Randomness, Computation and Mathematics

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Plan

- Since this is a general talk, I will give a basic lecture in this area, hopefully
- concentrating on recent themes.
- For more ... there are nice books in the registration area.
- Apologies to the experts.

Randomness

- How should we understand randomness?
- Can we generate randomness?
- What does this mean anyway?
- Can we *quantify* the amount of randomness?
- What does randomness do as a *computational resource*?

The great men

- Turing 1950:

  > *" An interesting variant on the idea of a digital computer is a "digital computer with a random element." These have instructions involving the throwing of a die or some equivalent electronic process; one such instruction might for instance be, "Throw the die and put the-resulting number into store 1000." Sometimes such a machine is described as having free will (though I would not use this phrase myself)."*

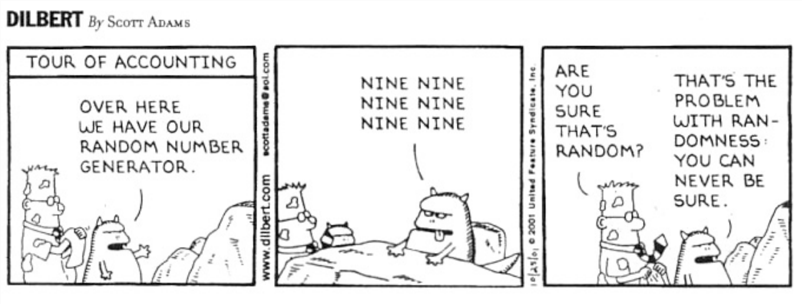
- von Neumann 1951:

  > *"Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin."*

But even earlier

> *"How dare we speak of the laws of chance?*
> *Is not chance the antithesis of all law?"*
>
> *— Joseph Bertrand, Calcul des Probabilités, 1889*

Intuitive Randomness


DILBERT By Scott Adams
TOUR OF ACCOUNTING
OVER HERE WE HAVE OUR RANDOM NUMBER GENERATOR.
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ARE YOU SURE THAT'S RANDOM?
THAT'S THE PROBLEM WITH RANDOMNESS: YOU CAN NEVER BE SURE.

Intuitive Randomness

Which of the following binary sequences seem random?

A 000000000000000000000000000000000000000000000000000000000000

B 001101001101001101001101001101001101001101001101001101001101

C 010001101100000101001110010111011100000001001000110100010101

D 001001101101100010001111010100111011001001100000001011010100

E 010101110110111101110010011010110111001101101000011011110111

F 011101111100110110011010010000111111001101100000011011010101

G 000001100010111000100000000101000010110101000000010000000100

H 010100110111101101110101010000010111100000010101110101010001

Intuitive Randomness

Non-randomness: increasingly complex patterns.

A 000000000000000000000000000000000000000000000000000000000000

B 001101001101001101001101001101001101001101001101001101001101

C 010001101100000101001110010111011100000001001000110100010101

D 001001101101100010001111010100111011001001100000001011010100

E 010101110110111101110010011010110111001101101000011011110111

F 011101111100110110011010010000111111001101100000011011010101

G 000001100010111000100000000101000010110101000000100000000100

H 010100110111101101110101010000010111100000010101110101010001

Intuitive Randomness

Randomness: bits coming from atmospheric patterns.

A 000000000000000000000000000000000000000000000000000000000000

B 001101001101001101001101001101001101001101001101001101001101

C 010001101100000101001110010111011100000001001000110100010101

D 001001101101100010001111010100111011001001100000001011010100

E 010101110110111101110010011010110111001101101000011011110111

F 011101111100110110011010010000111111001101100000011011010101

G 000001100010111000100000000101000010110101000000010000000100

H 0101001101111011011101010100000101111000000010101110101010001

Intuitive Randomness

Partial Randomness: mixing random and nonrandom sequences.

A 000000000000000000000000000000000000000000000000000000000000

B 001101001101001101001101001101001101001101001101001101001101

C 010001101100000101001110010111011100000001001000110100010101

D 001001101101100010001111010100111011001001100000001011010100

E 010101110110111101110010011010110111001101101000011011110111

F 011101111100110110011010010000111111001101100000011011010101

G 000001100010111000100000000101000010110101000000100000000100

H 010100110111101101110101010000010111100000010101110101010001

Intuitive Randomness

Randomness relative to other measures: biased coins.

A 000000000000000000000000000000000000000000000000000000000000

B 001101001101001101001101001101001101001101001101001101001101

C 010001101100000101001110010111011100000010010001101000010101

D 001001101101100010001111010100111011001001100000001011010100

E 010101110110111101110010011010110111001101101000011011110111

F 011101111100110110011010010000111111001101100000011011010101

G 000001100010111000100000000101000010110101000000100000000100

H 010100110111101101110101010000010111100000010101110101010001

Algorithmic

We need a way to talk about laws: "computable statistical tests" or "effective statistical tests" so that we can make sense of our intuition.

(So that, the expected behaviour of a distribution aligns itself to the behaviour of a particular input.)

Three Approaches to Randomness at an Intuitive Level

- The statistician's approach: Deal directly with rare patterns using measure theory. Random sequences should not have effectively rare properties. (von Mises, 1919, finally Martin-Löf 1966)
- Computably generated null sets represent effective statistical tests.
- The coder's approach: Rare patterns can be used to compress information. Random sequences should not be compressible (i.e., easily describable) (Kolmogorov, Levin, Chaitin 1960-1970's).
- Kolmogorov complexity; the complexity of σ is the length of the shortest description of σ.
- The gambler's approach: A betting strategy can exploit rare patterns. Random sequences should be unpredictable. (Solomonoff, 1961, Scnhorr, 1975, Levin 1970)
- No effective martingale (betting) can make an infinite amount betting on the bits.

The statisticians approach

- von Mises, 1919. A random sequence should have as many 0's as 1's. But what about 10101010101010.....
- von Mises idea: If you select a subsequence $\{a_{f(1)}, a_{f(2)}, \dots\}$ (e.g. $f(1) = 3, f(2) = 10, f(3) = 29,000$, so the 3rd, the 10th, the 29,000 th etc) then the number of 0's and 1's divided by the number of elements selected should tend to $\frac{1}{2}$. (Law of Large Numbers)
- But what selection functions should be allowed?
- Church, 1940 computable selections.
- Ville, 1939 showed no countable selection possible. Essentially not enough statistical tests.

Turing

- I remark that Turing was very interested in normality and absolute normality.
- Numbers are normal base d if they obey the frequency considerations : the number of i's that occur is $\frac{1}{d}$. Alsolutely normal if normal to any base.
- These concepts go back to Borel and his contemporaries.
- Schmidt (1960) was the first to construct absolutely normal numbers.
- Normality is a primitive form of randomness related to automata, and will be discussed by Veronica Becher later this week, and nice article in the proceedings.
- Interestingly, many recent advances in additive number theory based on supposing the primes are "random" but we don't understand,....returing to our story.

von Mises

- Recapping, the first person to look seriously at the notion of a random individual sequence was Richard von Mises (1919).
- Let $f : \omega \to \omega$ be an increasing injection, a selection function.
- Then a random X should satisfy the following.

$$\lim_{n\to\infty} \frac{\{m \mid m \leq n \wedge X(f(m)) = 1\}}{n} = \frac{1}{2}.$$

- Later Church 1940 said use (partial) computable f.

Ville's Theorem

Let $S(\alpha, n)$ denote the number of 1's in the first n bits of α and similarly S_f for the selected places.

Theorem (Ville's Theorem 1939)

Let E be any countable collection of selection functions. Then there is a sequence $\alpha = \alpha_0\alpha_1\ldots$ such that the following hold.

1. $\lim_n \frac{S(\alpha,n)}{n} = \frac{1}{2}$.
2. *For every $f \in E$ that selects infinitely many bits of α, we have* $\lim_n \frac{S_f(\alpha,n)}{n} = \frac{1}{2}$.
3. *For all n, we have* $\frac{S(\alpha,n)}{n} \leq \frac{1}{2}$.

The problem is 3 since it says you always get more 0's than 1's.

Martin-Löf randomness

- Martin-Löf, 1966 suggests using shrinking effective null sets as representing effective tests-abstract tests. Basis of modern effective randomness theory.

Definition (Martin-Löf)

1. A *Martin-Löf test* is a sequence $\{U_n\}_{n\in\omega}$ of uniformly Σ^0_1 classes such that $\mu(U_n) \leq 2^{-n}$ for all n.
2. A class $C \subset 2^\omega$ is *Martin-Löf null* if there is a Martin-Löf test $\{U_n\}_{n\in\omega}$ such that $C \subseteq \bigcap_n U_n$.
3. A set $A \in 2^\omega$ is *Martin-Löf random* if $\{A\}$ is not Martin-Löf null.

The computational paradigm

- Can think of a machine $U(\tau) = \sigma$ as the information of the bits of τ describing σ.
- The length of the shortest τ is the U-Kolmogorov complexity of σ, $C_U(\sigma)$.
- σ is random if $C_U(\sigma) \geq |\sigma|$.
- Have universal machines and can define an optimal C up to a constant.
- Intentional meaning is not quite right as τ provides τ and $|\tau|$ bits of information.
- This is avoided using telephone numbers, prefix-free complexity K.

K-randomness

- Prefix freeness gets rid of the use of length as extra information:
- Notice that prefix-freeness means that the domain of the machines has measure.
- The Coding Theorem (Levin-Gács) says that "Occam's Razor=Bayes' Theorem" in that if $Q(\sigma) = -\log(\mu\{\tau \mid U(\tau) = \sigma\}$, then $Q(\sigma) = K(\sigma)$.
- I remark that other forms are possible, such as process complexity and monotone, which acts continuously.
- Eg. $U(\sigma) = tau$ and $U(\sigma') = \tau'$ and $\sigma \prec \sigma'$ implies $\tau \prec \tau'$. (process)
- Not true for e.g. monotone or process complexity for continuous sample spaces (Gács, then Day).

K-randomness

- (Levin, Chaitin) α is K- random if there is a c s.t.

$$\forall n(K(\alpha \upharpoonright n) > n - c).$$

Theorem (Schnorr)

X is K-random iff X is ML-random.

- I remark that other forms are possible, such as process complexity, which acts continuously.
- Eg. $U(\sigma) = tau$ and $U(\sigma') = \tau'$ and $\sigma \prec \sigma'$ implies $\tau \prec \tau'$. (process)

Martingales

- von Mises again. This time think about predicting the next bit of a sequence. Then you bet on the outcome. You should not win!
- (Levy) A martingale is a function $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ such that for all σ,
$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$
- the martingale *succeeds* on a real α, if $\limsup_n F(\alpha \upharpoonright n) \to \infty$.

- Think of betting on a sequence where you know that every 2nd bit is 1. Then every second bit you could double your stake. This martingale exhibits exponential growth and that can be used to characterize computable reals.
- Ville proved that null sets correspond to success sets for martingales. They were used extensively by Doob in the study of stochastic processes.

- A supermartingale is a function $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ such that for all σ,
$$f(\sigma) \geq \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$
- Schnorr showed that Martin-Löf randomness corresponded to effective (super-)martingales failing to succeed.
- f as being effective or computably enumerable if $f(\sigma)$ is a c.e. real, and at every stage we have effective approximations to f in the sense that $f(\sigma) = \lim_s f_s(\sigma)$, with $f_s(\sigma)$ a computable increasing sequence of rationals.

All coincide

Theorem (Schnorr)

A real α is Martin-Löf random iff no effective (super-)martingale. succeeds on α.

Major Themes

- Computational power of randoms
- Information theory and characterizing computability.
- Reflections in analysis, ergodic theory etc.
- Calibrating randomness.

Randoms should be computationally weak

- We now know that there are two kinds of randoms, those which resemble Chaitin's $\Omega = \sum_\sigma 2^{-K(\sigma)}$ and more typical ones.

- There has been a lot of popular press about the "number of knowledge" etc, which is random, but has high computational power.

- We would theorize randoms to be stupid: computationally weak.
- For all X there is a random Y with $X \leq_T Y$. (Kučera-Gács)

Stephan's Theorem

Theorem (Stephan)

If X is random and X has enough computational power to compute a $\{0,1\}$-valued function f such that for all e, $f(e) \neq \varphi(e)$, (ie X is PA) then X computes the halting problem.

- Stupidity Tests
- There are two ways to convince someone you are stupid:
- The first people pass the stupidity test as they are so smart that they know how to be stupid, the second really are stupid.
- That is, with sufficient randomness, randomness begins to resemble order. This is kind of remarkable. We are still trying to understand it.
- In music it is quite difficult to distinguish between aleatoric (or chance) and totally serial (based on a pattern) music.

- What this means is that if X is $\emptyset'$-random (ie random relative to the halting problem) then it is already computationally weak.
- Recent work by Bienvenu and others look at adding statements asserting certain strings are random to logical systems. Again, as expected, this is not a way around the incompleteness phenomenon. (Except in the resource bounded case.)
- Barmpalias, Lewis, and Ng have shown that each PA degree is the join of two randoms, a remarkable result.
- This theme had realizations as to aligning randomness with weaker notions of computing fixed point free functions, and things like $K(X \upharpoonright h(n)) \geq n$. and "autocomplex" degrees.

Calibrating randomness

- When is X more random than Y? When is X somewhat random?
- One way is to vary the tests or gales. Stronger tests mean stronger randomness.
- Examples : Schnorr randomness (means that $\mu(V_e) = 2^{-e}$), computable randomness (means that computable martingales).
- Intricate dance with Turing degrees, Sample theorem: if **a** is not computationally powerful in terms of its jump (**a** is not high) (Nies, Stephan, Terwijn) then in **a** these randomness notions all coincide. That is A is MLR iff Schnorr random iff computably random.
- Varying oracles. $n+1$-randomness equals randomness relative to $\emptyset^{(n)}$. (Miller-Yu) if $A \leq B$ are random and B is n-random, so is A.

- Many reducibilities and measures of relative randomness. Eg $Y \leq_K X$ means $K(Y \upharpoonright n) \leq K(X \upharpoonright n) + c$ for all n. $Y \leq_{LR} X$ means every real Y can derandomize X can also.
- Sample theorem. $\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$ is Chaitins' Omega. Seems to depend on the machine, but in the same way as for the halting problem.

Theorem (Slaman-Kučera)

A left-c.e. real is random iff it is Solovay complete.

- $A \leq_S B$ roughly means that effectively approximating B allows us to B-tightly effectively approximate A.
- Another: $\emptyset^{(n)}$-randomness is definable in terms of K. (Bienvenu, Muchnick, Shen, Vereshchagin)

Effective Dimensions

- Fractional dimension: Caratheordory, Hausdorff etc.
- (Lutz) An s-gale is a function $F : 2^{<\omega} \mapsto \mathbb{R}$ such that
$$F(\sigma) = 2^{-s}(F(\sigma 0) + F(\sigma 1)).$$
- The basic idea here is that not betting on one outcome or the other is bad.
- Usually, decide that we are not prepared to favour one side or the other in our bet. Thus we make $F(\sigma i) = F(\sigma)$ at some node σ.In the case of an s-gale, then we will be unable to do this, without automatically losing money due to inflation.

- Lutz has shown that effective Hausdorff dimension can be characterized using these notions.
- It is not important exactly what the definition is but we get the following.
- (Lutz, Hitchcock) For a class X the following are equivalent:
  - (i) $\dim(X) = s$.
  - (ii) $s = \inf\{s \in \mathbb{Q} : X \subseteq S[d]$ for some s-gale $F\}$.
  - (iii) $s = \inf\{s \in \mathbb{Q} : X \subseteq S_{2^{(1-s)n}}[d]$ for some martingale $d\}$.
- An equivalent characterization due to Lutz is $\liminf_{n\to\infty} \frac{K(X\upharpoonright n)}{n}$.

- Lutz comment:
- “Informally speaking, the above theorem says the the dimension of a set is the most hostile environment (i.e. most unfavorable payoff schedule, i.e. the infimum s) in which a single betting strategy can achieve infinite winnings on every element of the set.”
- While Schnorr did not do any of this, he did look at exponential orders. He comments:
- “To our opinion the important statistical laws correspond to null sets with fast growing orders. Here the exponentially growing orders are of special significance.”

Themes

- Can be used for aperiodic tiling (Levin, Shen, Vereshchagin etc)
- Can have for all m, n, $K(X[m, m+n])$ is high (That is $\frac{K(X[m,m+n])}{n} \geq 1 - \epsilon$).
- Simpson recently used effective dimension for new results in Symbolic dynamics namely, classical dimension equals the entropy (generalizing a difficult result of Furstenburg 1967).
- Very close relationship between ergodic theory and randomness e.g.

Theorem (Hochman and Meyerovitch)

The values of entropies of subshifts of finite type over $\mathbb{Z}^d$ for $d \geq 2$ are exactly the complements of halting probabilities.

- Lutz, Mayordomo and others: use resource bounded versions to measure things like NP.

- The easiest way to make something of Hausdorff dimension $\frac{1}{2}$ is to take a random and "thin it."
- Is this the only way?

Theorem (Miller)

There is a real X of (effective) Hausdorff dimension $\frac{1}{2}$ such that every $Y \equiv_T X$ has Hausdorff dimension $\leq \frac{1}{2}$.

- Extracting randomness is hard.
- However, with two independent sources, it is possible to get a Y computable from both of them to within ϵ of random (Zimand).
- 0,1 law for effective packing dimension.
- (Mayordomo) Packing dimension $\limsup_{n\to\infty} \frac{K(X\upharpoonright n)}{n}$.

Lowness

- X is low for random means Y random iff Y is X-random.
- X is K-trivial iff for all n, $K(X \upharpoonright n) \leq K(n) + c$, for all n.

Theorem (Chaitin)

X is computable iff for all n, $C(X \upharpoonright n) \leq C(n) + c$ for all n. That is, C-trivial=computable.

A trivial story

- $A = \{\langle e, n\rangle : \exists s(W_{e,s} \cap A_s = \emptyset \wedge \langle e, n\rangle \in W_{e,s}$ and $\wedge \sum_{\langle e,n\rangle \leq j \leq s} 2^{-K(j)[s]} < 2^{-(e+2)})\}$.
- This A is K-trivial, non-computable.

Theorem (Downey, Hirschfeldt, Nies, Stephan)

If A is K-trivial then A solves Post's problem. That is, a one line description of a Turing incomplete set.

Theorem (Nies)

1. *A is K-trivial iff*
2. *A is low for randomness iff*
3. *$A \leq X$ for some A-random X (with Hirschfeldt)*
4. *A is low for K meaning $K^A = K$.*

Themes

- K-trivials are ubiquitous with maybe 15 characterizations.
- Bring to the fore themes of traceing.
- Variations have been used to solve longstanding questions in computability theory and in logic.
- Newer variations with weaker initial segment properties.
- First fundamentally enumerable property. No forcing.

Examples

- The solution to Post's problem due to Kučera turns out to essentially be K-trivial.
- Natural ideal. (Barmialias-Downey, no exact pair)
- Kučera-Slaman : For every Scott set F and every noncomputable set X in F, there is a Y in F such that X and Y are Turing incomparable.
- A is K-trivial implies it is "almost computable" in that there is a computable h, such that for any partial function $f \leq_T A$, we can enumerate $\{W_{g(e)} \mid e \in \omega\}$ with $|W_{g(e)}| \leq h(e)$ and $f(e) \in W_{h(e)}$. (Nies)
- Lead to "strongly jump traceable" and such that the sets A with $\emptyset'$ sjt relative to A have the property that there are c.e. sets X below all of them. Solves a question about pseudo-jump operators going back to Jockusch-Shore. (Downey-Greenberg)

- Still hoping for a combinatorial characterization e.g. A is low for K iff A is jump traceable at all orders h with $\sum_{n\to\infty} 2^{-h(n)} < \infty$.
- A is low for Schnorr iff A is computably traceable meaning that all functions $g \leq_T A$ can be approximated with a $\{D_{g(n)} \mid n \in \omega\}$ and $|D_{g(n)}| \leq n+1$. (Terwijn-Zambella, then Bedregal, Kjoss-Hanssen, Nies, Stephan)
- No noncomputable X is low for computable randomness. (Nies)
- Many other variations.

Measures and their random reals

- Given $X \not\equiv_T \emptyset$ is there a measure relative to which X is random?
- Well clearly we can concentrate measure on X, but the answer is still yes even if that is not allowed (Reimann-Slaman).
- For continuous measures,

Theorem (Reimann and Slaman)

The class NCR is countable.

- Kind of remarkable, given that it would seem that randomness only needs a few quantifiers.
- Also true for NCR_k, never continuously k-random. The for all k needs Borel Determinacy. This reversal is difficult and used metamathematical techniques.

Derivatives

- An old program of Demuth is that functions should behave well at random points.
- Continuity=computability relative to some oracle. Differentiability=some level of randomness.
- For example, monotone functions are differentiable at computably random points. (Brattka, Miller, Nies)
- Reflects the fact that ergodicity is a finite form of Lebesgue's theorem, a la Terry Tao's blog.
- The idea is that a Denjoy derivative looks like a martingale.

Computing from random strings

- We have already seen the work of Bienvenu, Shen, and others about using random axioms as a resource.
- Allender and others look at $R_Q = \{\sigma \mid Q(\sigma) \geq \frac{|\sigma|}{2}\}$, say. (and complexity Q)
- reduce with $\leq_m^P$.
- Earlier Muchnik proved that for Q_C is *tt*-complete.

Theorem (Allender, Buhrman, and Koucký)

$P = \cap_U \{A : A \leq_{dtt}^p R_{C_U}\} \cap COMP$

- Many possible interesting connections. Also Slaman's "low for speed" . That is X such that all DTIME classes in COMP relative to X remain the same.

Speculations

- Use randomness for understanding quantum physics.
- Can already buy it over the counter (Quandis) (see Calude and Svozil).
- Program is to figure out what is needed to make physics work.
- Is the universe granular? Is computability emergent?
- Can the universe manufacture randomness, computability, incomputability etc? Related to this morning's talk on BPP and BQP.
- Also left out applications in biology, music, etc.

Want to learn more?

- Calibrating randomness (BSL) Downey, Hirschfeldt, Nies Terwijn.
- Computability and Randomness, Nies OUP
- Algorithmic randomness and complexity, Downey and Hirchfeldt.

Thank You