,s+1}$ and $N_0$ not yet satisfied set $\sigma(s + 1),0) = 0$ and declare)terminal all $\mu \in 2^{<m}$ with $\mu \supset 1$ and $\mu \in C_s$. Initialize all $N_\tau$ and $R_\tau$ for $\tau \not\leq_L \infty$ and set $\delta(s + 1, s + 1) = \infty$. Then set $\sigma(s + 1, s)$ to be the rightmost nonterminal path in $C^0_{s+1}$. (*Remark*: (T1) It will be the case that $\sigma(s + 1, s) \supset 0$ and it will be the case that after this all versions of $R_j$ and $N_k$ $(k \geq 1)$ will now be in the initial states.) Declare $N_0$ as satisfied. Go to Step 2.

If either $N_0$ is satisfied or $0 \notin W_{0,s}$, set $\delta(s + 1,0) = f$ and $\sigma(s + 1,0) = 1$. (*Remark*: (T2) It will be the case that both 0 and 1 have nonterminal extensions in $C^0_{s+1}$.)

*Substage* $t = 2e$ $(e > 0)$ (*Attend* $N_e$) We can assume we are given $\sigma(s + 1, t - 1)$ and $\delta = \delta(s + 1, t - 1)$. Adopt the first case to pertain.

*Case 1.* $N_e$ is satisfied.

*Action.* Set $\delta(s + 1, t) = \delta(s + 1, t - 1)^\wedge\infty$ and $\sigma(s + 1, t) = \sigma(s + 1, t - 1)$. Go to stage $(s + 1, t + 1)$.

*Case 2.* $N_e$ has a follower $x(\tau, s)$ with guess $\tau \leq_L \delta$, and $x(\tau, s) \in W_{e,s}$, and $N_e$ as yet unsatisfied.

*Action.* Declare as terminal all strings $\pi \supset \mu^\wedge 1$ where $\mu^\wedge 1 = \gamma\langle x(\tau, s)\rangle$. (*Remark*: (T3) It will be the case that for all strings $\sigma$ nonterminal in $C^t_{s+1}$, $\sigma(x(\tau, s)) = 0$, so since $x(\tau, s) \in W_e$, we have won $N_e$.) Declare all versions of $N_e$ as satisfied. Initialize all $R_j$ for $j \geq e$. Set $\delta(s + 1, t) = \delta(s + 1, s + 1) = \tau$. Initialize all $N_{\hat\tau}$ and $R_{\hat\tau}$ for $\hat\tau \not\leq_L \tau$. For each $\rho \leq_L \tau$, if $p(\rho, s)^t$ is now terminated, reset $p(\rho, s)^{t+1}$ to the rightmost extension of length $s$ of $\mu^\wedge 0$. Go to Step 2.

*Case 3.* $N_e$ has a follower with guess $\delta$ but neither Case 1 nor Case 2 pertains.

*Action.* Set $\delta(s + 1, t) = \delta(s + 1, t - 1)^\wedge f$ and $\sigma(s + 1, f) = \sigma(s + 1, t - 1)$. Go to stage $(s + 1, t + 1)$. (*Remark*: (T4) As $N_e$ has a follower it follows that $t \neq s$.)

*Case 4.* $N_e$ has no follower with guess $\delta$ and $N_e$ is not satisfied and either $\sigma(s + 1, t - 1) \subseteq p(\delta, s)$ or $p(\delta, s) \subseteq \sigma(s + 1, t - 1)$.

*Action.* Let $\rho$ be the rightmost nonterminal string in $C^{t-1}_{s+1}$ with $\rho \supset \sigma(s + 1, t - 1)$. Declare $\rho^\wedge 0$ and $\rho^\wedge 1$ to be in $C^t_{s+1}$ and $s + 1$ $(= \mathrm{lh}(\delta^\wedge 0)$ to be a follower of $N_\delta$. Thus $x(\delta, s+1) = s+1$. Declare $\rho$ to have guess $\delta$. Let $\delta(s+1, s+1) = \delta$ and initialize all $N_\tau$ and $R_\tau$ for $\tau \not\leq_L \delta$. Go to Step 2.

*Case 5.* Otherwise. Set $\delta(s+1, t) = \delta(s+1, t-1)^\frown f$. If $t = s$ go to step 2. Otherwise go to stage $(s+1, t+1)$.

*Substage* $t = 2e+1$ (*Attend* $R_e$). We will be given $\sigma = \sigma(s+1, t-1)$ and $\delta = \delta(s+1, t-1)$. Adopt the first case to pertain.

*Case 1.* Either there have been no $\delta$-stages $\leq s$ or $\delta$ has been initialized since the last $\delta$-stage.

*Action.* Declare $(s+1, t)$ to be a $\delta^\frown f$-stage. Initialize all $N_\rho$ for $\rho \not\leq_L \delta$ and $R_\tau$ with both $\delta^\frown f <_L \tau$ and $\delta^\frown f \not\subseteq \tau$. If $\gamma\langle \delta, s+1\rangle^{t-1}$ is defined, set $\gamma\langle \delta, s+1\rangle^t = \gamma\langle \delta, s+1\rangle^{t-1}$. Otherwise find the rightmost string $\sigma$ of shortest length such that:

(i) both $\sigma^\frown 0$ and $\sigma^\frown 1$ have nonterminal extensions,
(ii) if there is a string $\hat{\sigma}$ with guess $\delta^-$, where $\delta = \delta^{-\frown} a$, then $\sigma \supset \hat{\sigma}$,
(iii) same as (ii) but with $\delta$ in place of $\delta^-$, and
(iv) for all $\hat{\sigma} \supseteq \sigma$ if $\hat{\sigma}$ has guess $\tau$ then $\delta^\frown f \leq_L \tau$.

Declare $\sigma$ to have guess $\delta^\frown f$ and let $p(\delta^\frown f, s) = \sigma^\frown 0$. Let $\gamma\langle \delta, s+1\rangle = \sigma = \sigma(s+1, t)$.

In either case, if $t = 2s$ go to Step 2. Otherwise, go to substage $t+1$.

*Case 2.* Not Case 1 (*Claim*: (T5) This implies $\gamma = \gamma\langle s, s+1\rangle^{t-1}$ is defined), and either all extensions of $\gamma$ have been terminated or there is an extension $\hat{\gamma}$ of $\gamma$ such that: (i) $\hat{\gamma}$ has a nonterminal extension, (ii) $\Phi_{e,s}(\hat{\gamma}, j)\downarrow$ for all $j \leq m(\delta, s+1)^{t-1}$, (iii) either $\delta(s+1, t-1) \subseteq \hat{\gamma}$ or $\hat{\gamma} \subseteq \sigma(s+1, t-1)$, and (iv) $\hat{\gamma}$ has guess $\delta^-$.

*Action.* Set $\sigma(s+1, t) = \hat{\gamma}$ if $\sigma(s+1, t-1) \subseteq \hat{\gamma}$ and $\sigma(s+1, t) = \sigma(s+1, t-1)$ otherwise. Declare $\gamma$ as $m(\delta, s+1)$-verified and if $\hat{\gamma}$ exists, declare $\hat{\gamma}$ to have guess $\delta^\frown\infty$. Find the rightmost extension $\gamma = \gamma^{-\frown}1$ of $\sigma(s+1, t)$ of length $s$ and declare it to have guess $\delta^\frown\infty$ too. If $\sigma(s+1, t) \supseteq p(\sigma^\frown\infty, s)^t$ or $p(\sigma^\frown\infty, s)^t$ is not defined, let $\gamma^-$ be $p(\sigma^\frown\infty, s)^{t+1}$, and declare both $\gamma^{-\frown}0$ and $\gamma^{-\frown}1$ to have guess $\delta^\frown\infty$. Terminate all other extensions of $\sigma(s+1, t)$ with guesses weaker than $\delta^\frown\infty$. Now we choose $\gamma\langle \delta, s+2\rangle^0$ as follows: If, for any initial segment $\tau \subseteq \delta$, we have already reset $\gamma\langle \tau, s+2\rangle^0$ to be different from $\gamma\langle \tau, s+1\rangle$, then for the longest such $\tau$ set $\gamma\langle \delta, s+2\rangle^0 = \gamma\langle \tau, s+2\rangle^0$. Otherwise, if all strings with nonterminal extensions of guess $\subseteq \delta$ already have $m$-verified initial segments, choose $\gamma\langle \delta, s+2\rangle^0$ as $\sigma$ chosen as in Case 1 with the additional proviso that $\sigma$ is not yet $m$-verified.

In any case, let $\delta(s+1, t) = \delta^\frown\infty$ and go to stage $(s+1, t+1)$.

*Case 3.* Otherwise.

*Action.* Let $\delta(s+1, t) = \delta^\frown f$. If there is a $\tau^\frown\infty \subseteq \delta$ with $\mathrm{lh}(\tau) = 2t+1$ for some $t$, then for the longest such $\tau$, it will be the case that $\gamma\langle \tau, s+2\rangle^0 = \gamma$ has been reset. Set $\gamma\langle \delta, s+2\rangle^0 = \gamma$. Otherwise $\gamma\langle \delta, s+1\rangle = \gamma\langle \delta, s+1\rangle^{t-1}$. In any case $\sigma\langle s+1, t\rangle = \sigma(s+1, t-1)$. If $t = 2s$ go to Step 2, otherwise go to substage $t+1$.

*Step 2.* For all nonterminal paths $\sigma$ in $C^t_{s+1}$ of length $s$ with (currently) no extensions of length $s+1$, put $\sigma\hat{\ }0$ in $C_{s+1}$.

*Verification* (Sketch). It is relatively easy to verify by induction on the construction that all the claims (T1)–(T5) made in the construction hold, and that $N_e$ requires attention at most once as there is at most one string $\sigma$ in $C_s$ with $\sigma(x) = 1$ if $x$ follows $N_e$ (and this is made terminal if $x \in W_{e,s}$). Let $TP$ denote the true path of the strategy tree. One can see by induction that if $\delta\hat{\ }\infty \subset TP$ and $\mathrm{lh}(\delta) = 2e$ then $W_{e,s}(x) = 1$ for some $x$ with $p(x) = 0$ for all paths $x$ on $[C]$. Now if $\delta\hat{\ }f \subset TP$ then we need to show that $x = \lim_s x(\delta, s)$ exists and $\gamma = \lim_s \gamma\langle\delta, s\rangle$ exists with $\gamma = \mu\hat{\ }1$, and $\mathrm{lh}(\gamma) = x$. This is a standard induction.

Then one argues that $\gamma$ is the preferred place to build. Of course the key observation needed is that the result of the above is $[C]$ with only one rank one point. To see this, suppose we have that $\sigma$ is on infinitely many paths in $[C]$ but if $\mathrm{lh}(\tau) \leq \mathrm{lh}(\sigma)$ and $\tau \not\subseteq \sigma$ then $\tau$ has only finitely many extensions in $[C]$. We now show one of $\sigma\hat{\ }0$ or $\sigma\hat{\ }1$ has these same properties. Now if both $\sigma\hat{\ }0$ or $\sigma\hat{\ }1$ are on $C_s$ at some $s$, then they are associated with some $N_j$. If ever one of $\sigma\hat{\ }0$ or $\sigma\hat{\ }1$ is terminated then we are done, so without loss of generality we can suppose both have extensions on $[C]$.

By our construction which of $\sigma_1 = \sigma\hat{\ }0$ or $\sigma_2 = \sigma\hat{\ }1$ can have the rank one extension depends only on $N_j$ and higher priority $R_e$. If at some stage some highest priority $R_e$ gets stuck on some $(\gamma, m_e)$ extending one of $\sigma_1$ or $\sigma_2$ (say $\sigma_1$) then we will stop forming permanent splittings of $\sigma_2$. It may very well be that the construction is performed above $\sigma_2$ infinitely often, but this will only be to verify the appropriate $R_e$, for $e' < e$. The only *splittings* that survive will be those above $\sigma_1$. Therefore if $s$ is the stage where we got stuck on $(\gamma, m_e)$, there can be at most $s$ paths in $[C]$ extending $\sigma_2$, since the only splittings can be those *already present* at stage $s$. Finally if no $R_e$ of higher priority gets stuck on some $(\gamma, m_e)$ extending one of $\sigma_1$ or $\sigma_2$, then $N_j$ will have the correct guess. As neither $\sigma_1$ or $\sigma_2$ gets terminated, it follows that $N_j$ will ask $A$ to extend $\sigma_2$. Thus $\sigma_1$ will get only one extension in $[C]$, as all the potential others get cancelled. Hence $\sigma\hat{\ }0$ or $\sigma\hat{\ }1$ cannot both be extended to rank one or more points.

By Jockusch [5], we know that the degrees consisting of a single *tt*-degree are exactly the hyperimmune-free degrees. As we noted earlier, Cenzer and Smith proved that all hyperimmune degrees contain unranked points. This suggests the conjecture that the degrees containing only ranked points are exactly the hyperimmune-free degrees. Unfortunately, this conjecture is easily destroyed. These are only countably many ranked points, and yet there are uncountably many hyperimmune-free degrees. In fact there are hyperimmune-free degrees containing no ranked points below $0''$ as we now see. To see this, in view of Jockusch's Theorem [5], it suffices to construct a hyperimmune-free unranked point. To do this one meets the previous $R_e$ requirements and also

$$P_e\colon \text{if } A \text{ is a member of } [T_e] \text{ then } [T_e] \text{ is uncountable.}$$

Here $T_e$ denotes the $e$-th $\Pi^0_1$ class. Now we meet $R_e$ by the standard forcing with perfect closed sets (*à la* Miller and Martin [8]) and the $P_e$ by diagonalization. We know how to meet $R_e$ in this setting: given a perfect tree $Q_g$ find a perfect subtree $Q_{g+1}$ where either $\Phi_e(P;m)\downarrow$ for all $P \in [Q_{g+1}]$ or $(\exists m)(\Phi_e(P;m)\uparrow)$ for all $P \in [Q_{g+1}]$. To meet $P_e$ we are given $Q_n$ a perfect tree and we ask if every node on $Q_n$ has an extension on $[T_e]$ for if $[Q_n] \subseteq [T_e]$. If the answer is yes, we need do nothing as $T_e$ is uncountable. If the answer is no, we take a string $\sigma$ on $Q$ such that no extension of $\sigma$ lies in $T_e$. We then let $Q_{n+1}$ be the perfect subtree of $Q_n$ above $\sigma$. Thus we have

**Theorem** *There is a hyperimmune free nonrecursive unranked point of degree $\leq 0''$.*

It would be very interesting to find some sort of characterization of the degrees containing unranked points. But this seems hard. The basic machinery of the construction of our main result is quite versatile. The basic idea attempting to force divergence can be used for other $\Pi_1^0$ class constructions. For instance the method can be extended to show that for $n \in \omega - \{0\}$ *there is a hyperimmune free set of rank n and hence a degree all of whose members have rank n*. It seems conceivable that this will extend to all $\alpha < \omega_1^{ck}$. Variations on the method have been used by the author to construct a *perfect* $\Pi_1^0$ *class, all of whose members are recursive or of minimal degree*. An open question here is whether there is a perfect $\Pi_1^0$ class, all of whose members have either minimal degree or have r.e. nonrecursive degree. Finally, the construction used for the main result is a full approximation one. It would be nice to have one where the set was obtained by direct forcing with $\Pi_1^0$ classes.

**Acknowledgments** – The author was partially supported by a US/NZ binational grant and the Mathematical Sciences Research Institute. He wishes to thank Doug Cenzer and Carl Jockusch for useful conversions on $\Pi_1^0$ classes and ranked points, and he particularly thanks Carl for a suggestion that simplified the verification of the main theorem.

## REFERENCES

[1] Cenzer, D., P. Clote, R. Smith, R. Soare and S. Wainer, "Members of countable $\Pi_1^0$ classes," *Annals of Pure and Applied Logic*, vol. 31 (1986), pp. 145-163.

[2] Cenzer, D., R. Downey, C. Jockusch, and R. Shore, "Countable thin $\Pi_1^0$ classes," in preparation.

[3] Cenzer, D. and R. Smith, "The ranked points of a $\Pi_1^0$ set, *The Journal of Symbolic Logic*, vol. 54 (1989), pp. 975-991.

[4] Clote, P., "On recursive trees with a unique infinite branch," *Proceedings of the American Mathematical Society*, vol. 93 (1985), pp. 335-342.

[5] Jockusch, C., "Relationships between reducibilities," *Transactions of the American Mathematical Society*, vol. 142 (1969), pp. 229-237.

[6] Jockusch, C. and R. Soare, "$\Pi_1^0$ classes and degrees of theories," *Transactions of the American Mathematical Society*, vol. 173 (1972), pp. 33-56.

[7] Jockusch, C. and R. Soare, "Degrees of members of $\Pi_1^0$ classes," *Pacific Journal of Mathematics*, vol. 40 (1972), pp. 605-616.

[8] Miller, W. and D. Martin, "Degrees of hyperimmune sets," *Zeitschrift für Mathmatische Logik und Grundlagen der Mathematik*, vol. 14 (1968), pp. 159–166.

[9] Odifreddi, P., *Classical Recursion Theory*, North Holland, Amsterdam, 1989.

[10] Soare, R., *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, 1987.

*Mathematics Department*
*Victoria University of Wellington*
*P.O. Box 600*
*Wellington, New Zealand*