The Finite Intersection Property and Computability Theory

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FIP

- One equivalent of the axiom of choice
- A family of sets $\mathcal{F} = \{A_i \mid \in Q\}$ has finite intersection property iff for all finite $F \subset Q$, $\cap_{i \in F} A_i \neq \emptyset$.
- The principal says: Any collection of sets has a maximal subfamily with FIP.
- We investigate the computability of this.
- First began by Dzharfarov and Mummert.

- The first thing to notice is that it depends on whether you consider the family as set or a sequence
- If as a set then $\emptyset'$ is easily codable into a sequence and the theorem is equivalent to ACA_0. (Namely, have a set $B = B_e$ such that it is initially empty, and if $e \in \emptyset'[s]$ henceforth intersect it with everything, so it must be included. $\emptyset'$ can clearly figure things out.)
- Interesting if a sequence, so that A_1, A_2, A_3 is different from A_2, A_3, A_1.
- Similarly $\bar{D}_2$IP for for all pairs $A_i \cap A_j \neq \emptyset$. (DM notation)

Definition

Say that **a** is FIP iff for all computable collections of sets, **a** can compute a solut to the FIP problem.

A Basic Result

Theorem (Dzharfarov and Mummert)

There is a computable collection of sets with no c.e. subfamily with FIP. So $\mathbf{0}$ is not FIP, or even $\bar{D}_2$IP.

1. Meet $R_e : W_e$ is not an index for a maximal FIP family.
2. In the below I will use $A_i, \ldots$ and X_e, B_i etc. Of course these are all the same and really are $W_{f(j)}$ for a computable f given by the s-m-n theorem, and I am really concerned with the index $f(i)$. Also we will ensure that each nonzero set has a unique idetifier in it, so these are really streams of numbers under consiferderation.
3. Use a trap set X_e.
4. Begin with $A_0, A_1, \ldots$. Wait for W_e to respond.
5. Start intersecting X_e "in the back" . If W_e enumerates it win with finite injury.

Theorem (Dzharfarov and Mummert)

If $\mathbf{a}$ is $\bar{D}_2 IP$ then it is hyperimmune. (i.e. not computably dominated for those under 35)

Theorem (Dzharfarov and Mummert)

If $\mathbf{a} \neq \mathbf{0}$ is c.e. then $\mathbf{a}$ is FIP.

Theorem (Dzharfarov and Mummert)

If $\mathbf{a}$ is $\emptyset'$-hyperimmune then it is FIP.

- The c.e. noncomputable case below $C \neq_T \emptyset$.
- We are building $A_0, A_1, \dots A_n$.
- We want to put some element B into this family (with truncation), as we have seen B intersect $A_0, \dots, A_j$, the first position determined by B's index.
- We then place a permitting challenge to C. If later we see C permit j, we change the family to $A_0, \dots A_j, B$.
- When B meets $A_{j+1}[s]$ place another challange on B.
- The $\emptyset'$-hyperimmune is because $\emptyset'$ knows if we ever want to put things in, and infinitely often the C can decode this.
- It might seem that the c.e. case would also work for Δ^0_2 C, but it fails for a nonuniform reason.
- An earlier promise for a C-configuration might force some D_1 into the sequence which might be disjoint from the B we are attempting to put in. (board)

Theorem (DM)

There is a computable nontrivial family such that every maximal subfamily with $\bar{D}_2IP$ has hyperimmune degree.

(proof)[DDGT] We will define a computable family of the form

$$\{A_e^i : e \leq i\} \cup \{B_e : e \in \omega\}.$$

We will call sets A_e^i and B_e with subscript e "e-sets". We will ensure the following hold.

- Every A_e^i is nonempty.
- B_e is nonempty iff $\phi_e(e)\downarrow$, and contains only numbers larger than the stage when $\phi_e(e)$ converges.
- If $i \neq e$, then every nonempty e-set intersects every nonempty i-set.
- For all $i, j \geq e$, A_e^i intersects A_e^j.
- A_e^i intersects B_e iff $\phi_e(x)\downarrow$ for all $x \leq i+1$. Moreover, the intersection only contains elements larger than the least stage s such that $\phi_e(x)\downarrow[s]$ for all $x \leq i+1$.

We can assume the nonempty sets also code their indices, so that for every subfamily $\mathcal{C} = \{C_n \mid n \in \omega\}$ which does not contain the empty set, we can compute from C_n which set A^i_e or B_e is equal to C_n.
Let $\mathcal{C}$ be a maximal subfamily with $\bar{D}_2$IP, and let $\mathcal{C}_s$ denote $\{C_n \mid n \leq s\}$. Since $\mathcal{C}$ does not contain the empty set, for each e, if $B_e \notin \mathcal{C}$, then $A^i_e \in \mathcal{C}$ for every $i \geq e$, since A^i_e intersects every nonempty set in our family, except perhaps B_e.
Now if $\phi_e(x)$ is total, then B_e must be in the family. From the family, we can compute the least number q with $q \in B_e \cap A^x_i$ for $x \geq e$, and this will exceed $\phi_e(x)$. We need to make the function essentially coding this total whether or not ϕ_e is total.

Let g be defined by

$$g(x) = (\mu s)\forall e \leq i \leq x\ A_e^i \in \mathcal{C}_s \vee B_e \in \mathcal{C}_s.$$

Let f be defined by

$$f(x) = (\mu n)\forall i, j \leq g(x)\ C_i \cap C_j \cap [0, n] \neq \emptyset.$$

Observe that $f \leq_T \mathcal{C}$.
We will show f is not majorized by any computable function. Suppose ϕ_e is total. Then every e-set intersects every nonempty set in the family we built, so the maximal subfamily $\mathcal{C}$ must contain B_e and every A_e^i. Let $x \geq e$ be minimal such that A_e^x appears after B_e in $\mathcal{C}$. We claim $f(x) > \phi_e(x)$. Notice $g(x)$ bounds the position that B_e appears. If $x = e$, then $B_e \cap [0, f(x)]$ is nonempty and therefore $f(x) > \phi_e(e)$. If $x > e$, then $g(x)$ also bounds the position A_e^{x-1} appears, and therefore $B_e \cap A_e^{x-1} \cap [0, f(x)]$ is nonempty. Thus $f(x) > \phi_e(x)$.

1-Generices, again

Theorem (DDGT)

If **a** *bounds a 1-generic then* **a** *is FIP.*

The main idea: Think about the proof that if **a** is c.e. then it is FIP. If we want to add some B to $A_0, A_1, \ldots$, then we put up a permitting challenge to **a**a and if permission occurs slot B in, and truncate the family.
If we need to add some B in then it will be dense in the construction so a permission occurs. For a 1-generic construction, for finite partial families, we will see such B occur and challenge generics to include B by the enumeration of a c.e. set of strings (thinking of sequences as strings, and the family as coding the generic). If this is dense then the generic will meet the condition.

In more detail:
Suppose that X is 1-generic. Let $\{A_n : n \in \omega\}$ be a nontrivial family of sets. Without loss of generality, we may assume $A_0 \neq \emptyset$. Given $f : \omega \to \omega$, we define a function g recursively as follows:

- $g(0) = 0$
- Suppose we have defined $g \upharpoonright n$. To define $g(n+1)$, look for the least $m \leq n+1$ different from $g(0) \ldots g(n)$ such that $A_m \cap \bigcap_{x \leq n} A_{g(x)}$ contains a number smaller than $f(n+1)$. If there is such an m, define $g(n+1) = m$. Otherwise, define $g(n+1) = 0$.

This defines a functional $\Psi : \omega^\omega \to \omega^\omega$. We define Ψ^σ for $\sigma \in \omega^{<\omega}$ in the usual way, noting that $|\Psi^\sigma| = |\sigma|$
In DDGT, we prove that if X is 1-generic, and if $g = \Psi^{p_X}$, where p_X is the principal function of X, then $\{A_{g(n)} : n \in \omega\}$ is a maximal subfamily of $\{A_n : n \in \omega\}$ with FIP.

By construction, for all N, $\bigcap_{n<N} A_{g(n)}$ is nonempty, as we only allow g to take a new value not already in its range when we see a witness to nonempty intersection. Thus the subfamily $\{A_{g(n)} : n \in \omega\}$ has FIP. Suppose it is not a maximal subfamily with FIP, and let m be minimal such that m is not in the range of g, but $\{A_m, A_{g(n)} : n \in \omega\}$ has FIP. Let

$$W = \{\sigma : \exists n \, \Psi^{p_\sigma}(n) = m\}$$

where p_σ is the element of ω^k, where k is the number of 1s in σ, such that $p_\sigma(i)$ gives the position of the ith 1 in σ. Then no initial segment of X can be in W, since m is not in the range of g. However, every initial segment of X can be extended to an element of W. Let σ be an initial segment of X such that the range of Ψ^{p_σ} contains every number less than m in the range of g, and for every number i less than m not in the range of g, the range of Ψ^{p_σ} contains some $j_1 \ldots j_k$ such that

$$A_i \cap A_{j_1} \cap \ldots \cap A_{j_k} = \emptyset.$$

Such a σ exists by the minimality of m.

Now, for any initial segment τ of X extending σ,

$$A_m \cap \bigcap_{n<|p_\tau|} A_{\Psi^{p_\tau}(n)} \neq \emptyset.$$

Therefore, extending τ by sufficiently many 0s followed by a 1 (such that the number of 0s bounds some element of this intersection) gives a string in W. This contradicts the 1-genericity of X.

The Δ^0_2 Case

Theorem (DDGT)

If X is Δ^0_2 and of FIP degree, then X computes a 1-generic.

The theorem is aided by the fact that there is a universal family.

Theorem (DDGT)

There is a computable instance of FIP named $\mathcal{U}$ which is universal in the sense that any maximal solution for $\mathcal{U}$ computes a maximal solution for every other computable instance of FIP. Further, this reduction is uniform—if $\mathcal{A}$ is a computable instance of FIP, then from an index for $\mathcal{A}$, one can effectively obtain an index for a reduction that computes a maximal solution for $\mathcal{A}$ from a maximal solution for $\mathcal{U}$. Thus FIP for $\mathcal{U}$ is Medvedev-above all other computable FIPs.

The idea for the proof is "intersect a lot, in a recoverable way."

The Δ^0_2 case

- Given Q of FIP degree, we build 1-generic $G \leq_T Q$, and a family. (NB nonuniformity or use the recursion theorem)
- At some stage have $X_0, X_1, \ldots$ and $G \leq_T Q[s]$.
- Want to make G meet V_e, say. Use a auxilairy set $B = B_e$.
- Make it meet, say, $X_0, \ldots, X_e$ (but not the rest) (A permitting challenge). Repeat with X_{e+1} etc.
- If at some stage we get permission, then want to have, say, $X_0, \ldots, X_j, B_e$ want to block this from going back (For the principle all families representing the same collections of sets should give the same 1-generic) using bocker $Z_{e,j}$

The general case

Theorem (DDGT)

There is a minimal FIP $\mathbf{a}$ *in* Δ^0_3*.*

The proof is a tricky full approximation argument.

- Image we have so far $A_0, A_1, \ldots, A_n$ and wish to slot in B_2, postion determined by index and "state".
- Presumably we have enumerated some description of $\Phi^{\langle A_0, A_1, \ldots, A_n \rangle}(j)$ for $j \leq p$.
- We can move $A_0 A_1 B \ldots$ for one step seeking agreeing computations.
- Then we can go back. If B stops intersecting, then who cares? If B intersects more, repeat.
- If we get a split we can change state.
- A split must generate equivalent families. $A_0 A_1 B \ldots A_j$ and $A_0 A_1 \ldots A_j B$ and this forces lots of pain when interactions are considered.

- Notably, priorities ensure that you need to force many splits before you believe "split", as places for entry of high priority sets.
- These are "left hanging" which is why the trees are partial.
- That is, we might have $A_0 A_1 B \ldots A_j$ and $A_0 A_1 \ldots A_j B$, being the place where we promise we would introduce C, but this intersection might never occur, so we force another split (at least).
- Matters can be arranged to make sure that the first splits split with the second, arguing about uses.
- Then we would work on the second split unless we need to introduce C.
- Interactions are intricate.

Finite variations

- Do the same but use only families of finite sets.
- Computably true if given as either canonical finite sets, or with a bound on the number.
- FIP is computably true (look at the big intersection)
- If only finite and weak indices:

Theorem (DDGT)

$\overline{D}_2 IP_{finite}$ and Δ_2^0 iff it bounds a 1-generic.

- The proof is similar but uses more initialization and priority.

Thank You