Logic and Invariants

Rod Downey
Victoria University
Wellington, New Zealand
Dedicated to Gaven Martin for his 60th Birthday

Albany, January 2019

Overview

- Mathematics is replete with "invariants."
- Think: dimension, rank, Ulm sequences, spectral sequences, etc, etc.
- What is an invariant? I recognize one when I see it.
- How to show that no invariants are possible? How to quantify how complex invariants must be if they have them?

No Invariants

- We concentrate on isomorphism.
- What is the use of an invariant, like e.g. dimension, Ulm invariants, etc.
- Arguably, they should make a classification problem easier.
- For example, one invariant for isomorphism type of a class of structures e.g. vector spaces over $\mathbb{Q}$ is the isomorphism type, but that's useless.
- We choose dimension as it completely classifies the type.
- How to show NO invariants?
- We give one answer in the context of computable mathematics, and mention some other approaches using logic.

A First Pass

- Stuff beyond my ken.
- If we consider models of a first order theory T, then structures like vector spaces over F of, say, cardinality $\aleph_0$ have only a countable number of models because of the invariants, things like trees have many more $2^{\aleph_0}$.
- Shelah formalized all of this by showing that

Theorem (Dichotomy Theorem)

For a complete theory T, either the number of models of cardinality κ is always 2^κ for all uncountable κ, or the number is "small". (Shelah $I(T, \aleph_\xi) < \beth_{\omega_1}(|\xi|)$, Hrshovsky and others have refined this.)

- Moreover, to prove this he describes a set of "invariants" roughly corresponding to dimension or "rank" (yes; matroids rear their ugly heads) that control the number of models of that cardinality. ("does not fork over")

Reductions

- All the methods below use reductions.
- A reduces to B ($A \leq B$) means that a method for solving B gives one for solving A.
- Typically, there is a function f such that for all instances x, $x \in A$ iff $f(x) \in B$. (meaning "yes" instances go to "yes" instances).
- Example from classical mathematics: map square matrices to determinants. A=nonsingular matrices and B nonzero reals.
- Important that the function f should be "simpler" than the problems in question.
- For classical computability theory, f is computable. For complexity theory, f might be poly-time.

Method 2

- We leave outer space, and concentrate on "normal" things.
- We can think of problems having isomorphism types as corresponding to "numbers" corresponding to equivalence classes (i.e. isomorphism types).
- Thus a problem A reduces to a problem B if I can map the isomorphism types correspnding to A to those of B. So determining if two B-instances are isomorphic gives the ability to do this for A. That is (in the simplest form) xAy iff $f(x)Bf(y)$.
- This is called Borel cardinality theory.
- Why? What is a reasonable choice for functions f? Answer: f should be Borel (at least when studying equivalence relations on Polish spaces-complete metrizable with countable dense set).
- Classical mathematics regards countable unions and intersections of basic open sets as "building blocks."

Examples

- All on ω^ω.
- Identity $E_=$.
- Vitali operation: E_1 $\overline{x} =^* \overline{y}$ iff they agree for almost all positions. $E_= <_B E_1$ and E_1 captures the complexity of rank one torsion free groups (more later).
- E_∞ the maximal. For example trees. There are also algebraic problems here such as the orbits of the 2 generator free group $\mathbb{Z}^2$ acting on $2^{\mathbb{Z}^2}$.
- This is an area of significant resent research (Hjorth, Thomas, Kechris, Pestov) and is still ongoing.

Method 3-Refining things

- As a logician I am more interested in deeper understanding of complexity.
- The plan is to understand invariants computationally.
- Invariants should make problems *simpler*.
- Let's interpret this as computationally simpler.

Computable mathematics

- Arguably Turing 1936: Computable analysis.
- Mal'cev 1962 A computable abelian group is computably presented if we have $G = (G, +, 0)$ has $+$ and $=$ computable functions/relations on $G = \mathbb{N}$.
- When can an abelian group be computably presented? (Relative to an oracle) Is there any reasonable answer?
- Do different computable presentations have different computable properties?
- Mal'cev produced examples presentations of $\mathbb{Q}^\infty$ that were not computably isomorphic, as we see later.
- Along with Rabin and Frölich and Shepherdson, began the theory of presentations of computable structures, though arguably back to Emmy Noether, Kronecker as recycled in van der Waerden (first edition).
- See Matakides and Nerode "Effective Content of Field Theory".

Why should we care?

- If we are interested in actual processes on algebraic structures then surely we need to understand the extent to which they are algorithmic.
- Effective algorithmics requires more detailed understanding of the model theory. Witness the resurrection of the study of invariants despite Hilbert's celebrated "destruction" of the programme.
- The Hilbert basis (or nulstellensatz) theorem(s) are fine, but suppose we need to calculate the relevant basis.
- Examples of this include the whole edifice of combinatorial group theory. The theory of Gröbner bases etc. New constructions in combinatorics, algebra, etc.
- As we will see a backdoor into establishing classical results about the existence/nonexistence of invariants in mathematics. Computability is used to establish classical result.
- Establishing calibrations of complexity of algebraic constructions.... reverse mathematics.

Σ^1_1-completeness?

- The halting problem is Σ^0_1. This means it can be described by an existential quantifier on numbers around a computable predicate. "There is a stage s where the e-th machine with input y halts in at most s steps"
- Showing that a problem A is Σ^0_1 complete means that there is a computable f such that for each instance I of a Σ^0_1 problem B, I can compute $f(I)$ which is an instance of A such that I is a yes for B iff $f(I)$ is a yes for A. A is the "most complex" Σ^0_1 problem.
- For example, the word problem for finitely presented groups, can be Σ^0_1 complete for a finitely presented group.
- If a problem can be expressed as a finite number of alternations of number quantifiers, it is called arithmetical, "Δ^0_n" for some n.
- For example: is φ_x total? provably needs an alternation of quantifiers.

- Some problems are too complex for this. Classical isomorphism of infinite structures: "There is a function such that ...."
- If we allow function quantifiers, we put a "1" on top.
- Thus we enter the realm of second order logic.
- Note now we are searching through the uncountably many possible functions $f : \mathbb{N} \to \mathbb{N}$.

$$\exists f \forall x, y (f(x+y) = f(x) + f(y)).$$

- Analogously, a problem is Σ^1_1 complete if every other Σ^1_1 problem reduces to it.

Consequences of Σ_1^1-completeness

- The idea of an invariant is that is ought to make the problem simpler.
- Classical isomorphism is always in the class Σ_1^1.
- Invariants make this easier, you would expect. Dimension in a vector space makes the problem Δ_3^0.
- The point is that a Σ_1^1-completeness result result means that the cannot be reasonable invariants for the isomorphism problem.

Torsion-free abelian groups

- In general the isomorphism problem is very complex:

Theorem (Downey and Montalbán-J. Algebra)

The isomorphism problem for torsion-free abelian groups is Σ_1^1 complete.

- That is a computational "proof" that there cannot be invariants.
- As explained in the DM paper, group theorists try to understand finitely presented groups via spectral sequences, one called the integral homology sequence (Stallings etc)
- The above result, combined with one of Baumslag, Dyer and Miller shows that deciding if two finitely presented groups have the same 3rd members of this sequence is already Σ_1^1 complete!!
- This methodology understands invariant theory computationally.
- Also used by Slaman and Woodin, Friedman, Stanley, Knight and others in many other settings.

Other examples

- (Melnikov TAMS) Compact connected Polish abelian groups are unclassifible.
- Here you need to be careful how you define computability in uncountable structures.
- (Bazhenov, Kalimullin, Melnikov, and Ng) The isomorphism for automatic structures is unclassifiable.
- Solves a 1994 problem of Khoussainov and Nerode. Uses methods from our new work on online structures.
- (Harrison-Trainor, JML) Decidably presentable structures are unclassifiable.

Better algebraic classes

- Okay, so the general classification problem is intractable, then can we measure how complex the classically "understood" classes are.
- Recall that if G is a torsion-free then G embeds into $\oplus_{i \in F}(\mathbb{Q}, +)$. The cardinality of the least such F is called the (Prüfer) rank of G.
- Khisamiev proved that there is an effective embedding. (That is if G is a computable torsion-free abelian group then G can be computably embedded into a computable copy of $\oplus_{i \in F}(\mathbb{Q}, +)$.)

Rank One Groups

- The only groups we understand well are the rank one groups (and certain mild generalizations) If $g \in G$, define $t(g) = (a_1, a_2, \dots)$ where $a_i \in \{\infty\} \cup \omega$ and represents the maximum number of times p_i divides g. Say that $t(g) = t(h)$ if they are $=^*$, meaning that they must be ∞ in the same places, but otherwise are finitely often different. Thus we can write $t(G)$.
- For example, a divisible group would have $(\infty, \infty, \dots)$ as its type.

Theorem (Baer, Levi)

For rank 1 torsion-free abelian groups, $G \cong H$ iff they have the same type.

- One corollary is that if we consider $T(G) = \{\langle x, y\rangle \mid x \leq t(G)_y\}$, then G is computably presentable iff $T(G)$ is Σ^0_1. (Mal'tsev)

Two Corollaries

- Our goal is understanding the isomorphism problem.
- A structure is called computably categorical iff any two computable copies are computably isomorphic.
- G is a computably categorical torsion-free abelian group iff it has finite rank.
- (Coles, Downey and Slaman-Bull LMS) The isomorphism type each torsion free abelian group of infinite finite rank can be decoded from a specific Turing degree computing the halting problem. (Specifically, it has "first jump degree".)
- Anderson, Kach, Melnikov, Solomon have a refinement of this for general torsion-free groups.
- Melnikov and Ng have solved Mal'cev's 60yo problem showing that a computable torsion abelian group has one or infinitely many computable copies, up to computable isomorphism.

Algorithmic dimension

- The point here is that long ago in the 1990's Goncharov proved that there are computable structures with exactly 2 computable copies in the isomorphism type.
- (Hirschfeldt, Khoussainov, Slinko, Shore) showed how to do this for various normal classes like graphs, partial orderings, groups.
- Lots of recent work here.

The infinite rank case

- It could be hoped that if G has infinite rank, then $G \cong \oplus_{i\in\omega} H_i$ with H_i of rank one.
- Alas, this is not true, however, there is a class of groups for which this is true, called completely decomposable for which this does happen.
- What about categoricity for such groups?
- We cannot hope for computable categoricity, but can hope for things "higher up".

The homogeneous case

- If $G \cong \bigoplus H$ for a fixed H then G is called homogeneous

Theorem (Downey and Melnikov-J. Algebra)

Homogeneous computable torsion free abelian groups are Δ^0_3 categorical.

- The proof relies on a new notion of independence called S-independence generalizing a notion of Fuchs to sets S of primes.
- B, a set of elements, is S-independent (in G) iff for all $p \in S$ and $b_1, \ldots, b_k \in G$,

$$p \Big| \sum_{i=1}^{k} m_i b_i \text{ implies } p | m_i \text{ for all } i.$$

- This bound is tight.

But when can it be Δ^0_2 categorical?

- Recall that a set S is called semilow if $\{e \mid W_e \cap S \neq \emptyset\} \leq \emptyset'$.
- Semilow sets allow for a certain kind of local guessing, and arose in (i) automorphisms of the lattice of computably enumerable sets (Soare) and in (ii) computational complexity as non-speedable ones. (Soare, Blum-Marques, etc.)

Theorem (Downey and Melnikov-J. Algebra)

G is Δ^0_2 categorical iff the type of H consists of only 0's and ∞'s and the position of the 0's is semilow.

- The proof is tricky and splits into 5 cases depending on "settling times".
- We remark that this is one of the very few known examples of when Δ^0_2 categoricity of structures has been classified.

The general completely decomposable case

Theorem (Downey and Melnikov-TAMS)

A completely decomposable G is Δ^0_5 categorical. The bound is tight.

The proof uses methods from the homogeneous case, plus some new ideas. The sharpness is a coding argument. For sharpness we use copies of $\oplus_{i\in\omega}\mathbb{Z}\oplus\oplus_{i\in\omega}\mathbb{Q}^{(p)}\oplus\oplus_{i\in\omega}\mathbb{Q}^{(q)}$., where $p\neq q$ primes and $\mathbb{Q}^{(r)}$ denotes the additive group of the localization of $\mathbb{Z}$ by r. Then a relation θ on this group which is decidable in one copy and very bad in another.
With some extra work we can also prove the following. We don't know if the bound is sharp here.

Corollary (Downey and Melnikov-TAMS)

The index set of completely decomposable groups is Σ^0_7.

Thank You