The Finite Intersection Property and Computability Theory

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One equivalent of the axiom of choice

A family of sets $\mathcal{F} = \{ A_i \mid \in Q \}$ has finite intersection property iff for all finite $F \subset Q$, $\cap_{i \in F} A_i \neq \emptyset$.

The principal says: Any collection of sets has a maximal subfamily with FIP.

We investigate the computability of this.

First began by Dzharfarov and Mummert.
The first thing to notice is that it depends on whether you consider the family as **set** or a **sequence**

- If as a **set** then $\emptyset'$ is easily codable into a sequence and the theorem is equivalent to ACA$_0$. (Namely, have a set $B = B_e$ such that it is initially empty, and if $e \in \emptyset'[s]$ henceforth intersect it with everything, so it must be included. $\emptyset'$ can clearly figure things out.)
- Interesting if a **sequence**, so that $A_1, A_2, A_3$ is different from $A_2, A_3, A_1$.
- Similarly $\bar{D}_2$IP for for all pairs $A_i \cap A_j \neq \emptyset$. (DM notation)

**Definition**

Say that $a$ is FIP iff for all computable collections of sets, $a$ can compute a solution to the FIP problem.
Theorem (Dzhariyarov and Mummert)

There is a computable collection of sets with no c.e. subfamily with FIP. So $\mathbf{0}$ is not FIP, or even $\overline{D}_2$IP.

1. Meet $R_e : W_e$ is not an index for a maximal FIP family.
2. In the below I will use $A_i, \ldots$ and $X_e, B_i$ etc. Of course these are all the same and really are $W_{f(j)}$ for a computable $f$ given by the s-m-n theorem, and I am really concerned with the index $f(i)$. Also we will ensure that each nonzero set has a unique identifier in it, so these are really streams of numbers under consideration.
3. Use a trap set $X_e$.
4. Begin with $A_0, A_1, \ldots$. Wait for $W_e$ to respond.
5. Start intersecting $X_e$ “in the back”. If $W_e$ enumerates it win with finite injury.
Theorem (Dzhafarov and Mummert)

If $a$ is $\bar{D}_2 IP$ then it is hyperimmune. (i.e. not computably dominated for those under 35)

Theorem (Dzhafarov and Mummert)

If $a \neq 0$ is c.e. then $a$ is FIP.

Theorem (Dzhafarov and Mummert)

If $a$ is $\emptyset'\text{-hyperimmune}$ then it is FIP.
The c.e. noncomputable case below $C \neq_T \emptyset$.

We are building $A_0, A_1, \ldots A_n$.

We want to put some element $B$ into this family (with truncation), as we have seen $B$ intersect $A_0, \ldots, A_j$, the first position determined by $B$'s index.

We then place a permitting challenge to $C$. If later we see $C$ permit $j$, we change the family to $A_0, \ldots A_j, B$.

When $B$ meets $A_{j+1}[s]$ place another challenge on $B$.

The $\emptyset'$-hyperimmune is because $\emptyset'$ knows if we ever want to put things in, and infinitely often the $C$ can decode this.

It might seem that the c.e. case would also work for $\Delta^0_2 C$, but it fails for a nonuniform reason.

An earlier promise for a $C$-configuration might force some $D_1$ into the sequence which might be disjoint from the $B$ we are attempting to put in. (board)
Theorem (DM)

There is a computable nontrivial family such that every maximal subfamily with $\overline{D}_2 IP$ has hyperimmune degree.

(proof)[DDGT] We will define a computable family of the form

$$\{A^i_e : e \leq i\} \cup \{B_e : e \in \omega\}.$$  

We will call sets $A^i_e$ and $B_e$ with subscript $e$ “$e$-sets”. We will ensure the following hold.

- Every $A^i_e$ is nonempty.
- $B_e$ is nonempty iff $\phi_e(e) \downarrow$, and contains only numbers larger than the stage when $\phi_e(e)$ converges.
- If $i \neq e$, then every nonempty $e$-set intersects every nonempty $i$-set.
- For all $i, j \geq e$, $A^i_e$ intersects $A^j_e$.
- $A^i_e$ intersects $B_e$ iff $\phi_e(x) \downarrow$ for all $x \leq i + 1$. Moreover, the intersection only contains elements larger than the least stage $s$ such that $\phi_e(x) \downarrow [s]$ for all $x \leq i + 1$. 

We can assume the nonempty sets also code their indices, so that for every subfamily $\mathcal{C} = \{C_n \mid n \in \omega\}$ which does not contain the empty set, we can compute from $C_n$ which set $A^i_e$ or $B_e$ is equal to $C_n$.

Let $\mathcal{C}$ be a maximal subfamily with $\overline{D}_2$IP, and let $\mathcal{C}_s$ denote $\{C_n \mid n \leq s\}$. Since $\mathcal{C}$ does not contain the empty set, for each $e$, if $B_e \notin \mathcal{C}$, then $A^i_e \in \mathcal{C}$ for every $i \geq e$, since $A^i_e$ intersects every nonempty set in our family, except perhaps $B_e$.

Now if $\phi_e(x)$ is total, then $B_e$ must be in the family. From the family, we can compute the least number $q$ with $q \in B_e \cap A^x_i$ for $x \geq e$, and this will exceed $\phi_e(x)$. We need to make the function essentially coding this total whether or not $\phi_e$ is total.
Let $g$ be defined by

$$
g(x) = (\mu s) \forall e \leq i \leq x \ A^i_e \in C_s \lor B_e \in C_s.
$$

Let $f$ be defined by

$$
f(x) = (\mu n) \forall i, j \leq g(x) \ C_i \cap C_j \cap [0, n] \neq \emptyset.
$$

Observe that $f \leq_T C$.

We will show $f$ is not majorized by any computable function. Suppose $\phi_e$ is total. Then every $e$-set intersects every nonempty set in the family we built, so the maximal subfamily $C$ must contain $B_e$ and every $A^i_e$. Let $x \geq e$ be minimal such that $A^x_e$ appears after $B_e$ in $C$. We claim $f(x) > \phi_e(x)$. Notice $g(x)$ bounds the position that $B_e$ appears. If $x = e$, then $B_e \cap [0, f(x)]$ is nonempty and therefore $f(x) > \phi_e(e)$. If $x > e$, then $g(x)$ also bounds the position $A^{x-1}_e$ appears, and therefore $B_e \cap A^{x-1}_e \cap [0, f(x)]$ is nonempty. Thus $f(x) > \phi_e(x)$.
Theorem (DDGT)

If \( a \) bounds a 1-generic then \( a \) is FIP.

The main idea: Think about the proof that if \( a \) is c.e. then it is FIP. If we want to add some \( B \) to \( A_0, A_1, \ldots \), then we put up a permitting challenge to \( aa \) and if permission occurs slot \( B \) in, and truncate the family. If we need to add some \( B \) in then it will be dense in the construction so a permission occurs. For a 1-generic construction, for finite partial families, we will see such \( B \) occur and challenge generics to include \( B \) by the enumeration of a c.e. set of strings (thinking of sequences as strings, and the family as coding the generic). If this is dense then the generic will meet the condition.
In more detail:
Suppose that $X$ is 1-generic. Let $\{A_n : n \in \omega\}$ be a nontrivial family of sets. Without loss of generality, we may assume $A_0 \neq \emptyset$. Given $f : \omega \to \omega$, we define a function $g$ recursively as follows:

- $g(0) = 0$
- Suppose we have defined $g \upharpoonright n$. To define $g(n + 1)$, look for the least $m \leq n + 1$ different from $g(0) \ldots g(n)$ such that $A_m \cap \bigcap_{x \leq n} A_{g(x)}$ contains a number smaller than $f(n + 1)$. If there is such an $m$, define $g(n + 1) = m$. Otherwise, define $g(n + 1) = 0$.

This defines a functional $\Psi : \omega^\omega \to \omega^\omega$. We define $\Psi^\sigma$ for $\sigma \in \omega^{<\omega}$ in the usual way, noting that $|\Psi^\sigma| = |\sigma|$.

In DDGT, we prove that if $X$ is 1-generic, and if $g = \Psi^{p_X}$, where $p_X$ is the principal function of $X$, then $\{A_{g(n)} : n \in \omega\}$ is a maximal subfamily of $\{A_n : n \in \omega\}$ with FIP.
By construction, for all \( N \), \( \bigcap_{n < N} A_g(n) \) is nonempty, as we only allow \( g \) to take a new value not already in its range when we see a witness to nonempty intersection. Thus the subfamily \( \{ A_g(n) : n \in \omega \} \) has FIP.

Suppose it is not a maximal subfamily with FIP, and let \( m \) be minimal such that \( m \) is not in the range of \( g \), but \( \{ A_m, A_g(n) : n \in \omega \} \) has FIP. Let

\[
W = \{ \sigma : \exists n \Psi^{p_\sigma}(n) = m \}
\]

where \( p_\sigma \) is the element of \( \omega^k \), where \( k \) is the number of 1s in \( \sigma \), such that \( p_\sigma(i) \) gives the position of the \( i \)th 1 in \( \sigma \). Then no initial segment of \( X \) can be in \( W \), since \( m \) is not in the range of \( g \). However, every initial segment of \( X \) can be extended to an element of \( W \). Let \( \sigma \) be an initial segment of \( X \) such that the range of \( \Psi^{p_\sigma} \) contains every number less than \( m \) in the range of \( g \), and for every number \( i \) less than \( m \) not in the range of \( g \), the range of \( \Psi^{p_\sigma} \) contains some \( j_1 \ldots j_k \) such that

\[
A_i \cap A_{j_1} \cap \ldots \cap A_{j_k} = \emptyset.
\]

Such a \( \sigma \) exists by the minimality of \( m \).
Now, for any initial segment $\tau$ of $X$ extending $\sigma$,

$$A_m \cap \bigcap_{n < |p_\tau|} A_{\psi_{p_\tau}(n)} \neq \emptyset.$$ 

Therefore, extending $\tau$ by sufficiently many 0s followed by a 1 (such that the number of 0s bounds some element of this intersection) gives a string in $W$. This contradicts the 1-genericity of $X$. 
The $\Delta^0_2$ Case

**Theorem (DDGT)**

If $X$ is $\Delta^0_2$ and of FIP degree, then $X$ computes a 1-generic.

The theorem is aided by the fact that there is a universal family.

**Theorem (DDGT)**

There is a computable instance of FIP named $\mathcal{U}$ which is universal in the sense that any maximal solution for $\mathcal{U}$ computes a maximal solution for every other computable instance of FIP. Further, this reduction is uniform—if $A$ is a computable instance of FIP, then from an index for $A$, one can effectively obtain an index for a reduction that computes a maximal solution for $A$ from a maximal solution for $\mathcal{U}$. Thus FIP for $\mathcal{U}$ is Medvedev-above all other computable FIPs.

The idea for the proof is “intersect a lot, in a recoverable way.”
The $\Delta^0_2$ case

- Given $Q$ of FIP degree, we build 1-generic $G \leq_T Q$, and a family. (NB nonuniformity or use the recursion theorem)
- At some stage have $X_0, X_1, \ldots$ and $G \leq_T Q[s]$.
- Want to make $G$ meet $V_e$, say. Use a auxiliary set $B = B_e$.
- Make it meet, say, $X_0, \ldots, X_e$ (but not the rest) (A permitting challenge). Repeat with $X_{e+1}$ etc.
- If at some stage we get permission, then want to have, say, $X_0, \ldots, X_j, B_e$ want to block this from going back (For the principle all families representing the same collections of sets should give the same 1-generic) using bocker $Z_{e,j}$
The general case

Theorem (DDGT)

*There is a minimal FIP $a$ in $\Delta_3^0$.*

The proof is a tricky full approximation argument.
- Image we have so far $A_0, A_1, \ldots, A_n$ and wish to slot in $B_2$, postion determined by index and “state”.
- Presumably we have enumerated some description of $\Phi^{\langle A_0, A_1, \ldots, A_n \rangle}(j)$ for $j \leq p$.
- We can move $A_0 A_1 B \ldots$ for one step seeking agreeing computations.
- Then we can go back. If $B$ stops intersecting, then who cares? If $B$ intersects more, repeat.
- If we get a split we can change state.
- A split must generate equivalent families. $A_0 A_1 B \ldots A_j$ and $A_0 A_1 \ldots A_j B$ and this forces lots of pain when interactions are considered.
Notably, priorities ensure that you need to force many splits before you believe “split”, as places for entry of high priority sets.

These are “left hanging” which is why the trees are partial.

That is, we might have $A_0A_1B \ldots A_j$ and $A_0A_1\ldots A_jB$, being the place where we promise we would introduce $C$, but this intersection might never occur, so we force another split (at least).

Matters can be arranged to make sure that the first splits split with the second, arguing about uses.

Then we would work on the second split unless we need to introduce $C$.

Interactions are intricate.
Finite variations

- Do the same but use only families of finite sets.
- Computably true if given as either canonical finite sets, or with a bound on the number.
- FIP is computably true (look at the big intersection)
- If only finite and weak indices:

**Theorem (DDGT)**

$\overline{D}_2 IP_{finite}$ and $\Delta^0_2$ iff it bounds a 1-generic.

- The proof is similar but uses more initialization and priority.
Thank You