# *Randomness and Computability 2: Reals*

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# Motivation

- What is “random”?
- How can we calibrate levels randomness? Among randoms?, Among non-randoms?
- How does this relate to classical computability notions, which calibrate levels of computational complexity?

# LAST TIME

- We defined various notions of compressibility for strings: $C$ (plain complexity), $K$ (prefix free complexity) $Km$ (monotone complexity), $KM$, universal semimeasures etc.
- Examined how they related, and basic facts about them.
- Example the complexity of the overgraphs.
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- Schnorr's Theorem (i)=(ii).

# Martin-Löf randomness:

- Recall that a Martin-Löf test is a uniformly c.e. sequence $U_1, U_2, \ldots$ of c.e. open sets s.t.

$$\forall i(\mu(U_i) \leq 2^{-i}).$$

(Computably shrinking to measure 0)

**Definition**

$\alpha$ is Martin-Löf random if for every Martin-Löf test,

$$\alpha \notin \bigcap_{i>0} U_i.$$

**Theorem (Schnorr)**

*A is ML random iff for all n, $K(A \upharpoonright n) \geq^+ n$. (Iff $Km(A \upharpoonright n) =^+ n$.)*

## Chaitin's $\Omega$

- We've sen a random real is random, but are there explict examples?
- The most famous random real is

$$\Omega = \mu\,\mathrm{dom}(M) = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|},$$

a "halting probability."

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### Theorem (Chaitin)

*$\Omega$ is random.*

- Proof. We use Kraft-Chaitin: We build a Kraft-Chaitin set with coding constant $c$ given by the recursion theorem. If, at stage $s$, we see $K_s(\Omega_s \upharpoonright n) < n - c - 1$, enumerate $\langle n - c, \Omega_s \upharpoonright n\rangle$ into KC, and hence $\Omega \upharpoonright n \neq \Omega_s \upharpoonright n$.

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- Method is "quanta recycling"=charging our costs to the opponent ($U$).

## $\Omega$ and Halting

- Solovay looked at basic properties of $\Omega$, in terms of computability. e.g.
- Let $D_n = \{x : |x| \leq n \wedge U(x)\downarrow\}$.

**Theorem (Solovay)**

$K(D_n) =^+ n$.

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(i) $K(D_n|\Omega \upharpoonright n) = O(1)$. *(Indeed $D_n \leq_{wtt} \Omega \upharpoonright n$ via a weak truth table reduction with identity use.)*

(ii) $K(\Omega \upharpoonright n|D_{n+K(n)}) = O(1)$.

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- (i) is easy. Wait till $\Omega_s =_{\mathrm{def}} \sum_{U(\sigma)\downarrow[s]} 2^{-|\sigma|}$ is correct on its first $n$ bits. Then we can compute $D_n$.
- (ii) is rather difficult, and is in 5 lectures, or DH book.

# Extending Schnorr's Theorem

**Theorem (Miller and Yu, The Ample Excess Lemma)**
*$\alpha$ is Martin-Löf random iff $\sum_{n\in\mathbb{N}} 2^{n-K(\alpha\restriction n)} < \infty$.*

# EXTENDING SCHNORR'S THEOREM

### THEOREM (MILLER AND YU, THE AMPLE EXCESS LEMMA)

*$\alpha$ is Martin-Löf random iff $\sum_{n\in\mathbb{N}} 2^{n-K(\alpha\restriction n)} < \infty$.*

- This says that whilst the K-complexity is above $n$, mostly it is "pretty far" from n. (Proof in 5 Lectures, or DH)

### THEOREM (MILLER AND YU)

*Suppose that $f$ is an arbitrary function with $\sum_{m\in\mathbb{N}} 2^{-f(m)} = \infty$. Suppose that $\alpha$ is random. Then there are infinitely many $m$ with $K(\alpha \restriction m) > m + f(m) - O(1)$.*

# Plain Complexity again

- In spite of the fact that we have this natural characterization in terms of $K$ or $Km$, it was a longstanding question whether there was a plain complexity characterization of randomness.
- It was known that there were sufficient conditions on on $C(\alpha \upharpoonright n)$ to guarantee randomness. To wit:

**Definition**

Say that it is Kolmogorov random if there are infinitely many n with $C(n) \geq n - O(1)$.

**Theorem (Martin-Löf, Solovay)**

*Almost every real is Kolmogorov random.*

# $n$-RANDOMNESS

- In the same way as the arithmetical hierarchy,

### DEFINITION (KURTZ, SOLOVAY)

(I) A $\Sigma^0_n$ test is a computable collection $\{V_n : n \in \mathbb{N}\}$ of $\Sigma^0_n$ classes such that $\mu(V_k) \leq 2^{-k}$.
(II) A real $\alpha$ is $\Sigma^0_n$-random or $n$-random iff it passes all $\Sigma^0_n$ tests.
(III) One can similarly define $\Pi^0_n$, $\Delta^0_n$ etc tests and randomness.
(IV) A real $\alpha$ is called arithmetically random iff for any $n$, $\alpha$ is $n$-random.

# Kurtz or weak randomness

- Thinking about $\Pi^0_n$ randomness, we get a randomness notion called **Kurtz *n*-randomness** or weak *n*-randoness meaning that it passes all $\Pi^0_n$ tests, or, equivalently the real is in all $\Sigma^0_n$ sets of measure 1.

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- Weak 1-randomness is often regarded as a genericity notion.
- Weak 2-randomnness has especial signifince in that it corresponds to generalized ML tests, meaning that $\mu(U_n) \to 0$ but we have no effective convergence rate.

# Kurtz's Theorem

- We use open sets to define Martin-Löf randomness.
- Consider: the $\Sigma_2^0$ class consisting of reals that are always zero from some point onwards. It is not equivalent to $\cup\{[\sigma] : \sigma \in W\}$ for any $W$.
- Kurtz showed that $n$-randomness is the same as $n$ randomness relative to open classes.

**Theorem (Kurtz)**

*$n+1$-randomness = 1-randomness relative to $\emptyset^{(n)}$.*

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THEOREM (DOWNEY, NIES, WEBER, YU+HIRSCHFELDT)

*$A$ is weakly 2-random iff $A$ is ML random and $\deg(A)$ and $\mathbf{0}'$ form a minimal pair.*

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*$A$ is weakly 2-random iff $A$ is ML random and deg$(A)$ and $\mathbf{0}'$ form a minimal pair.*

- Also there is a $n+1$-random set $\Omega^{(n+1)}$ namely $\Omega^{\emptyset^{(n)}}$ which is computably enumerable relative to $\emptyset^{(n)}$.
- NOTE it is NOT CEA($\emptyset^{(n)}$). But $\Omega^{(n)} \oplus \emptyset^{(n)} \equiv_T \emptyset^{(n+1)}$.

# Warning

- Similar relativization work for Schnorr, computable, etc (we later meet) randomness. BUT not for weak randomness.
- It is NOT true that weak-2-randomness (meaning being in every $\Sigma^0_2$ class of measure 1) is the same as being Kurtz random over $\emptyset'$. This is a genericity notion. 2-generics have this property.
- The best we can do is: $n \geq 2$, $\alpha$ is Kurtz $n$-random iff $\alpha$ is in every $\Sigma_2^{\emptyset^{(n-2)}}$-class of measure 1.
- weak 2-randomness is the same as "Martin-Löf randomness with no effective convergence" In fact, weak 2-randomness might best be described as strong 1-randomness.

# A hierarchy

**Theorem**

(i) *(Kurtz) Every n-random real is Kurtz n-random.*

(ii) *(Kurtz) Every Kurtz $n+1$-random real is n-random.*

(iii) *(Kurtz, Kautz) All containments proper.*

- In thse lectures I won't be able to discuss the long history of results relating $n$-randomness to the arithmetical hierarchy.
- The first result in this area was
- Define $P(A) = \mu\{X : W_e^X = A\}$, where $e$ is a universal index for the halting problem. This is the probability that $A$ is enumerated.

**Theorem (de Leeuw, Moore, Shannon, Shapiro, 1956)**

$P(A) \geq 0$ *iff A is computably enumerable.*

**Corollary (Sacks)**

$\mu\{X : A \leq_T X\} > 0$, *iff A is computable.*

- The proof uses the "majority vote" technique: Assume $P(A) > 0$.
- For some $e$, $D_e = \{X : A = W_e^X\}$ has positive measure.
- There is a string $\sigma$ such that the relative measure of $D_e$ above $\sigma$ is greater than $\frac{1}{2}$. (Lebesgue Density Theorem)
- Let the oracles extending $\sigma$ vote on membership in $D_e$
- Pu $n$ into $A$ if more than half (by measure) say so. This enumerates $A$.

- The almost all theory of degrees is well-behaved, and in fact is decidable (Stillwell).
- Similarly $\leq_T$ and randomness. Examples:

**Theorem (Kurtz)**
*If $A$ is $n+1$-random then $A$ is $GL_n$.*

**Theorem (Kučera, Gács)**
*If $A$ is noncomputable, then there is a random $B$ with $A \leq_{wtt} B$.*

**Theorem (Miller and Yu)**
*If $A \leq_T B$ are random and $B$ is $n$ random, then $A$ is $n$ random.*

**Theorem (van Lambalgen's Theorem)**
*$A \oplus B$ is random iff $A$ is $B$-random and $B$ is $A$-random.*

**Theorem (Kurtz)**
*If $A$ is 2-random then $A$ is hyperimmune.*

**Theorem (Stephan)**
*If $A$ is random and has PA degree then $\emptyset' \leq_T A$.*

- That is random randoms are computationally weak.

# 2-RANDOMNESS

- There are some relative natural examples of $n$-randoms using methods akin to Post's Theorem and index sets (Becher-Figueira). However, there are some really unexpected characterizations also of 2-randoms.
- Recall that the maximum a string of length $n$ can be is
  (I) $C(\sigma) = n - O(1)$.
  (II) $K(\sigma) = n + K(n) - O(1)$.

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THEOREM (SOLOVAY)

*(ii) implies (i), but not conversely.*

- Say that a real $\alpha$ is strongly Chaitin random iff there are infinitely many n with $K(\alpha \upharpoonright n) \geq n + K(n) - O(1)$.
- Recall $\alpha$ is Kolmogorov random if there are infinitely many n with $C(\alpha \upharpoonright n) \geq n - O(1)$.
- Fundamental question: are they the same?

# Kolmogorov randomness

**Theorem (Nies-Terwijn-Stephan, Miller)**

*2-randomness=Kolmogorov randomness (!).*

- Proof We fix a universal machine U which is universal and prefix-free for all oracles. Suppose that A is not 2-random. Thus, for each c there is an n with

$$K^{\emptyset'} A \upharpoonright n) < n - c.$$

- We build a plain machine M. On an input $\sigma$, $M$ tries to parse $\sigma$ as $\tau\beta$, with $\tau$ in the domain of $U^{\emptyset'}$. Note that as $K^X$ is prefix-free for all oracles X, there is at most one $\tau \prec \sigma$.
- Let $s = |\sigma|$.
- First it assumes that $s$ is sufficiently large that $H_s$ is correct on the use of $A \upharpoonright n$. It assumes that It then uses $\emptyset'_s$ as an oracle, to compute (if anything) $\tau \prec \sigma$ with $U^{\emptyset'_s}(\tau) \downarrow$ .
- If there is one, $M$ outputs $U^{\emptyset'_s}(\tau)\beta$. From some time onwards, upon input $\nu A[n+1, m]$ with $U^{\emptyset'}(\nu) = A \upharpoonright n$, this will be $A \upharpoonright m$.
- Thus $C(A \upharpoonright m)$ is bounded away from m.

- The other direction. (Miller, NST)
- Recall from Lecture 1 that a compression function acts like $U^{-1}$.
- Recall that we defined $F : \Sigma^* \mapsto \Sigma^*$ to be a compression function if for all $x$ $|F(x)| \leq C(x)$ and $F$ is 1-1.
- Recall also that since they forma $\Pi^0_1$ class, there is a compression function $F$ with $F' \leq_T \emptyset'$. (NST's idea)
- Namely, consider the $\Pi^0_1$ class of functions $|\widehat{F}(\sigma)| \leq C(\sigma)$.
- The main idea is that most of the basic facts of plain complexity can be re-worked with any compression function. For a compression function $F$ we can define $F$-Kolmogorov complexity: $\alpha$ is $F$-Kolmogorov random iff $\exists^\infty n(F(\alpha \upharpoonright n) > n - O(1))$.

- (NST) If $Z$ is 2-random relative a compression function $F$, then $Z$ is Kolmogorov $F$-random.
- Now we can save a quantifier using a low compression function.
- This still leaves strongly Chaitin random reals. Question are they 3-random, 2-random or something else. Note that the same approach won't work because both sides change. (To wit: $F(\alpha \upharpoonright n) = n + F(|n|) - d$. Could to this if there was a low compression function with $K(\sigma) > K(\tau)$ implies $F(\sigma) > F(\tau)$ but this is surely false.)

## Randomness and plain complexity

- Finally Miller and Yu provided a plain complexity characterization of Martin-Löf randomness.

### Theorem (Miller and Yu)

*$x$ is Martin-Löf random iff $(\forall n)\ C(x \upharpoonright n) \geq n - g(n) \pm O(1)$, for every computable $g\colon \omega \to \omega$ such that $\sum_{n \in \omega} 2^{-g(n)}$ is finite.*

# Martingales

- von Mises again. This time think about predicting the next bit of a sequence. Then you bet on the outcome. You should not win!

## Definition (Levy)

(i) A martingale is a function $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ such that for all $\sigma$,
$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$

(ii) The martingale *succeeds* on a real $\alpha$, if $\limsup_n F(\alpha \upharpoonright n) \to \infty$.

- Think of betting on sequence where you know that every 2nd bit was 1. Then every second bit you could double you stake. This martingale exhibits exponential growth and that can be used to characterize computable reals.

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- Ville proved that null sets correspond to success sets for martingales. They were used extensively by Doob in the study of stochastic processes.

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$$f(\sigma) \geq \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$
- Schnorr showed that Martin-Löf randomness corresponded to effective (super-)martingales failing to succeed.
- $f$ as being effective or computably enumerable if $f(\sigma)$ is a c.e. real, and at every stage we have effective approximations to $f$ in the sense that $f(\sigma) = \lim_s f_s(\sigma)$, with $f_s(\sigma)$ a computable increasing sequence of rationals.

# Schnorr, Again

### Theorem (Schnorr)

*A real $\alpha$ is Martin-Löf random iff no effective (super-)martingale. succeeds on $\alpha$.*

- The proof uses a basic fact about (super-)martingales.
- (Kolmogorov's inequality)
  - (I) Let $f$ be a (super-) martingale. For any string $\sigma$ and prefix-free set $X \subseteq \{x : \nu \preceq x\}$,

$$2^{-|\nu|} f(\nu) \geq \sum_{x \in X} 2^{-|x|} f(x).$$

  - (II) Let $S^k(f) = \{\sigma : f(\sigma) \geq k\}$, then

$$\mu(S^k(f)) \leq f(\lambda) \frac{1}{k}.$$

- That is the stake must be shared fairly at level $n$.

- Proof of Schnorr's Theorem: We show that test sets and martingales are essentially the same. (Ville effectivized). Firstly suppose that $f$ is an effective (super-)martingale.
- Let $V_n = \cup\{\beta : f(\beta) \geq 2^n\}$.
- $V_n$ is a c.e. open set and $\mu(V_n) \leq 2^{-n}$ by Kolmogorov's Inequality.
- Thus $\{V_n : n \in \mathbb{N}\}$ is a Martin-Löf test.
- And $\alpha \in \cap_n V_n$ iff $\limsup_n f(\alpha \upharpoonright n) = \infty$.
- Hence a martingales succeeds on $\alpha$ iff it fails the derived test.

- The other direction.
- Build a martingale from a Martin-Löf test. Let $\{U_n : n \in \mathbb{N}\}$ be a Martin-Löf test.
- We represent $U_n$ by extensions of a prefix-free set of strings $\sigma$, and whenever such a $\sigma$ is enumerated into $\cup_{n,s} U_n^s$, increase $F(\sigma)[s]$ by one.
- To maintain the martingale nature of $F$, we also increase $F$ by 1 on all extensions of $\sigma$, and by $2^{-t}$ on the substring of $\sigma$ of length $(|\sigma| - t)$.)

# Schnorr randomness

- One could argue that to be algorithmically random, Martin-Löf's definition is too strong.
- For instance, $\alpha$ is ML-random iff no c.e. Martingale succeeds on $\alpha$. (That is the betting startegy $F : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ is a c.e. function.)
- Schnorr argued that *ML* randomness is intrinsically c.e. not defeating “effectively”= computably given objects.
- Schnorr proposed two notions of more computable randomness.

# More effective randomness

## Definition (Schnorr)

(i) A martingale $f$ is called computable iff $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ is a computable function with $f(\sigma)$ (the index of functions representing the effective convergence of) a computable real. (That is, we will be given indices for a computable sequence of rationals $\{q_i : i \in \mathbb{N}\}$ so that $f(\sigma) = \lim_s q_s$ and $|f(\sigma) - q_s| < 2^{-s}$.)

(ii) A real $\alpha$ is called computably random iff for no computable martingale succeeeds on it.

## Definition (Schnorr)

(i) A Schnorr test is a Martin-Löf test $U_i : i \in \omega$ such that $\mu(U_i) = 2^{-i}$.

(ii) $\alpha$ is Schnorr random iff $\alpha \notin \cap_i U_i$ for all Schnorr tests $\{U_i\}$.

- There is a machine characterization of Schnorr randomness, solving an old question of Ambos-Spies and others.
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- For example $\Omega = \lim_s \Omega_s = \sum_{U(\sigma)\downarrow[s]} 2^{-|\sigma|}$.
- Computably enumerable reals play the same role in this theory as computably enumerable set do in classical computalility theory, and will be deal with in more detail later.

**Theorem (Downey and Griffiths)**

*$\alpha$ is Schnorr random iff for all computable prefix free machines M, there is a c such that for all n,*

$$K_M(\alpha \upharpoonright n) \geq n - c.$$

- There are also machine characterizations of computable and Kurtz randomness in terms of variant machines. (Downey, Griffiths, Reid, LaForte, Merkle, Mihailovic, Slaman)

- Schnorr used martingales and a kind of forcing argument to prove that there are Schnorr random reals that are not Martin-Löf random.
- Soon we will show that all c.e. random reals are Turing complete.
- (Downey-Griffiths) All Schnorr random c.e. reals are of “high” c.e. degree.
- (Downey-Griffiths) There are c.e. reals that are Schnorr random that have incomplete $T$-degree.
- (Downey, Griffiths and Reid) Each c.e. degree contains a left c.e. Kurtz random real.
- (Kurtz) Every hyperimmune degree contains a Kurtz random real.

- (Downey-Griffiths-LaForte, Nies-Stephan-Terwijn) All high c.e. degrees contain Schnorr random c.e. reals.
- NST have a stronger result for computably random left c.e. reals and high degrees. (soon)

# MARTINGALE CHARACTERIZATIONS

- (Wang) A real $\alpha$ is Kurtz random iff there is no computable martingale $F$ and nondecreasing function $h$, such that for almost all $n$,
$$F(\alpha \upharpoonright n) > h(n).$$
- (Schnorr) We say that a computable martingale strongly succeeds on a real $x$ iff there is a computable unbounded nondecreasing function $h : \mathbb{N} \mapsto \mathbb{N}$ such that $F(x \upharpoonright n) \geq h(n)$ infinitely often.
- (Schnorr) A real $x$ is Schnorr random iff no computable martingale strongly succeeds on $x$.

# THE FULL CHARACTERIZATION

- Martin-Löf implies computable implies Schnorr implies Kurtz. (randomness)
- The following very attractive result gives the full picture.

THEOREM (NIES, STEPHAN AND TERWIJN)

*For every set $A$, the following are equivalent.*

(I) *$A$ is high.*

(II) *$\exists B \equiv_T A$, $B$ is computably random but not Martin-Löf random.*

(III) *$\exists C \equiv_T A$, $C$ is Schnorr random but not computably random.*

*Moreover, for c.e. degrees, the examples can be chosen to be c.e.*

# OUTSIDE THE HIGH DEGREES

**THEOREM (NIES, STEPHAN AND TERWIJN)**

*Suppose that a set A is Schnorr random and does not have high degree. (That is, $A' \not\geq_T \emptyset' i)'$. Then A is Martin-Löf random.*

**THEOREM (NIES, STEPHAN, TERWIJN)**

*Suppose that A is of hyperimmune-free degree. Then A is Kurtz random iff A is Martin-Löf random.*

# von Mises strikes back

- There has been a lot of work recently on nonmonotonic selection, and nonmonotomic martingales, which might address Schnorr's critique.
- Briefly, we get to select position $f(0), f(1), \ldots$ and bet on thse bits, but now the selction on the places can be nonmonotonic.
- Important open question (Muchnik, Uspensky, Semenov)
- Is randomness relative to computable nonmonotonic supermartingales the same as Martin-Löf randomness. (also see MMNRS)

# Calibrating Randomness

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- One way is to use initial segment measures of realtive randomness.
- Satisfies: If $\beta \le \alpha$ then $\exists c\ (\forall n\ (K(\beta \upharpoonright n) \le K(\alpha \upharpoonright n) + c))$.

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- Can also use $C$, and others.

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- Of course this is open to question, and we could also suggest other programs such as using tests and maybe effective Hölder transformations (for instance) to attempt such a calibration. These are unexplored.

# One example: Solovay reducibility

- We talk about the halting problem, whereas of course we really mean $\mathrm{HALT}_U$ for a universal $U$. But... they are all the same (Myhill)

### Definition (Solovay)

($\alpha \leq_S \beta$) $\alpha$ is Solovay or domination reducible to $\beta$ iff there is a constant $d$, and a partial computable $\varphi$, such that for all rationals $q < \beta$

$$\varphi(q) \downarrow \wedge\, d(\beta - q) > |\alpha - \varphi(q)|.$$

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- Intuitively, however well I can approximate $\beta$, I can approximate $\alpha$ just as well. Clearly $\leq_S$ implies $\leq_T$.

# The Kučera-Slaman Theorem

- $\Omega$ is a halting probability. Define $A$ to be a left c.e. real iff $A$ is a halting probability, equivalently $A = \lim_s a_s$ for some computable increasing sequence of rationals $\{a_s\}$.
- A left c.e. real is $\Omega$-like if it dominates all left c.e. reals. (Solovay) Any $\Omega$-like real is random
- Solovay proved that $\Omega$-like reals posessed many of the properties that $\Omega$ posessed. He remarks:
  "It seems strange that we will be able to prove so much about the behavior of $K(\Omega \upharpoonright n)$ when, a priori, the definition of $\Omega$ is thoroughly model dependent. What our discussion has shown is that our results hold for a class of reals (that include the value of the universal measures of ...) and that the function $K(\Omega \upharpoonright n)$ is model independent to within $O(1)$."

THEOREM (CALUDE, HERTLING, KHOUSSAINOV, WANG)

*If a left c.e. real is $\Omega$-like then it is an $\Omega$-number. That is, a halting probability.*

THEOREM (KUČERA-SLAMAN)

*If a left c.e. real is random then it is $\Omega$-like.*

- Proof: Suppose that $\alpha$ is randomi, left c.e. and $\beta$ is a left c.e. real. We need to show that $\beta \leq_S \alpha$. We enumerate a Martin-Löf test $F_n : n \in \omega$ in stages.
  Let $\alpha_s \to \alpha$ and $\beta_s \to \beta$ computably and monotonically. We assume that $\beta_s < \beta_{s+1}$.
- At stage $s$ if $\alpha_s \in F_n^s$, do nothing, else put $(\alpha_s, \alpha_s + 2^{-n}(\beta_{s+1} - \beta_{t_s}))$ into $F_n^{s+1}$, where $t_s$ denotes the last stage we put something into $F_n$.
- One verifies that $\mu(F_n) < 2^{-n}$. Thus the $F_n$ define a Martin-Löf test. As $\alpha$ is random, there is a $n$ such that for all $m \geq n$, $\alpha \notin F_m$. This shows that $\beta \leq_S \alpha$ with constant $2^n$.

- The structure of left c.e. reals under $\leq_S$ is a dense USL, where join is induced by $+$ and $[\Omega]$ is the only join inaccessible. (Downey, Hirschfeldt, Nies)
- Undecidable (Downey, Hirschfeldt, LaForte)
- $\leq_K$ studied Mainly by Miller and Yu
- Lots of work remaining here.

# Hausdorff Dimension

- Yet another way to calibrate randomness is to use effective Hausdorff dimension.
- Recall $A$ is Schnorr random iff no computable martingale strongly succeeds, meaning that for some nondecreasing computable $h$, with $h(n) \to \infty$, the martingales succeeds $h$-quickly.
- Schnorr called the function $h$ and order.

- If $F$ is a martingale and $h$ is an order the *h-success* set of $F$ is the set:

$$S_h(F) = \{\alpha : \limsup_{n\to\infty} \frac{F(\alpha \upharpoonright n)}{h(n)} \to \infty\}.$$

- Thus, A real $\alpha$ is Schnorr random iff for all computable orders $h$ and all computable martingales $F$, $\alpha \notin S_h(F)$.
- Exponential orders offer a special place in this subject.

### Definition (Lutz)

An *s-gale* is a function $F : 2^{<\omega} \mapsto \mathbb{R}$ such that

$$F(\sigma) = 2^s(F(\sigma 0) + F(\sigma 1)).$$

- The basic idea here is that not betting on one outcome or the other is bad.
- Usually, decide that we are not prepared to favour one side or the other in our bet. Thus we make $F(\sigma i) = F(\sigma)$ at some node $\sigma$.In the case of an $s$-gale, then we will be unable to do this, without *automatically losing money due to inflation*.

- Lutz has shown that effective Hausdorff dimension can be characterized using these notions.
- It is not important exactly what the definition is but we get the following.

THEOREM (LUTZ, MAYORDOMO)

*For a class $X$ the following are equivalent:*

(I) $\dim(X) = s$.
(II) $s = \inf\{s \in \mathbb{Q} : X \subseteq S[d]$ *for some s-gale F*$\}$.
(III) $s = \inf\{s \in \mathbb{Q} : X \subseteq S_{2^{(1-s)n}}[d]$ *for some martingale d*$\}$.

- Lutz comment:
- “Informally speaking, the above theorem says the the dimension of a set is the most hostile environment (i.e. most unfavorable payoff schedule, i.e. the infimum $s$) in which a single betting strategy can achieve infinite winnings on every element of the set.”
- While Schnorr did not do any of this, he did look at exponential orders. He comments:
- “To our opinion the important statistical laws correspond to null sets with fast growing orders. Here the exponentially growing orders are of special significance.”

- For instance if $\Omega = a_1 a_2 \ldots$ then $a_1 0 a_2 0 \ldots$ has dimension $\frac{1}{2}$.
- No time to talk about results here.
- Also packing and box counting and other dimensions.
- Currently there is a lovely open question, which more or less asks if the only way to constuct fractional dimension is to effectively decompose a random: Suppose that $A$ has dimension $\frac{1}{2}$. Is there a $B \leq_T A$ with $B$ random?
- best partial result here is by Nies and Terwijn for $\leq_{wtt}$ .
- Jan Reimann's and Sebastiaan Terwijn's Theses.