Recent Progress in Algorithmic Randomness

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The Basic Refs are van Lambalgen's Thesis, Solovay's unpublished notes, and Li-Vitanyi. Also a new book "to appear" by Downey and Hirschfeldt prelim version on my home page.

And *Calibrating Randomness* (with Hirschfeldt, Nies and Terwijn) for BSL, soon on my web page.

Some Computability-Theoretical Aspects of Reals and Randomness, to appear, in a *Lecture Notes in Logic* volume edited by Cholak. et. al.

Some of the papers can be found in

`www.mcs.vuw.ac.nz/`

`research/math-pubs.shtml`

Nies home page, Hirschfeldt's home page.

Motivation

- What is "random"?
- How can we calibrate levels randomness? Among randoms?, Among non-randoms?
- How does this relate to classical computability notions, which calibrate levels of computational complexity?
- Von Mises, Church, Solomonoff, Levin, Gács, Chaitin, Kolmogorov, Shannon, etc.

Notation

- Real is a member of Cantor space 2^ω with topology with basic clopen sets $[\sigma] = \{\sigma\alpha : \alpha \in 2^\omega\}$ whose measure is $2^{-|\sigma|}$.
- for uniformity, a real is always nonrational.
- Strings = members of $2^{<\omega} = \{0, 1\}^*$.

Three views of effective randomness

1 Measure-Theoretical:

- Random means no distinguishing features.
- In effective terms:
  - Avoids all effective sets of measure 0.

2. Algorithmic:

- Random means hard to describe, incompressible: e.g. 1010101010.... (10000 times) would have a short program.
- In effective terms:
  - Initial segments cannot be output by short programs.

3. Other views: e.g. random means unpredictable.

- martingales=betting strategies (Doob etc)

– $F : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$, with $F(\sigma) = \frac{F(\sigma 1)+F(\sigma 0)}{2}$.

– F succeeds on α iff $\limsup_n F(\alpha \upharpoonright n) \rightarrow \infty$.

- Think of a real where every 3rd bit was 1. You could win betting on the bits.
- In effective terms, no effective martingale succeeds on α.

Martin-Löf randomness:

- A *c.e. open set* is one of the form $\bigcup_i (q_i, r_i)$ $(= \{[\sigma] : \sigma \in W\})$ where $\{q_i : i \in \omega\}$ and $\{r_i : i \in \omega\}$ are c.e..

- A *Martin-Löf test* is a uniformly c.e. sequence $U_1, U_2, \ldots$ of c.e. open sets s.t.

$$\forall i(\mu(U_i) \leq 2^{-i}).$$

(Computably shrinking to measure 0)

- α is *Martin-Löf random* if for every Martin-Löf test,

$$\alpha \notin \bigcap_{i>0} U_i.$$

Solovay Randomness

- We call a real α *Solovay random* iff for all c.e. sets of open intervals $\{I_n : n \in \omega\}$, with $\sum_n |I_n| < \infty$, $\alpha \in I_n$ for at most finitely many n.
- (Solovay) α is Martin-Löf random iff α is Solovay random.

Kolmogorov Complexity

- Capture the incompressibility paradigm.
- A string σ is random iff the only way to describe it is by hardwiring it. (Formalizing the Berry paradox)
- For a fixed machine N, we can define
- The *Kolmogorov complexity* $C(\sigma)$ of $\sigma \in \{0,1\}^*$ with respect to N, is $|\tau|$ for the shortest τ s.t. $N(\tau)\downarrow = \sigma$. (Kolmogorov)

- A real is N-random iff $C_N(\sigma) \geq |\sigma|$.
- A machine U is called universal iff for all N, there is a d such that for all σ, $C_U(\sigma) \leq C_N(\sigma) + d$.
- They exist (Kolmogorov). Hence there is a notion of Kolmogorov randomness for strings up to a constant.
- Proof: We can enumerate the Turing machines $\{M_e : e \in \mathbb{N}\}$. Define

$$U(1^e 0\sigma) = M_e(\sigma).$$

This particular coding gives $C(\tau) \leq M_e(\tau) + e + 1$.

- (Plain Counting Theorem-Kolmogorov) $|\{\tau : C(\tau) \leq |\tau| - d\}| \leq O(1)2^{|\tau|-d}$.
- Proof: pigeonhole principle.
- Thus plain complexity is a *combinatorial fact* This is important when we look at compression functions later.

A first attempt for *reals*

- The above works for *strings*.
- For *reals* problems occur because a string τ gives "$|\tau| + \tau$" much information.
- First try α, a real, is random iff for all n, $C(\alpha \upharpoonright n) \geq n - d$.
- (Martin-Löf) NO such real exist!

- Why? Take any α. Then, as a string $\alpha \upharpoonright n$ corresponds to some number which we can interpret as a string using llex ordering: $\alpha \upharpoonright n$ is the m-th string.

- Now consider the program that does the following. It takes a strings ν, interprets its length $m_\nu = |\nu$ as a string, $\sigma = \sigma_m$ and outputs $\sigma\nu$.

- Apply this to the string τ whose length is m th code of $\alpha \upharpoonright n$.

- The output would be much longer, and would be $\alpha \upharpoonright m + n$, with input having length m. Thus $C(\alpha \upharpoonright m + n) < m + n - O(1)$.

- This phenomenom is fundamental in

our understanding of Kolmogorov complexity and is called *complexity oscillations.*

- There are several known ways to get round this problem to cause only to get the information provided by the *bits* of the strings.

Prefix free

universal computers

- Levin, Gács, Chaitin.
- Computers have alphabet $\{0,1\}$.
- A computer M is *prefix-free* if

$$(M(\sigma)\downarrow \ \wedge \ \sigma' \supsetneq \sigma) \Rightarrow M(\sigma')\uparrow .$$

- A prefix-free M is *universal* if for every prefix-free N there is a c s.t.

$$N(\sigma)\downarrow \to \exists\tau(|\tau| \le |\sigma| + c \ \wedge$$

$$M(\tau)\downarrow = N(\sigma)).$$

- Fix a universal prefix-free machine M.

Levin-Chaitin randomness:

- Prefix freeness gets rid of the use of length as extra information:
- The *prefix-free complexity* $K(\sigma)$ of $\sigma \in \{0,1\}^*$ is $|\tau|$ for the shortest τ s.t. $M(\tau)\downarrow = \sigma$.
- Note now $K(\sigma) \leq |\sigma| + K(|\sigma|) + d$, about $n + 2\log n$, for $|\sigma| = n$.

- (Counting Theorem-Chaitin) $|\{\tau : |\sigma| = |\tau| = n \wedge K(\sigma) \leq K(\tau) + d - c\}| \leq O(2^d)2^{n+K(n)-c}$.

- α is *random* if there is a c s.t.

$$\forall n(K(\alpha \upharpoonright n) > n - c).$$

  This happens if there is a c such that for infinitely many n, $C(\alpha \upharpoonright n) > n - c$.

Schnorr's Theorem

Theorem[Schnorr]

Levin-Chaitin random $\iff$ Martin-Löf random.

So we know that we are on a reasonable idea since the notions coincide.

The proof and Kraft-Chaitin

- Theorem(Kraft)

  (i) If A is prefix-free then $\sum_{n \in A} 2^{-|n|} \leq 1$.

  (ii) (Kraft-Chaitin) Let $d_1, d_2, \cdots$ be a computably enumerable collection of lengths (possibly with repetitions), with *targets* σ_i, called an *axiom* $\langle d_i, \sigma_i \rangle$. Then $\sum 2^{-d_i} \leq 1$ iff we can compute a prefix-free machine M with domain members τ_i and $|\tau_i| = d_i$, and $M(\tau_i) = \sigma_i$.

Lots of random reals

- $\mu\{A : A \text{ random }\} = 1.$
- Consider the Σ_2^0 class $\{A : \exists k \forall n K(A \upharpoonright n > n - k\}$ contains all random reals.
- Hence there are ones of low Turing degree (low basis theorem) and hyperimmune free degree.
- There are ones of all jumps and even Δ_2^0 ones of all jumps (Kučera, Downey-Miller)

Extending Schnorr's Theorem

- (Miller and Yu) α is Martin-Löf random iff $\sum_{n\in\mathbb{N}} 2^{n-K(\alpha\upharpoonright n)} < \infty$.
- This says that whilst the K-complexity is above n, mostly it is "pretty far" from n.
- (Miller and Yu) Suppose that f is an arbitrary function with $\sum_{m\in\mathbb{N}} 2^{-f(m)} = \infty$. Suppose that α is 1-random. Then there are infinitely many m with $K(\alpha \upharpoonright m) > m + f(m) - O(1)$.

Monotone Complexity

- Levin's original idea here was to try to assign a complexity to the *real itself.* That is, think of the complexity of the real as the shortest machine that outputs the real. Hence now we are thinking of machines that take a program σ and might perhaps output a real α. (Nonsense unless α is computable)
- The following definition can be applied to Turing machines with potentially infinite output, and to discrete ones mapping strings to strings. In this definition, we regard

$M(\sigma)\downarrow$ to mean that at some stage s, $M(\sigma)\downarrow[s]$.

- We say that a machine M is *monotone* if its action is continuous. That is, for all $\sigma \preceq \tau$, if $M(\sigma)\downarrow$ and $M(\tau)\downarrow$ then

$$M(\sigma) \preceq M(\tau).$$

- Levin's (standard) monotone complexity Km is defined as follows. Fix a universal monotone machine U.

$$Km(\sigma) = \min\{|\tau| : \sigma \preceq U(\tau)\}.$$

- (Levin's Theorem) A is Martin-Löf random iff $Km(A \upharpoonright n) > n - O(1)$.

Computably enumerable reals:

- α is *c.e.* if there is a computable sequence of rationals

$$q_0 < q_1 < \ldots \longrightarrow \alpha.$$

Equivalently, the lower cut of α is a c.e. set of rationals.

- α is *computable* if there is a computable f s.t.

$$\forall n(\alpha - q_{f(n)} < 2^{-n}).$$

- Being a c.e. real is *not* the same as being the characteristic function of a c.e. set. (Soare)

- Chaitin's Omega
- The most famous c.e. real is

$$\Omega = \mu \ \mathrm{dom}(M) = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|},$$

the "halting probability."

- c.e. reals are the relevant effective sets for randomness (as they are the measures of domains of prefix free machines) in the same way that c.e sets are the central objects in classical computabiltity theory.

- Ω is random.
- Proof. We use Kraft-Chaitin: We build a Kraft-Chaitin set with coding constant c given by the recursion theorem. If, at stage s, we see $K_s(\Omega_s \restriction n) < n - c - 1$, enumerate $\langle n - c, \Omega_s \restriction n \rangle$ into KC, and hence $\Omega \restriction n \neq \Omega_s \restriction n$.

Plain Complexity again

- In spite of the fact that we have this natural characterization in terms of K or Km, it was a longstanding question whether there was a plain complexity charcaterization of randomness.
- the relationship between plain and prefix-free complexities is complicated.
- (Solovay)

$$K(x) = C(x) + C^{(2)}(x) + \mathcal{O}(C^{(3)}(x)).$$

$$C(x) = K(x) - K^{(2)}(x) + \mathcal{O}(K^{(3)}(x)).$$

- The 3's are *sharp*
- The maximum complexity a string of length n can have is (i) $C(\sigma) = n - O(1)$. (ii) $K(\sigma) = n + K(n) - O(1)$.
- (Solovay) (ii) implies (i), but not conversely.
- Say that a real is *strongly Chaitin random* iff there are infinitely many n with $K(\alpha \restriction n) \geq n + K(n) - O(1)$.
- Say that it is *Kolmogorov random* if there are infinitely many n with $C(n) \geq n - O(1)$.
- (Solovay) They exist.

- Fundamental question: are they the same?

n-randomness

- This "all" relativizes, so we can define Martin-Löf randomness relative to a set B, and n-randomness relative to $\emptyset^{(n-1)}$ which, due to the work of Kurtz, is the same as randomness for tests with Σ^0_n *classes* as tests.
- Thus, we can define 2-randomness as 1-randomness relative to the halting problem.

Kolmogorov randomness

- Theorem Nies-Terwijn-Stephan, Miller 2-randomness=Kolmogorov randomness. (!)
- The other direction. (Miller, NST)
- A compression function acts like U^{-1}.
- We say that $F : \Sigma^* \mapsto \Sigma^*$ is a compression function if for all x $|F(x)| \leq C(x)$ and F is 1-1.
- Nies, Stephan, and Twerijn There is a compression function F with $F' \leq_T \emptyset'$.
- Consider the Π^0_1 class of functions $|\widehat{F}(\sigma)| \leq C(\sigma)$.

- The main idea is that most of the basic facts of plain complexity can be re-worked with any compression function. For a compression function F we can define F-Kolmogorov complexity: α is F-Kolmogorov random iff $\exists^\infty n(F(\alpha \upharpoonright n) > n - O(1))$.
- (NST) If Z is 2-random relative a compression function F, then Z is Kolmogorov F-random.
- Now we can save a quantifier using a low compression function.

1-randomness and plain complexity

- Theorem (Miller and Yu) x is Martin-Löf random iff $(\forall n)\, C(x \upharpoonright n) \geq n - g(n) \pm O(1)$, for every computable $g\colon \omega \to \omega$ such that $\sum_{n\in\omega} 2^{-g(n)}$ is finite.

Measures of Relative Randomness

- A pre-ordering $\leq$ on reals is a *measure of relative randomness* if it satisfies the *Solovay property*:

$$\text{If } \beta \leq \alpha \text{ then}$$
$$\exists c\ (\forall n\ (K(\beta \upharpoonright n) \leq K(\alpha \upharpoonright n) + c)).$$

- Notice that if α is random and $\alpha \leq \beta$ then by Schnorr's Theorem, β is random too.

- Can also use C, and others.

- The idea is that *if* we can characterize *randomness* by initial segment complexity, then we oght to be able to calibrate *randomness* by comparing initial segment complexities.

Solovay Reducibility

- We talk about *the* halting problem, whereas of course we really mean HALT_U for a universal U. But... they are all the same (Myhill)

- Solovay introduced a reduction to address this for randomness.

- $(\alpha \leq_S \beta)$ α is Solovay or domination reducible to β iff there is a constant d, and a partial computable φ, such that for all rationals $q < \beta$

$$\varphi(q) \downarrow \wedge\ d(\beta - q) > |\alpha - \varphi(q)|.$$

- Intuitively, however well I can approximate β, I can approximate α just as well. Clearly $\leq_S$ implies $\leq_T$.
- S-reducibility is a measure of relative randomness (Solovay)
- This follows by : Let d be given. Then there is a constant $c = c(d)$ such that for all n: if σ and τ have length n and $|\sigma - \tau| < 2^{-n+d}$, $K(\sigma) + c > K(\tau)$.

Only one random c.e. real

- A c.e. real is Ω-*like* if it dominates all c.e. reals.
- (Solovay) Any Ω-like real is random.
- Proof : By Schnorr.

Solovay proved that Ω-like reals posessed many of the properties that Ω posessed. He remarks:

"It seems strange that we will be able to prove so much about the behavior of $K(\Omega \upharpoonright n)$ when, a priori, the definition of Ω is thoroughly model dependent. What our discussion has shown is that our results hold for a class of reals (that include the value of the universal measures of ...) and that the function $K(\Omega \upharpoonright n)$ is model independent to within $O(1)$."

- **Theorem** (Calude, Hertling, Khoussainov, and Wang) If a c.e. real is Ω-like then it is an Ω-number. That is, a halting probability.

- Proof: We have $\Omega \leq_S \alpha$ with enumerations $\Omega_s \to \Omega$, $\alpha_s \to \alpha$

- We know that if we use a stage by stage approximation, then essentially $c(\alpha - \alpha_s) \geq \Omega - \Omega_s$.

- Use Kraft-Chaitin. If $U(\tau) = \sigma$ then $2^{-|\tau|}$ enters the domain of $U - U_s$. We re-cycle this by defining $M(\tau) = \sigma$ with $2^{-|\tau|}$ entering $\alpha - \alpha_s$, this keeping $K_M(\sigma) \leq K_U(\sigma) + c$. etc.

Kučera-Slaman Theorem

- **Theorem** [Kučera-Slaman] If a c.e. real is random then it is Ω-like.
- ie all random c.e. reals are the "same" and are halting probabilities. (even though it might be possible for it to be as high as $n + 2\log n$ all oscillations occur at the "same" n's.)

- Proof: Suppose that α is random and β is a c.e. real. We need to show that $\beta \leq_S \alpha$. We enumerate a Martin-Löf test $F_n : n \in \omega$ in stages. Let $\alpha_s \to \alpha$ and $\beta_s \to \beta$ computably and monotonically. We assume that $\beta_s < \beta_{s+1}$.

- At stage s if $\alpha_s \in F_n^s$, do nothing, else put $(\alpha_s, \alpha_s + 2^{-n}(\beta_{s+1} - \beta_{t_s}))$ into F_n^{s+1}, where t_s denotes the last stage we put something into F_n.

- One verifies that $\mu(F_n) < 2^{-n}$. Thus the F_n define a Martin-Löf test. As α is random, there is a n such that for all $m \geq n$, $\alpha \notin F_m$. This shows that $\beta \leq_S \alpha$ with constant 2^n.

Structure

- The c.e. reals using $\leq_S$ forms a upper semilattice, called the Solovay degrees.
- (Downey, Hirschfeldt, Nies) (i) $+$ induces a join
  (ii) It is distributive
  (iii) dense
  (iv) $[\Omega]$ is the only join inaccessible element.

- (Downey, Hirschfeldt, Laforte) The structure of the S-degrees of c.e. reals has an undecidable theory.
- S-reducibility is a measure of relative randomness, but not the only one, and it has some problems.
- However, structure of $\leq_K$ and $\leq_C$, except on c.e. reals, is largely unknown. On the randoms, we lack techniques.

Results on $\leq_K$

- *Define* $A \leq_K B$ to mean $K(A \restriction n) \leq K(B \restriction n) + O(1)$, all n.
- Yu, Ding, Downey $\mu(\{A \leq_K A\}) = 0$. Hence uncountably many K degrees.
- Yu, Ding In fact $2^{\aleph_0}$.
- (Miller and Yu) for almost all pairs $A|_K B$.
- (Miller and Yu) for all $n \neq m$, $\Omega^{(n)}|_K \Omega^{(m)}$.
- (Miller and Yu) However, there are random A, B with $B <_K A$.

- (Miller and Yu) Each K-degree of a random countable. Unknown if there is an uncountable K-degree.
- (Csima and Montalbán) There are minimal pairs of K-degrees.

A unified approach

- (von Lambalgen reducibility) For $x, y \in 2^\omega$, write $x \leq_{vL} y$ if $(\forall z \in 2^\omega)\ x \oplus z$ is 1-random $\implies$ $y \oplus z$ is 1-random.
- is the same as Define $y \leq_{\mathcal{L}R} x$ if $(\forall z \in 2^\omega)\ z$ is 1-x-random $\implies$ z is 1-y-random, on the randoms.
- inspired by van Lambalgen: $A \oplus B$ is random iff A is B-random and B is A-random.
- (Miller and Yu) If $x \leq_{vL} y$ and x is n-random, then y is n-random.

- (Miller and Yu) If $y \leq_T x$ and y is 1-random, then $x \leq_{vL} y$.
- (Miller and Yu) If $m \neq n$, then $\Omega^{\emptyset^{(m)}}$ and $\Omega^{\emptyset^{(n)}}$ have no upper bound in the vL-degrees.
- (Miller and Yu) If x is n-random and $y \leq_T x$ is 1-random, then y is n-random.
- (Miller and Yu)
  $x \leq_K y \implies x \leq_{vL} y$.
- (Hence)(Yu, Ding, Downey)
  $\mu(\{\beta : \beta \leq_K \alpha\}) = 0$.
- Proof: If β is 1-α-random, then $\beta \not\leq_{vL} \alpha$ and hence, since $\mu(\{\beta : \beta$ is 1-α-random $\}) = 1$, we get

$\mu(\{\beta : \beta \leq_K \alpha\}) = 0$, seince $\leq_K$ implies $\leq_{vL}$.

K-trivial reals:

- (Solovay) There exist noncomputable reals α such that for all n

$$K(\alpha \upharpoonright n) \leq K(1^n) + c.$$

- These are called K-trivial reals.
- Note that Chaitin proved that if $K(\alpha) \leq K(1^n) + c$, then α is Δ^0_2. It was earlier by Loveland proved that $C(\alpha \upharpoonright n) \leq C(1^n) + c$ *for all* n, *implies* α *is computable.*

- Such A can be c.e. *sets*. (DHNS, and others)

- Solovay's 1974 proof is very complicated. Here is a simplified version proving a stronger result.

- (DHNS) There is a c.e. noncomputable set A such that for all n

$$K(A \upharpoonright n) \leq K(n) + \mathcal{O}(1).$$

- Let

$$A_{s+1} = A_s \cup \{x : W_{e,s} \cap A_s = \emptyset \wedge x \in W_{e,s}$$

$$\wedge \sum_{x \leq j \leq s} 2^{-K(1^j)[s]} < 2^{-(e+1)}\}.$$

- (DHNS) K-trivial reals are never of high degree, so this is an injury free solution to Post's problem.

Nies Theorems

Nies (and Hirschfeldt) has some deep material here:

- Every K-trivial is bounded by a K-trivial c.e. set.
- Every K-low is superlow, and "tracable". (and hence (Chaitin) there are only countably many.)
- K-trivial = low for K.
- K-trivials are closed under T-reducibility and form the only known natural Σ^0_3 ideal in the Turing degrees.

- The are bounded above by a low_2 degree.

Cereals are red herrings

- In some sense, the c.e. reals and Ω make us think of randomness as "like" the halting problem, and more random=more information. This seems false.
- Stephan has shown that if $\boldsymbol{a}$ is a Turing degree containing a 1-random real which has enough information to compute a $\{0,1\}$ valued diagonal function then $\boldsymbol{a} \geq \mathbf{o}'$. (If $\boldsymbol{a}$ is 1-random and PA then $\mathbf{o}' \leq \boldsymbol{a}$.)

- Miller proved the following remarkable result. Say α is "pseudo-low" if $(\exists^\infty n)[K(n) \leq K^\alpha(n) + O(1)]$.
- The intuition is that α is so computationally useless that it gives no help in computing such n.
- (Miller) α is random and pseudo-low if α is 3-random.
- (DHNM) also if $A \leq_T B$ and A, B are random with B n-random, then A is also n-random. Thus randomness is a *lowness* property.

Relativizing Chaitin's Halting Probability

- Look at Ω^A as an operator.
- Care is needed as to exactly what this means.
- Note this is CE but not CEA.
- e.g. $\Omega|_T\Omega^\Omega$, indeed they form a minimal pair.
- Hoped to solve Martin's conjecture
- (DHNM) Alas there are $A =^* B$ such that Ω^A and Ω^B are relatively

random and hence Turing incomparible.

- (DHNM) Every 2-random is Ω^B for some B. (so maybe ce reals are not red herrings)
- Compare with Kurtz's Theorem : every 2-random is properly CEA.
- Omega operators are lower semicontinuous but not continuous, and moreover, that they are continuous exactly at the 1-generic reals.

Martingales

- martingales=betting strategies (Doob etc)

- $F : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$, with $F(\sigma) = \frac{F(\sigma 1)+F(\sigma 0)}{2}$.
- F succeeds on α iff $\limsup_n F(\alpha \restriction n) \to \infty$.
- Think of a real where every 3rd bit was 1. You could win betting on the bits.
- also have *supermartingales* where we only ask $F(\sigma) \geq \frac{F(\sigma 1)+F(\sigma 0)}{2}$.
- Supermartingales have an advantage that they can be enumerated, and

there is a multiplicatively optimal minimal one. (Levin)

- (Schnorr) a real is 1-random iff no computably enumerable (super-)martingale succeeds on it. Here F above should have a computable sequence of approximations $F_s \to F$.

- Proof; (Idea) Let $U_k : k \in \omega$ be a Martin-Löf test. Initially define $F_0(\sigma) = 1$ for all σ. If we see $[\sigma] \in U_k[s]$ raise $F_{s+1}(\widehat{\sigma}) = F_s(\widehat{\sigma}) + 1$ for all $\sigma \preceq \widehat{\sigma}$. Propogate this down to λ losing powers of 2. (fig) Then $A \in \cap_k U_k$ iff $F(A \upharpoonright n) \to \infty$.

  The other direction is similar. When $F(\sigma) > 2^k$ put $[\sigma] \in U_k$.

Hausdorff dimension

- 1895 Borel, Jordan
- Lebesgue 1904 measure
- In any n-dimesnional Euclidean space, Carathéodory 1914

  $$\mu^s(A) = \inf\{\sum_i |I_i|^s : A \subset \cup_i I_i\},$$

  where each I_i is an interval in the space.
- 1919 Hausdorff s fractional. and refine measure 0.
- For $0 \leq s \leq 1$, the s-measure of a clopen set $[\sigma]$ is

  $$\mu_s([\sigma]) = 2^{-s|\sigma|}.$$

- Lutz has the folllwijg characterization of effective Hausdorff dimension:
- An s-*gale* is a function $F : 2^{<\omega} \mapsto \mathbb{R}$ such that

$$F(\sigma) = 2^s(F(\sigma 0) + F(\sigma 1)).$$

Similarly we can define s-supergale, etc.

- Theorem (Lutz) For a class X the following are equivalent:
  - (i) $\dim(X) = s$.
  - (ii) $s = \inf\{s \in \mathbb{Q} : X \subseteq S[d]$ for some s-gale $F\}$.
- Lutz says the folllowing:

"Informally speaking, the above theorem says the the dimennsion of a set is the *most hostile environment* (i.e. most unfavorable payoff schedule, i.e. the infimum s) in which a single betting strategy can *achieve infinite winnings* on every element of the set."

- Thm Lutz, Mayordomo: The Hausdorff dimension of a real α is

$$\liminf_{n\to\infty} \frac{K(\alpha \restriction n)}{n} = (\liminf_{n\to\infty} \frac{C(\alpha \restriction n)}{n})$$

Dimensions of strings

- Lutz has introduced a method of assigning dimensions to strings.
- liminf $\frac{K(\alpha \restriction n}{n}$,
- equivalently, the infimum over all s of the values of $d^s(\alpha \restriction n)$.
- To discreteize this characterization, Lutz used three devices:

  (i) He replaced supergales by *termgales*, which resemble supergales, yet have modifications to deal with the terminations of strings. This is done first via s-termgales and then later by termgales, which are uniform families of s-termgales.

(ii) He replaced $\to \infty$ by a finite threshold.

(iii) He replaced optimal s-supergale by and optimal termgale.

- For $s \in [0, \infty)$, an s-*termgale* is a function d from the collection of terminated strings T to $\mathbb{R}^+ \cup \{0\}$, such that $d(\lambda) \leq 1$, and

$$d(\sigma) \geq 2^{-s}[d(\sigma 0) + d(\sigma 1) + d(\sigma \Box)].$$

  Here $\Box$ is a delimiting symbol, and has vanishing probability as $n \to \infty$.

- (i) A *termgale* is a family $d = \{d^s : s \in [0, \infty)\}$ of

s-termgales such that

$$2^{-s|\sigma|}d^s(\sigma) = 2^{-s'|\sigma|}d'(\sigma),$$

for all s, s' and $\sigma \in 2^{<\omega}$.

(ii) We say that a termgale is *constructive* or Σ^0_1, if d^0 is a Σ^0_1 function.

- Now introduce optimal termgales etc.
- filtering through discrete semimeasures and the Coding theorem, you get
- There is a constant $c \in \mathbb{N}$ such that for all $\sigma \in 2^{<\omega}$,

$$|K(\sigma) - |\sigma| \dim(\sigma)| \leq c.$$

Some ignored material

- supermartingale reducibility
- work on Shnorr, computable and other randomness notions.
- especially the beautiful lowness material of Kučera-Terwijn-Zambella, and of Nies.
- Stochasiticity and Miller, Nies, Stephan, Merkle, etc
- complexity of c.e. sets and Kummer, Michnik.

- Thank you.