When does a problem have a solution: A logician and computability-theorist's view

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Thanks

- Supported by the Marsden Fund of New Zealand.
- The New Zealand Institute of Mathematics and its Applications.

- From a set of basic instructions (ingredients) specify a mechanical method to obtain the desired result.
- Already you can see that I plan to be sloppy, but you should try to get the feel of the subject.
- I will try to have a general overview but will talk about some of my own work. Not to say that my work is the most important, but that I actually know something about it!

No. 1 – THE REVEREND JOHN MACFARLANE

(Reel)

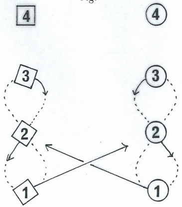
MUSIC *Bars*	DESCRIPTION
1– 8	1st woman dances a reel of three on the men's side with 2nd and 3rd men, while 1st man dances a reel of three on the women's side with 2nd and 3rd women. (Fig.) 1st couple finish in partner's place.
9–12	1st couple dance a half figure of eight round 2nd couple.
13–16	1st couple, joining both hands, dance four slip steps down the middle to third place and then set with hands still joined. (1st man sets to the left and then to the right.) 2nd and 3rd couples step up and 4th couple step in to meet on bars 15–16.
17–24	1st and 4th couples poussette.
25–28	2nd couple with 3rd couple and 4th couple with 1st couple dance right hands across once round to places.
29–32	2nd, 3rd, 4th and 1st couples turn partners once round with the left hand. Repeat with a new top couple.

This dance commemorates the 150th anniversary of the founding in Wellington of New Zealand's first Scots Church, later known as St. Andrews.

The Rev. John Macfarlane, the first minister of Martyr's Memorial Church, Paisley, arrived in New Zealand on 20th February, 1840 and he held the first service on the beach at Petone on Sunday 23rd February.

Devised by Gary W. Morris (New Zealand Branch).

Fig.



Top

ice when it reaches the mushy stage and every 30 minutes after that until it is ready to serve, to insure smoothness. Garnish with pitted black cherries.

CREAM FRITTERS

READY TRAY | Serves 4 to 6

4 egg yolks
¼ cup sugar
½ cup flour
Salt to taste
4 cups milk, scalded
1 teaspoon grated orange or lemon rind
1 egg, beaten
Breadcrumbs
2 tablespoons oil
2 tablespoons butter
Powdered sugar
2 tablespoons brandy or rum

Beat egg yolks and sugar in top of double boiler. Cook over low heat, stirring with wooden spoon until slightly thickened. Mix in ¼ cup flour, salt and gradually add milk. Simmer, stirring, until very thick. At no time allow to boil. Blend in rind.

Rinse a square dish or pan with cold water and pour in mixture to a depth of 2 inches. Chill until firm. Cut into squares or rectangular pieces 2 inches long. Dip in remaining flour, in egg and then in breadcrumbs. Brown gently on both sides in hot oil and butter. Serve sprinkled with sugar, and flame with heated brandy or rum.

FRIED RICOTTA

READY TRAY | Serves 8

½ pound macaroons
1 pound ricotta cheese
Pinch cinnamon
3 eggs
Breadcrumbs
¼ pound butter
Powdered sugar
Brandy

ALGORITHMS

- From a set of basic instructions (ingredients) specify a mechanical method to obtain the desired result.

Greatest Common Divisors

- The greatest common divisor of two numbers x and y is the biggest number that is a factor of both.
- For instance, the greatest common divisor, gcd(4,8) is 4. gcd(6,10)=2; gcd(16,13)=1.
- Euclid, or perhaps Team Euclid, (around 300BC) devised what remains the “best” algorithm for determining the gcd of two numbers.

Euclid's Algorithm

- To find gcd(1001,357).
- $1001 = 357 \cdot 2 + 287$
- $357 = 287 \cdot 1 + 70$
- $287 = 70 \cdot 4 + 7$
- $70 = 7 \cdot 10$
- 7=gcd(1001,357).

- Ingredients: numbers, $+$, $-$, $\times$, division.
- Operations : Combine in sensible ways.

- At school one sees the following algorithm for solving the quadratic equation.

$$ax^2 + bx + c = 0.$$

- The solutions are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.
- Example $4x^2 - 5x + 1 = 0$ gives $\frac{5+\sqrt{5^2-4\cdot4\cdot1}}{2\cdot4}$ which equals 1; and $\frac{5-\sqrt{5^2-4\cdot4\cdot1}}{2\cdot4}$ which equals $\frac{1}{4}$

ANALYSIS

- Ingredients: numbers, $+, - \times$, division, $\sqrt{}$, maybe cube roots, powers etc.
- Operations : Combine in sensible ways.

A GOOD QUESTION

- Can we do the same for degree 3, the “cubic”

$$ax^3 + bx^2 + cx + d = 0?,$$

  what about degree 4, etc.
- This was one of the many questions handed to us by the Greeks.
- The answer is yes for degree 3 and degree 4.

The Sorry Tale

- For degree 3 this was first proven by Ferro (1500).
- Ferro left it to his son-in-law Nave and pupil Fiore.
- Fiore challenged Tartaglia (in 1535) who then re-discovered the solution with a few days to spare, leaving Foire in ignomy.

ɔlo Fontana (Tartaglia), who discovered how to solve cubic

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- Fiore challenged Tartaglia (in 1535) who then re-discovered the solution with a few days to spare, leaving Foire in ignomy.
- Tartaglia also kept it secret, but told Cardano, who promised by his Christian faith to keep it secret, but....
- in 1545 Cardano published it in his great text Ars Magna
- Additionally Cardano published how to extend to degree 4, being discovered by a student Ferrari, (of whom the car company is surely named).


HIERONYMI CARDANI, PRÆSTANTISSIMI MATHEMATICI, PHILOSOPHI, AC MEDICI,

ARTIS MAGNÆ,

SIVE DE REGVLIS ALGEBRAICIS,

Lib. unus. Qui & totius operis de Arithmetica, quod OPVS PERFECTVM inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aût librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

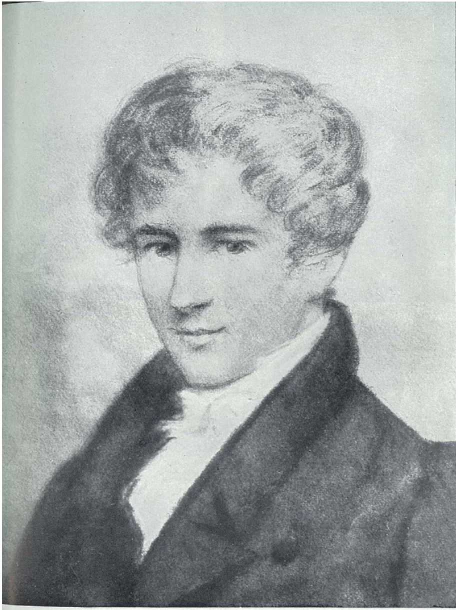
Figure 4: Title page of Cardano's *Ars Magna*.

square roots being chosen so that

$$\sqrt{-y_1} \times \sqrt{-y_2} \times \sqrt{-y_3} = -q.$$

22 Cardano, the first to publish solutions of cubic and quartic ations.

- Finally, in 1823, a young Norwegian mathematician, Abel proved that there is no recipe using the given ingredients for the degree 5 case, the quintic.



The only authentic portrait of Niels Henrik Abel, executed by the painter Görbitz in Paris in 1826

- Finally, in 1823, a young Norwegian mathematician, Abel proved that there is no recipe using the given ingredients for the degree 5 case, the quintic.
- (The paper was called “Memoir on algebraic purifications...” rather than “Memoir on algebraic equations...” due to a typsetting error.)
- (My favourite error in one of my own papers referred to a journal “Annals of Mathematical Logic” as “Animals of Mathematical Logic.”It made me think of some of my colleagues!)
- Nobody believed him, for a long time. (There had been an earlier announcement by Ruffini, which contained “gaps”.)

- Evariste Galois (1811-32) eventually gave a general methodology for deciding if a given degree *n* equation admits a solution with the ingredients described.

Portrait of Èvariste Galois age 15.

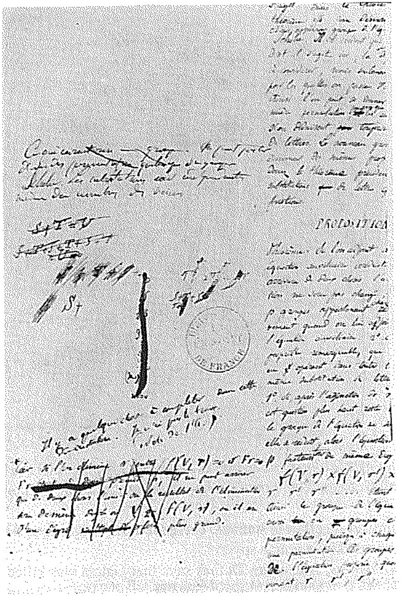


Figure 8: "I have no time" (*je n' ai pas le temps*) above deleted paragraph in lower left corner. But consider the context.

Galois

- Evariste Galois (1811-32) eventually gave a general methodology for deciding if a given degree n equation admits a solution with the ingredients described.
- This work laid the basis for group theory.
- Galois method is to associate a group with each equation, so that the equation is solvable in terms of the given ingredients (arithmetic operations and radicals) iff the group has a certain structure on its subgroups. This is one of the gems of mathematics.

Changing Ingredients

- It is not true to say that the quintic has no solution, just none with the given ingredients.
- We can add some new operations "elliptic functions" and show that there is a method of solving the general degree *n* equation.
- These operations are "mechanical" so there is a an algorithm for solving all such equations.

- David Hilbert, 1900, working from a background of 19th century determinism basically asked the question of whether mathematics could be finitely "mechanized".

David Hilbert, 1912 — one of a group of portraits of professors which were sold as postcards in Göttingen

The first three problems concerned the foundations of mathematics:

1. To prove Cantor's "continuum hypothesis" that any set of real numbers can be put into one-to-one correspondence either with the set of natural numbers or with the set of all real numbers (i.e., the continuum).
2. To investigate the consistency of the arithmetic axioms.
6. To axiomatize those physical sciences in which mathematics plays an important role.

"So far we have considered only questions concerning the foundations of the mathematical sciences. Indeed, the study of the foundations of a science is always particularly attractive, and the testing of these foundations will always be among the foremost problems of the investigator. Weierstrass said, 'The final objective always to be kept in mind is to arrive at a correct understanding of the foundations . . . . But to make any progress in the sciences the study of individual problems is, of course, indispensable.' In fact, a thorough understanding of its special theories is necessary to the successful treatment of the foundations of the science. Only that architect is in the position to lay a sure foundation for a structure who knows its purpose thoroughly and in detail."

The next four problems were selected from arithmetic and algebra:

7. To establish the transcendence, or at least the irrationality, of certain numbers.
8. To prove the correctness of an extremely important statement by Riemann that the zeros of the function known as the "zeta function" all have the real part $^1/_2$, except the well known negative integral real zeros.
13. To show the impossibility of the solution of the general equation of the 7th degree by means of functions of only two arguments.
16. To conduct a thorough investigation of the relative position of the separate branches which a plane algebraic curve of *n*th order can have when their number is the maximum . . . and the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space.

The last three problems came from the theory of functions:

19. To determine whether the solutions of "regular" problems in the calculus of variations are necessarily analytic.
21. To show that there always exists a linear differential equation of the Fuchsian class with given singular points and monodromic group.
22. To generalize a theorem proved by Poincaré to the effect that it is always possible to uniformize any algebraic relation between two variables by the use of automorphic functions of one variable.

"The problems mentioned," Hilbert told his audience, "are merely samples of problems; yet they are sufficient to show how rich, how manifold and how extensive the mathematical science is today; and the question is urged upon us whether mathematics is doomed to the fate of those other

HILBERT

- David Hilbert, 1900, working from a background of 19th century determinism basically asked the question of whether mathematics could be finitely "mechanized".
- Can we create an algorithm, a machine, into which one feeds a statement about mathematics or at least in a reasonable "formal system" and from the other end a decision emerges: true or false.
- Or, for a given formal system, can we produce a machine that would eventually emit all the "truths" of that system.
- Hilbert also proposed that we should prove the consistency of various formal systems of mathematics.

FORMAL SYSTEM

- It is not important what this is, save to say the type envisioned would be a bunch of axioms, saying things like
- for all numbers x, x+1 exists,
- for all numbers x and y $x + y = y + x$,
- and other “obvious truths.”
- plus rules of inference, like “if whenever P is true then Q is true, and whenever Q is true then R is true; then whenever P is true R is true.”
- induction.

Gödel

- Hilbert's dreams were forever shattered by a young mathematician, Kurt Gödel.

GÖDEL

- Hilbert's dreams were forever shattered by a young mathematician, Kurt Gödel.
- He prove the two incompleteness theorems.
- The first incompleteness theorem says that any sufficiently rich formal system has statements
- expressible in the system
- true of the system, but
- cannot be proven in the system.
- Secondly no sufficiently rich formal system can prove its own consistency.
- The collective intuition of a generation of mathematicians was wrong.

- Some rich systems are decidable by mechnaical methods.
- For example, Euclidian geometry. (Tarski, using quantifier elimination, and inventing model theory).

- Gödel's results had only weak penetration into the consciousnesses of so-called "working mathematicians", and far too much penetration into that of would-be philosophers and physicists.
- There are definitely now known "mathematical incompleteness of various systems"
- Here is one example: Kruskal's Theorem says that finite trees are well-partially ordered by topological embedding. (No infinite antichain)
- (Harvey Friedman) For all k there is an n so large that if $\{T_i : i \leq n\}$ are trees with $|T_i| < k \cdot i$ then for some $i < j$, T_i topologically embeds into T_j.
- This is statable in PA but not provable in any system that esentially does not prove the existence of uncountable sets. Friedman has examples equivalent to the existence of "Mahlo Cardinals"(i.e. their truth is equivalent to deciding what colour cheese the moon is made of)

- Around the same time, various people were working on formalizing what we might mean by “mechanical method.” (such as the above)

Kurt Gödel and Alfred Tarski, Princeton, New Jersey, 1962.

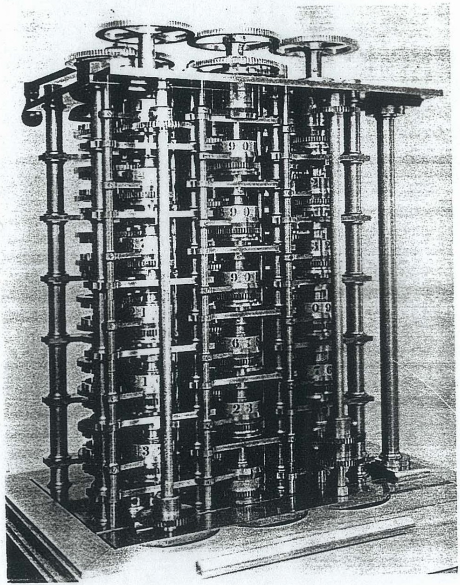


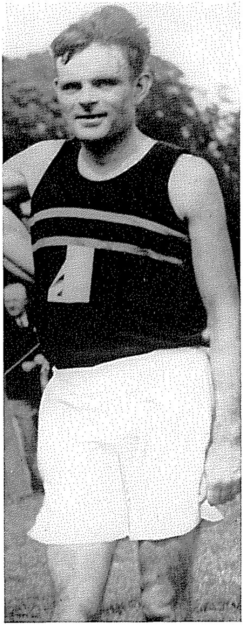
Plate 3. The analytical engine

CHURCH-TURING-KLEENE

- Around the same time, various people were working on formalizing what we might mean by "mechanical method." (such as the above)
- There were various models proposed, "lambda calculus" (Church) "partial recursive functions (Kleene)"
- The convincing model was that of Turing which are now called Turing Machines.
- Church's Thesis "All mechanically computable processes on the numbers can be simulated on a Turing Machine"

- Worked on Code cracking in the 2nd world war. (Enigma machine)

Aged 5



After a successful race. May, 1950

The Enigma Machine, employed by the Germans to encrypt classified and sensitive messages during World War II. (HultonArchive/Getty Images)

John von Neumann, Princeton, 1932

The von Neumanns starting the descent into the Grand Canyon on an excursion in the late 1940s: Klari, with visor, is fourth from front; Johnny, bareheaded and in city suit, is last, on the only mule facing the wrong way

- Worked on Code cracking in the 2nd world war. (Enigma machine)
- His fundamental paper, was part of the inspiration for the first computers, and strongly influenced John von Neumann.
- Much work e.g. on Colussus only recently declassified.
- Also the foundations of numerical analysis and ill-conditioning.

A Basic Undecidable Question

- Using the fact that all Turing machines can be enumerated we can use a beautiful argument of Cantor about differing sizes of infinite sets(!) to show that there is no algorithm to decide to following question.
- INPUT Turing machine number x and an input y.
  QUESTION Does the machine x halt on input y.

- (Proof. Suppose that we could decide this algorithmically. We can then use the decision procedure to construct a machine M that halts on input n if T_n does not halt on input n, and our machine M does *not* halt if machine T_n does halt on input n. Then M would be some machine T_m, but then $T_m(m)$ halts if and only if $M(m)$ halts iff $T_m(m)$ does not halt....)
- We code this problem into others.

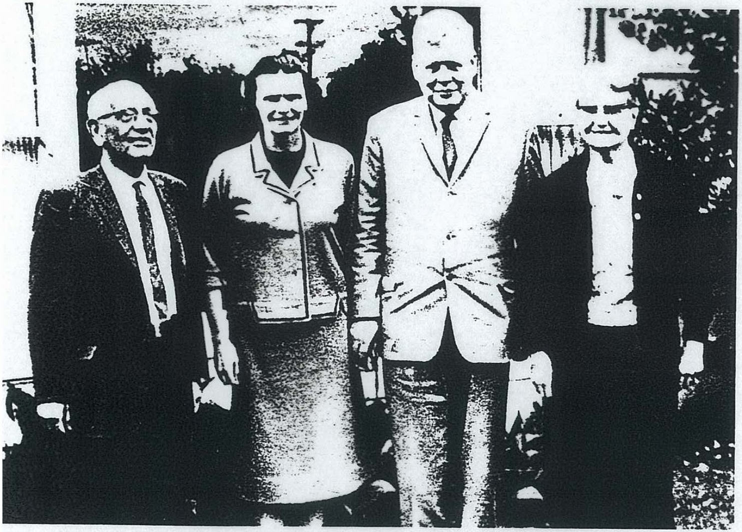
- Collatz-type functions. $f(x) = \frac{x}{2}$ if x is even, and $f(x) = 3x + 1$ if x odd.
- e.g. $f(3) = 10$ $f(f(3)) = 5$, get the sequence, 3,10,5,16,8,4,2,1
- Do you always get to 1? (Still open)
- General type of question : e.g $g(x) = 1/2x$ if x divisible by 4, $g(x) = 5x - 1$ if x has remainder 1 when divided by 4, etc.
- John Conway (1980's) showed that there is no general algorithm to decide
  INPUT A system like the above, and a number x.
  QUESTION Does x get back to 1?

Wang Tiles

- INPUT a set of square coloured tiles of the same size. Only same colour borders next to one another. QUESTION Can an initial configuration be extended to colour the plane?
- Wang in the 60's showed that there is no algorithm to decide this.

HILBERT'S 10TH PROBLEM

- INPUT A polynomial P in variables $x_1, ..., x_n$
  QUESTION Is there a positive solution to the equation $P = 0$?
- Matijasevich, after Julia Robinson in the 70's showed there is no algorithm to decide such questions.
- But there is now a polynomial whose only positive rational zeroes are the primes!



This shows myself, Julia Robinson, Raphael Robinson, and my wife.

$$
\begin{aligned}
Q(a,\ldots,z) = (k+2)\{1 &- [wz+h+j-q]^2 \\
&- [(gk+2g+k+1)(h+j)+h-z]^2 \\
&- [2n+p+q+z-e]^2 - [16(k+1)^3(k+2)(n+1)^2+1-f^2]^2 \\
&- [e^3(e+2)(a+1)^2+1-o^2]^2 - [(a^2-1)y^2+1-x^2]^2 \\
&- [16r^2y^4(a^2-1)+1-u^2]^2 \\
&- [((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2 \\
&- [n+1+v-y]^2 \\
&- [(a^2-1)l^2+1-m^2]^2 - [ai+k+1-l-i]^2 \\
&- [p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2 \\
&- [q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2 \\
&- [z+pl(a-p)+t(2ap-p^2-1)-pm]^2\}.
\end{aligned}
$$

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Recent Examples

- Recently it was shown by Braverman and Yampolsky (STOC, 2007) that Julia sets can be noncomputable, any halting problem being codable. (Also Blum-Smale-Shub, but that's another story.)
- Julia set: $z \mapsto z^2 + \alpha z$, where $\alpha = e^{2\pi i\theta}$.
- Nabutovsky and Weinberger (Geometrica Dedicata, 2003) showed that basins of attraction in differential geometry faithfully emulated certain computations. Refer to Soare Bull. Symbolic Logic.
- Remark: Earlier and ignored work by Lee Rubel on universal PDE's.

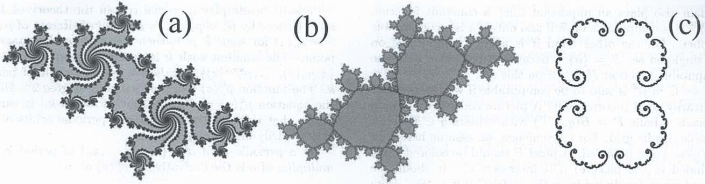

Figure 1: Examples of quadratic Julia sets J_p (black), and filled Julia sets K_p (gray); orbits that originate at white points escape to ∞; note that on picture (c) $K_p = J_p$, since K_p has empty interior

- The answer is inescapable: these diverse mathematical objects, tiles, Conway sequences, and polynomials can be used to simulate computations.

NO INVARIANTS

- Much of mathematics is concerned with classification of structures (groups rings, DE's etc) by invariants.
- Bases for vector spaces, Ulm invariants for abelian groups.
- How can we show that no invariants are possible?
- A computability theorists's view.

ANALYTIC=Σ^1_1

- The halting problem is called Σ^0_1 in that $\varphi_x(y)$ halts iff

$$\exists t \in \mathbb{N} \varphi_x(y)$$

  halts in t steps. (And $\varphi_x(y)$ halts in t steps is computable.) This is arithmetic, where the quantifier searches over $\mathbb{N}$.
- Almost all problems in normal mathematics are analytic.
- A is analytic or Σ^1_1 iff deciding $x \in A$ entails asking if there is a function f from $\mathbb{N}$ to $\mathbb{N}$ such that some computable relation holds for all $f(n)$.
- E.g. isomorphism is typically in Σ^1_1.

Σ_1^1 COMPLETENESS

- Many problems in Σ_1^1 are much easier. E.g. isomorphism for finitely presented groups is Σ_3^0. (Is there a matching of generators for which every equation in the first holds in the second?)
- If some problem is shown to be Σ_1^1 complete, then no simpler set of invariants is possible.
- E.g. (Downey and Montalbán) the problem of deciding if two finitely prsented groups have $H_i(G) \cong H_i(\hat{G}$ for $i \leq 3$ is Σ_1^1 complete.
- Uses the result that the isomorphism problem for computable torsion free Abelian groups is Σ_1^1 complete. (DM)

How to prove such a result?

- A tree is a downward subset of $\mathbb{N}^{<\mathbb{N}}$, the set of finite strings of natural numbers.
- A tree is well-founded if it has no infinite path.
- Core problem: Deciding of a tree is well-founded is Σ_1^1 complete. Deciding if two trees are isomorphic is Σ_1^1 complete. (Essentially Kleene, Harrison)

Theorem (DM)

There is a computable operator G, that assigns to each tree T a torsion-free group G(T), in a way that

1. *if* $T_0 \cong T_1$, *then* $G(T_0) \cong G(T_1)$,
2. *if* T_0 *is well-founded and* T_1 *is not, then* $G(T_0) \ncong G(T_1)$.

- Borel cardinality theory: equivalence relations (such as isomorphism) $E_1 \leq_B E_2$ iff there is a Borel mapping $f : E_1 \to E_2$ with $x \approx_{E_1} y$ iff $f(x) \approx_{E_2} f(y)$.
- Their idea is that any reasonable translation should be at least Borel.
- Invariants? e.g. finite rank torison free Abelian groups. $E_{\text{rank } i} <_B E_{\text{rank } i+1}$. Complete?
- From a computability point of view, all the same Σ^0_3.
- Structure largely unknown.

- Another example is provided by extending partial orderings.
- A linear extension $(P, \leq_L)$ of a partial ordering $(P, \leq_P)$ is a linear ordering such that whenever $x \leq_P y$, $x \leq_L y$.
- Theorem (Szpilrajn) every partial order has a linear extension.
- A well-partial ordering has a well ordered extension. (Bonnet, Corominas, Fraïssé, Jullien, and Pouzet, Galvin, Kostinsky, and McKenzie, etc for extendible types)
- Theorem (Slaman and Woodin) The collection of computable partial orderings with dense extensions is co-Σ_1^1 complete, and hence there are no reasonable invariants, answering a question of Łoś.

An aside

- One of the greatest theorems in mathematics is not widely known, and is due to Saharon Shelah.
- The realization is that orderings are very complex and if one can code them into a structure then that structure will have so many models in its isomorphism type that it will be impossible to "classify"
- Shelah proved the Dichotomy Theorem which says very very roughly, that either a class of models resembles a vector space and has a decent set of invariants "like a basis" generated by a relation called "forking" or it resembles a linear ordering and is unclassifiable.
- "Why am I so happy?" (AMS notices), Classification Theory and the Number of Non-isomorphic Models.
- There is a nice user friendly discussion of this in a Bull LMS paper by Wilfred Hodges "What is structure theory?" (1987)

- One modern incarnation of this story is to only look at efficient solutions to problems. The the title would be A complexity-theorists view.
- See e.g. Wigderson's ICM article
- But I have run out of time, so that is another story.

Thanks for your attention.