Colouring as a Mathematical Miniature

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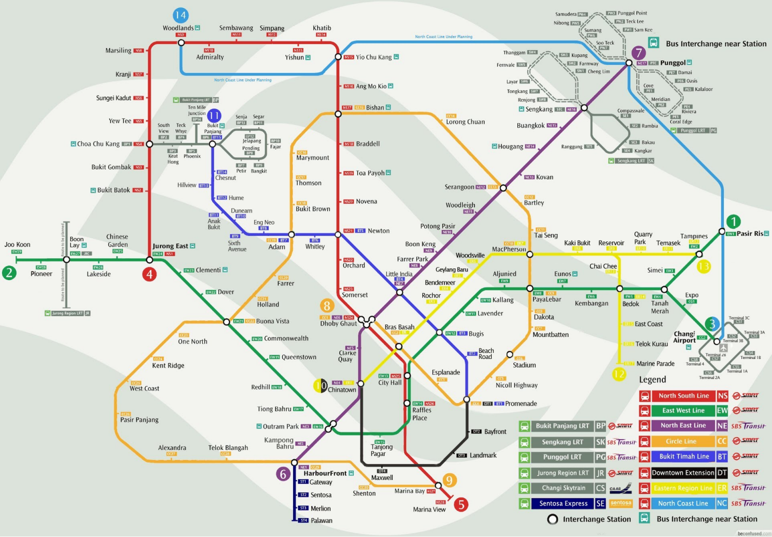
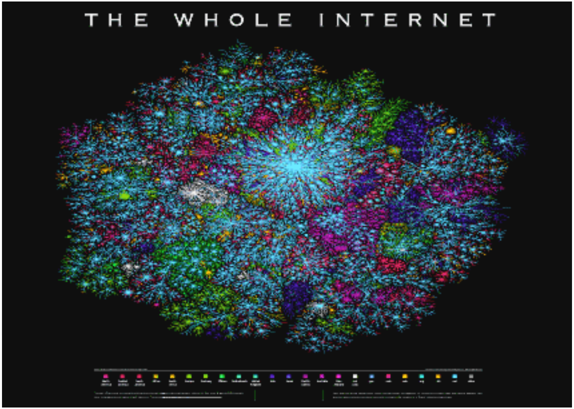
Graphs

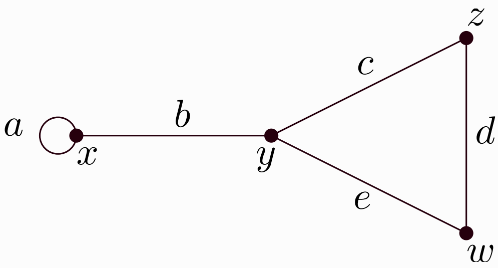
- Graphs are mathematical abstractions of many objects.
- They consist of a bunch of dots called vertices and lines possibly between them called edges.
- In this lecture we will colour them, and see why this is of interest.

Some examples




THE WHOLE INTERNET

Another



The set of vertices is $\{x, y, z, w\}$, and the set of edges is $\{a, b, c, d, e\}$. Thus, for example, b joins x to y. Note that the edge a joins x to itself. Any such edge is called a loop. Our graphs are simple and have no loops If loops or multiple edges are allowed then it is called a network.

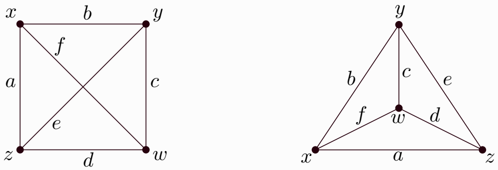


Figure: Two drawings of the same graph.

We regard these as the same, in math talk, they are isomorphic.
Terminology The number of edges from a vertex is called the degree of the vertex. The degree of the vertices above is 3. In the graph before there are some 2's and some 3's.
Finally, the graph is connected if I can get from any vertex to any other vertex along edges.

Euler

Graph theory began with the genius Leonard Euler.

Königsberg Bridge Problem

Can I travel over all the bridges exactly once?

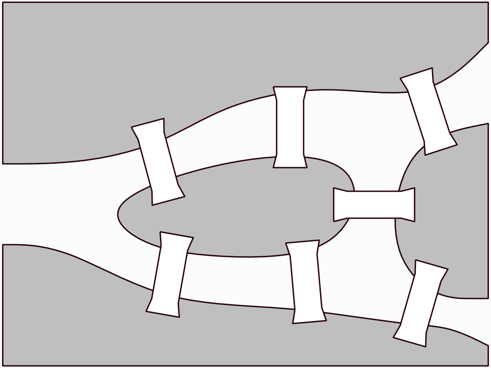


Figure: Königsberg Bridges.

Euler's Analysis

Euler realised that the route taken inside each landmass is completely irrelevant to the problem. So we may as well replace each of the four landmasses with a single vertex, and represent each bridge as an edge joining a pair of landmasses.

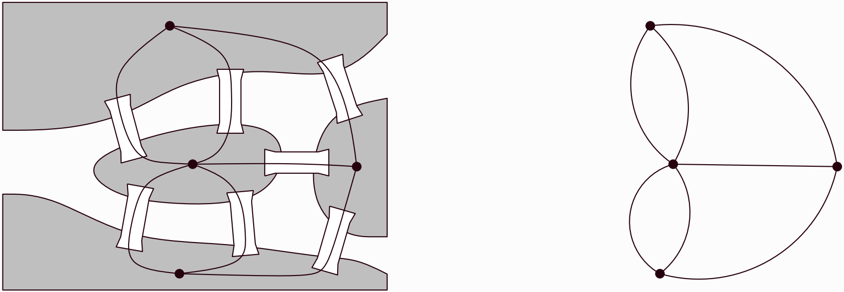


Figure: Representing Königsberg as a network.

Euler's Theorem

A path through all the edges exactly once returning to where you start is called an Euler Cycle.

Theorem (Euler)

A connected network has an Euler Cycle if and only if all the verteices have even degree.

Notice that this means that there is a very simple test to see if there is such a cycle. A quick algorithm.

Hamilton Cycle

What about the apparently similar problem of a traversal of the graph through every vertex exactly once? These are called Hamilton Cycles. Hamiltonian cycles are named after William Rowan Hamilton (1805–1865). He proposed (and sold!) a board game which involved finding such cycles in the graph (which is called the *dodecahedron graph*). The edges drawn with bold lines show a Hamiltonian cycle. (You may not be surprised to hear that the game was a commercial failure.)

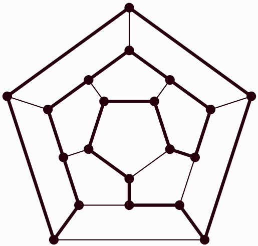


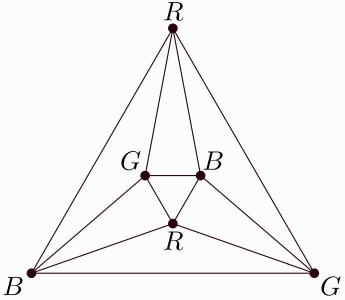
Figure: A Hamiltonian cycle in the dodecahedron graph.

Finding Hamilton Cycles

- Solve this efficiently and you can earn \$1,000,000.
- Destroy all cyber security.
- Revolutionize life on earth.
- Read "The Golden Ticket." (a populatization of the P vs NP problem, Princeton University Press.)

Colouring

A colouring of a graph (sometimes called a proper colouring) is an assignment of colours to the vertices in such a way that adjacent vertices never receive the same colour.



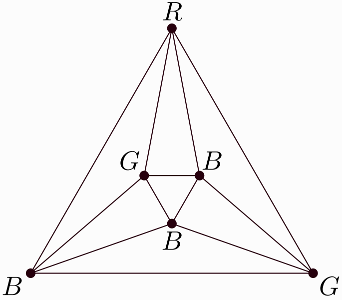


Figure: A proper colouring, and an improper colouring.

Definition

If a graph has a proper colouring that uses at most k colours, then we say

Planar Graphs and Maps

G is called planar if we can draw it on the plane with no edges crossing.
Maps can be converted into planar graphs.

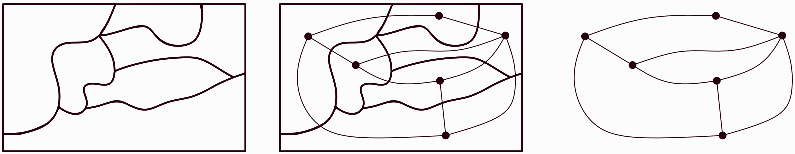


Figure: Translating a map into a planar graph.

A colouring of the map where no bordering countries have the same colour corresponds to a colouring of the vertices where no two adjacent vertices have the same colour.

The 4-Colour Theorem

The Four Colour Problem was proposed in 1852. It was brought to the attention of mathematicians by Augustus de Morgan (1806–1871). A proof was given by Alfred Kempe in 1879, and Tait in 1880 but eleven years later Percy Heawood showed that Kempe's proof had a mistake in it. The next year Peterson showed that Tait's proof was wrong.
It wasn't until 1976 that Kenneth Appel and Wolfgang Haken proved the result:

Theorem (Four Colour Theorem)

Every simple planar graph is 4-colourable.

5-Colour Theorem

We find it too difficult to prove this here, but the next result give the spirit.

Theorem (Heawood-1890's)

Every simple planar graph is 5-colourable.

We need a technical Lemma

Lemma

Let G be a simple connected planar graph. Then G has a vertex with degree at most five.

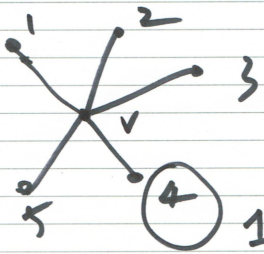
This is called an Unavoidable Configuration

I begin with the smallest planar graph that is not 5-colourable.

If I remove a vertex v of degree ≤ 5 I can 5-colour the rest.

So do this and 5-colour the smaller graph.

Can I put v back in?

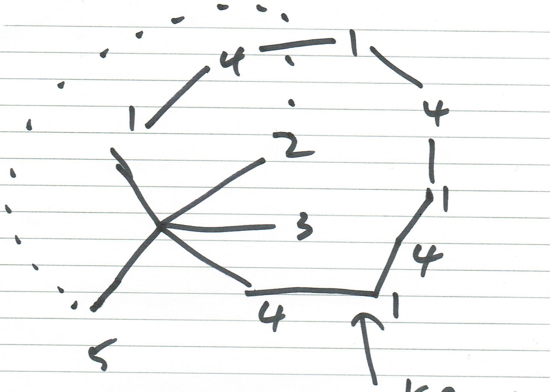


1?

Can I recolor

as $1 \leftrightarrow 4$

Smallest counterexample



Kempp chain

If NO !

The Appel-Haken Proof

Extremely controversial at the time.

Used similar elements. Unavoidable configurations, and a Kempp chain like process called discharging.

They wrote a program to automate parts of this. If the program got stuck it would modify the set of configurations.

Used several hundred hours on a super computer!

First proof that no person could check.

Subsequently checked by several other independent formal proof systems.

Specifically

- Prove that G contains at least one of 1476 unavoidable configurations. (To do this, assign each vertex a charge. Let the electrons flow around G (according to 487 discharging rules). If a vertex still has electrons that it cannot discharge, the reason must be that there is one of those 1476 configurations nearby.)
- Prove that each one of those 1476 unavoidable configurations is reducible it can be replaced with something smaller without affecting the chromatic number of G. (This part of the proof was carried out by a computer.) What they did was also modify the unavoidable configurations by hand each time it appeared the computer was stuck. There was no *a priori* reason this method would work in finite time!

Controvery

Haken and Appel:

> *"This leaves the reader to face 50 pages containing text and diagrams, 85 pages filled with almost 2500 additional diagrams, and 400 microfiche pages that contain further diagrams and thousands of individual verifications of claims made in the 24 lemmas in the main sections of text. In addition, the reader is told that certain facts have been verified with the use of about twelve hundred hours of computer time and would be extremely time-consuming to verify by hand. The papers are somewhat intimidating due to their style and length and few mathematicians have read them in any detail."*

Subsequent

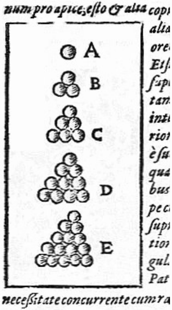
- Changed the idea of a proof.
- Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas [RSST] (19951997) gave an improved proof, using the same approach as Appel and Haken, but with 633 unavoidable configurations and 32 discharging rules.
- Georges Gonthier (2005) shows how to translate the whole proof into logic and have a computer formally verify the proof.

Formal Proof

There are several other results proven by formal proof.

Famously the Kepler Conjecture

Roughly the most effecient way to pack spheres is the way we do it.



From Wikipedia:

“In 1998 Thomas Hales, following an approach suggested by Fejes Tóth (1953), announced that he had a proof of the Kepler conjecture. Hales' proof is a proof by exhaustion involving the checking of many individual cases using complex computer calculations. Referees have said that they are "99% certain" of the correctness of Hales' proof, and now Kepler conjecture is accepted as a theorem. In 2014, the Flyspeck project team, headed by Hales, announced the completion of a formal proof of the Kepler conjecture using a combination of the Isabelle and HOL Light proof assistants.”

How Hard

2-colouring is easy
4-Colouring is easy ; Just say “yes’.

Theorem (Karp, 1972)

3-Colouring of planar graphs is computationally the same as finding a Hamilton cycle.

What about with help? You are colouring with a child and you colour a vertex and then the child, etc. Will they “help”.

Theorem (Kierstead, 1990)

33 colours suffice for colouring planar graphs with an uncooperative partner.

What else?

Clearly lots and lots of work on graphs.
For example, a graph is planar if it can be put on the sphere. The genus of a “manifold” is the number of handles it has on the sphere. A sphere has genus 0. Torus (doughnut) has genus 1. (Incidentally, Heawood proved that 7 colours suffice for a torus.)

The following has genus 2.

The following cannot be put on the plane but can be put on the torus.

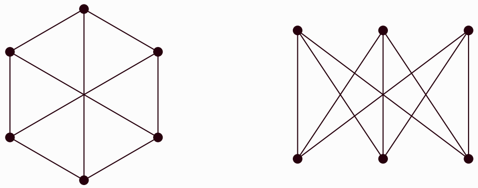


Figure: Two drawings of $K_{3,3}$.

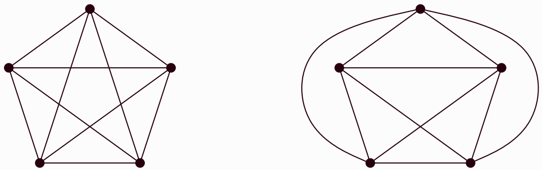


Figure: Two non-planar drawings of K_5.

Kuratowski and Friends

Theorem (Kuratowski's Theorem)

A graph is planar if and only if it does not have a "minor isomorphic to" K_5 or $K_{3,3}$.

Roughly speaking this means that G is non-planar if we can find a copy of one of these two graphs inside it. The pair K_5, $K_{3,3}$ is called an obstruction set for planarity. Testing for H a minor of G for fixed H is theoretically fast so this gives a fast algorithm.
BTW "theoretical" is very bad. Like $2^{2^{2^{\cdots}}}$ (about $100|H|$ high) is the constant and $|G|^3$. Challenges the idea that polynomial time=efficient

Robertson-Seymour Theorem

A wonderful recent theorem of Robertson and Seymour (Graph Minors 1-23, about 2,000 journal pages) proves as a consequence

Theorem (Robertson and Seymour, early 21st century)

For any genus g there is a finite obstruction set, and hence fast testing.

Remarkably, *we have no idea what the obstruction set are* (even for the torus).

Thus we have a method of proving there is an "efficient algorithm" with no idea what it is!

"This is not mathematics this is theology" (Kronecker, about a proof of Hilbert in the early 20th century.)

"This is not computer science, this is mathematical curiosity" (Dave Johnson, 1990's)

Thank You