Algorithmic Randomness and Complexity

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Rod dedicates this book to his wife Kristin, and Denis to Larisa.

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