Euclidean Domains and Euclidean Functions

Rod Downey

(Joint Work with Asher Kach)

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References

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Outline

The Division Algorithm...

Problem

Divide 18 into 218 (over $\mathbb{Z}$).

Answer.

Perform long division

$$
\begin{array}{r}
12 \\
18\,\overline{)\,218} \\
\underline{180} \\
38 \\
\underline{36} \\
2
\end{array}
$$

and so $218 = 12 \cdot 18 + 2$. $\square$

The Division Algorithm...

Problem

Divide $x + 2$ into $x^3 + 18x^2 + 2x + 18$ (over $\mathbb{Q}$).

Answer.

Perform long division

and so $x^3 + 18x^2 + 2x + 18 = (x^2 + 16x - 30)(x + 2) + 78.$ $\square$

Euclid's Algorithm...

Proposition

The algorithm

```
function gcd(a,b)
   if (a < b)
     swap(a,b)
   if (b == 0)
     return a
   return gcd(a − b,b)
```

computes the greatest common divisor of non-negative integers a and b.

Problem

Find the greatest common divisor of 18 and 10.

Answer.

Note $gcd(18, 10) = gcd(8, 10) = gcd(10, 8) = gcd(2, 8) = gcd(8, 2) = gcd(6, 2) = gcd(4, 2) = gcd(2, 2) = gcd(0, 2) = gcd(2, 0) = 2$. □

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Note $gcd(18, 10) = gcd(8, 10) = gcd(10, 8) = gcd(2, 8) = gcd(8, 2) = gcd(6, 2) = gcd(4, 2) = gcd(2, 2) = gcd(0, 2) = gcd(2, 0) = 2.$ $\square$

The Division Algorithm and Euclid's Algorithm...

Remark

In both $\mathbb{Z}$ and $\mathbb{Q}[X]$, the division algorithm (Euclid's algorithm) terminates because the dividend (either a or b) decreases in *size* at every step.

- Within $\mathbb{Z}$, the *size* of an integer is its magnitude.
- Within $\mathbb{Q}[X]$, the *size* of a polynomial is its degree.

Generalizing this requirement of remainders decreasing in *size* yields the (traditional) definition of a Euclidean domain.

Defining Euclidean Domains...

Definition

A commutative ring $\mathcal{R}$ is a *Euclidean ring* if there is a function $\phi : R^0 \to \mathbb{N}$ (where $R^0 := R \setminus \{0\}$) satisfying

$$(\forall a, d \in R^0)(\exists q \in R)\big[a + qd = 0 \text{ or } \phi(a + qd) < \phi(d)\big].$$

The function ϕ is termed a *Euclidean function* for $\mathcal{R}$.

If the ring is also an integral domain (i.e., there are no zero divisors) then it becomes a Euclidean Domain.

Euclidean Functions for $\mathbb{Z}$...

Example

The integers $\mathbb{Z}$ are a Euclidean domain.

Proof.

The functions

$$\begin{aligned}
\phi_1(z) &= |z| \\
\phi_2(z) &= \lceil \log_2 |z| \rceil \\
\phi_3(z) &= \begin{cases} |z| & \text{if } z \neq 5 \\ 13 & \text{otherwise} \end{cases}
\end{aligned}$$

are Euclidean functions for $\mathbb{Z}$.

Note that ϕ_3 serves as an example where the implication

$$x \text{ divides } y \quad \text{implies} \quad \phi(x) \leq \phi(y)$$

fails. □

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Euclidean without Euclidean functions

Definition

If $\mathcal{R}$ is a commutative ring (with 1), define a sequence of sets $\{R_n\}_{n\in\mathbb{N}}$ via recursion by

$$R_n := \{d \in R^0 : (\forall a \in R^0)(\exists q \in R)\,[a + dq = 0 \text{ or } a + dq \in R_{<n}]\}$$

where $R_{<n} = \bigcup_{m<n} R_m$ and $R^0 = R - \{0\}$.

Remark

Thus R_0 consists of the units, R_1 consists of those elements which exactly divide every other $a \in R^0$ or leave remainder a unit, etc. (NB if you read Samuel, $R_1 = R_2$ there)

Theorem (Motzkin 1949, Samuel 1971)

An integral domain $\mathcal{R}$ (resp. ring) is a Euclidean domain (resp. ring)if and only if $R^0 = \bigcup_{n\in\mathbb{N}} R_n$.

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The Least Euclidean Function $\phi_{\mathcal{R}}$

Definition (Motzkin 1949, Samuel 1971)

If $\mathcal{R}$ is a Euclidean domain (ring), define $\phi_{\mathcal{R}} : R^0 \to \mathbb{N}$ by

$$\phi_{\mathcal{R}}(d) = n$$

where n is least so that $d \in R_n$.

Theorem (Motzkin 1949, Samuel 1971)

If $\mathcal{R}$ is a Euclidean domain (resp. ring), the function $\phi_{\mathcal{R}}$ is a Euclidean function for $\mathcal{R}$. Moreover, it is the least Euclidean function for $\mathcal{R}$; i.e., if ϕ is a Euclidean function for $\mathcal{R}$, then $\phi_{\mathcal{R}}(d) \leq \phi(d)$ for all $d \in R^0$.

Consequently, the function $\phi_{\mathcal{R}}$ satisfies

$$\phi_{\mathcal{R}}(d) = \inf_{\phi} \phi(d)$$

where ϕ ranges over all Euclidean functions for $\mathcal{R}$.

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- I am sure that, as logicians, you immediately notice that it is unnecessary in the definition of a Euclidean ring that the range of the ranking function is $\mathbb{N}$.
- Any ordinal will do, and maybe even well founded partial orders.

Outline

Transfinite Euclidean Domains...

Definition

A commutative ring $\mathcal{R}$ is a transfinite Euclidean ring if there is a function $\phi : R^0 \to \mathrm{ON}$ (where $R^0 := R \setminus \{0\}$ and ON is the class of ordinals) satisfying

$$(\forall a, d \in R^0)(\exists q \in R)\big[a + qd = 0 \text{ or } \phi(a + qd) < \phi(d)\big].$$

The function ϕ is termed a *transfinitely-valued Euclidean function* for $\mathcal{R}$.

Theorem (Motzkin 1949)

An integral domain $\mathcal{R}$ is a transfinite Euclidean domain if and only if $R^0 = \bigcup_{\alpha \in ON} R_\alpha$. (Here we take unions at limit stages.)

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Properly Transfinite Euclidean Rings and Domains...

Question (Motzkin 1949; Samuel 1971; Schrieber 1985)

Is there an integral domain that is a properly transfinite Euclidean domain? I.e., is there an integral domain having transfinitely-valued Euclidean functions but not finitely-valued Euclidean functions?

Theorem (Samuel 1971)

The function $\phi(z) = \omega \cdot i + j + 1$ is a Euclidean function for $\mathbb{Z}$, where $z = \pm 2^i(2j+1)$.

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Making Euclidean Rings

Lemma (Samuel, 1971)

If R is a Euclidean ring and A is a multiplicatively closed subset, then $A^{-1}R$ is Euclidean.

- A product of a finite number of transfinite Euclidean rings is i transfinite Euclidean.
- This gives an example of a properly transfinite Euclidean *ring*

Theorem (Samuel 1971)

The commutative ring $\mathbb{Z} \oplus \mathbb{Z}$ has a transfinitely-valued Euclidean function (indeed, one with range $\omega^2 + \omega^2$) but no finitely-valued Euclidean function.

- To prove this, we prove *If R has a finite Euclidean function with the R_0 finite, then for all n, $\phi^{-1}(n)$ is finite*.
- Induction. If $R'_n = \cup_{j=0}^{n-1} \phi^{-1}(j)$ is finite, and $\phi(b) = n$, then then $R'_n \to R/Rb$ is surjective. The use algebra prove that in Noetherian ring the number of ideals with a given norm is finite, and hence the ideal Rb can only have a finite number of values. So only finitely many b.
- So if $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{N}$ is an Euclidean algorithm, and suppose $\phi((1,0)) = n$ then $R'_n \to (\mathbb{Z} \oplus \mathbb{Z})(0,1)$ is supposedly surjective, but the last ring is isomorphic to $\mathbb{Z}$ which is infinite.

- You would likely naturally think of working in some ring like $\mathbb{Q}[X_i : i \in \omega]$ and perhaps taking quotients.
- The ring $\mathcal{R} := \mathbb{Q}[X]$ is a Euclidean domain with $\phi_{\mathcal{R}}(X) = 1$. Unfortunately, this is not very useful as $\mathbb{Q}[X, Y]$ is not a Euclidean domain.
- The question is how to control rank (e.g. 2) in an extension ring. For instance you *could* try adding roots but even here the situation is kind of murky.

- What about adding, say, square roots?

Theorem (Samuel, 1971)

The only imaginary quadratic fields of the form $\mathbb{Q}(\sqrt{-d})$ for which the ring of integers is a Euclidean domain are for $d = 1, 2, 3, 7, 11$.

- It is known for positive d which $\mathbb{Q}(\sqrt{d})$ are Euclidean for the norm, which is a measure associated with number fields, (See Samuel) it is 16 cases.
- Even $\mathbb{Z}[\sqrt{d}]$ is hard, with $d = 14$ recently done!

Outline

- Anyway, we were motivated by trying to understand the computability theory of the situation and the reverse mathematics.
- To wit, to what extent is having a Euclidean algorithm the same as having a Euclidean function.

Computable Rings...

Definition

A ring $\mathcal{R}$ is *computable* if it has a *computable presentation*.

Definition

A *computable presentation* of an infinite ring $\mathcal{R} = (R : +, \cdot, 0, 1)$ is a bijection between R and $\mathbb{N}$ so that the operations of addition and multiplication are *computable* functions on $\mathbb{N}$.

Definition

A computable presentation of a ring is a computably presented Euclidean ring iff there is a computable ranking function from the elements onto $\mathbb{N}$. In the transfinite case we would need a notation for the ordinal.

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Examples of Computable Rings...

Example

The ring $\mathbb{Z}$ is a computable Euclidean ring.

Example

Let $S \subseteq \mathbb{N}$ be a (multiplicatively closed) *computably enumerable* set with $0 \notin S$ and $1 \in S$. Then the subset $\mathcal{R} \subseteq \mathbb{Q}$ consisting of those $a/b \in \mathbb{Q}$ with $b \in S$ is a computable ring. This uses the $A^{-1}R$ construction for $R = \mathbb{Z}$.

Proof.

Identify fractions $a/b \in \mathbb{Q}$ with natural numbers $n \in \mathbb{N}$ as they appear. □

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- The first person to prove anything about computable Euclidean domains was Schrieber in his PhD from Cornell.
- In summary he constructed a Euclidean domain which was a computable domain, and yet had no computable Euclidean function with range $\mathbb{N}$ (In fact coded $\emptyset'$) and showed the units need not be computable in a computable Euclidean domain.
- We extend these results by showing R_1 can be as complex as possible, removing the restriction $\mathbb{N}$, and examing the reverse mathematics of the situation. We also examine the possibility of Δ_2^0 Euclidean functions.
- The arguments are not difficult, but this is a situation where the question remaining are quite arresting.

Computing $\{z : \phi_{\mathcal{R}}(z) = 0\}$...

Theorem (Schrieber 1985)

There is a computable Euclidean domain $\mathcal{R} \subseteq \mathbb{Q}$ for which there is no "algorithm" to determine whether an element is a unit. In fact $R_0 \equiv_m \emptyset'$.

Proof.

Identify the Halting Problem K with a subset of the prime numbers. Have $\mathcal{R}$ consist of those fractions $a/b \in \mathbb{Q}$ with b a multiple of elements of K. Then p prime is a unit iff $p \in \emptyset'$. □

Proposition (Schrieber 1985)

Despite this, the Euclidean domain $\mathcal{R}$ has a computable Euclidean function ϕ.

Proof.

Write $z = \frac{a}{b} \in \mathbb{Q}$ in lowest terms, and then define $\phi(z) = a$. □

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Corollary in Reverse mathematics

- Reverse mathematics seeks to calibrate the proof theoretical strength of theorems of mathematics in terms of comprehension axioms (saying that something of a certain definable complexity exists) in second order arithmetic.
- I'll assume you are familiar with this.
- This usually goes hand in hand with computability theory. For example, usually showing that $\emptyset'$ is coded in a computable setting aligns to ACA_0. (Of course there are counterexamples, one of my favourites is Kruskal's Theorem on wqo of finite trees. It is "computably true" but is certainly not provable in RCA_0.
- The result above suggest that the existence of a minimal functions implies ACA_0.

- A priori, there is no reason for the minimal Euclidean function ϕ (in $\mathcal{M}$) to satisfy $\phi = \phi_{\mathcal{R}}$.

Lemma (RCA_0)

Fix a Euclidean domain $\mathcal{R}$ and a non-minimal finitely-valued Euclidean function ϕ for $\mathcal{R}$. Let α be the least ordinal for which there is a $T \in R$ with $\alpha = \phi_{\mathcal{R}}(T) < \phi(T)$. Then (fixing such a T)

$$\hat{\phi}(z) = \begin{cases} \phi(z) & \text{if } z \neq T \\ \phi_{\mathcal{R}}(T) & \text{if } z = T \end{cases}$$

is a finitely-valued Euclidean function for $\mathcal{R}$ and satisfies $\hat{\phi} \neq \phi$.

Proof.

Since $\hat{\phi}(T) = \phi_{\mathcal{R}}(T) < \phi(T)$, it is immediate that $\hat{\phi} \neq \phi$. As α was chosen minimal, for any $A \in R$, there exists a $Q \in R$ with $\phi(A + QT) < \phi(T)$ as $\phi(A + QT) = \phi_{\mathcal{R}}(A + QT) < \phi_{\mathcal{R}}(T) = \alpha$. □

Theorem

(RCA_0*) The statement*

MEF: *every Euclidean domain has a minimal Euclidean function*

proves ACA_0.

Proof

Fixing a set X in the model, we show X' exists. We consider the X-computable ring whose units are $\Sigma_1^0(X)$-complete constructed by relativizing Schrieber's construction of a computable subring of the rationals whose units are intrinsically Σ_1^0-complete. As noted in the introduction, the (relativized) ring $\mathcal{R}$ has a computable Euclidean function, namely $\phi(a/b) = a$. Thus, it is a Euclidean domain, i.e., it has a Euclidean function in the model.

Consequently, by MEF, we may fix a minimal Euclidean function ϕ for $\mathcal{R}$ (so that $\phi \leq \phi'$ for all ϕ' in the model). We argue that $\phi = \phi_{\mathcal{R}}$. If not, the function $\hat{\phi}$ of the lemma above is in the model as $\hat{\phi} \equiv_T \phi \equiv_T \emptyset$ and is a Euclidean function for $\mathcal{R}$. But then we would have $\hat{\phi} < \phi$, contradiction the minimality of ϕ. Thus it must be the case that $\phi = \phi_{\mathcal{R}}$. As $\phi_{\mathcal{R}}$ computes X', the model must be closed under the Turing Jump.

How complicated?

- As the set R_0 is Σ^0_1, being the collection of units
- The set R_n is Π^0_{2n} for $0 < n < \mathbb{N}$.
- If $\phi_{\mathcal{R}}$ is finitely-valued, then $\phi_{\mathcal{R}}$ is $\emptyset^{(\omega)}$-computable.
- In the transfinite case the same for n on some path in Π^1_1.
- So proof theoretically upper bounds for the finite case is ACA_0^+ and transfinitely $\Pi^1_1 - CA_0$.

Our Theorems I

Theorem

There is a Euclidean computable domain $\mathcal{R}$ having no transfinitely-valued computable Euclidean function ϕ. Moreover, every transfinitely-valued Euclidean function ϕ for $\mathcal{R}$ computes $\emptyset'$.

Theorem

There is a computable Euclidean domain $\mathcal{R}$ for which the set R_1 is Π^0_2-complete.

Our Theorems II

Theorem

There is a Euclidean computable domain $\mathcal{R}$ for which there is no finitely-valued $\emptyset'$-computable Euclidean function ϕ.

Theorem

There is a computable Euclidean domain $\mathcal{R}$ having no computable finitely-valued Euclidean function but having a computable transfinitely-valued Euclidean function.

The underlying ring

- We use extensions of the ideas of Schrieber.

Definition (Schrieber 1985)

If K is a field and $\{X_i\}_{i\in\mathbb{N}}$ is a set of variables, denote by $K\left\langle X_i\right\rangle_{i\in\mathbb{N}}$ the commutative ring of reduced fractions p/q with $p, q \in K[X_i]_{i\in\mathbb{N}}$ and $X_i \not| \ q$ for all i.

- Thus, every element x of the commutative ring $K\left\langle X_i\right\rangle_{i\in\mathbb{N}}$ is the product of a monomial m and a unit u.

Why is this good?

Theorem (Schrieber 1985)

The function $\phi(x) = \phi(mu) := \deg(m)$*, where m is a monomial and u is a unit, is the least Euclidean function for* $K\langle X_i\rangle_{i\in\mathbb{N}}$*. In particular,* $K\langle X_i\rangle_{i\in\mathbb{N}}$ *is a Euclidean domain.*

- All the Euclidean domains we construct will be of the form $K\langle X_i\rangle_{i\in\mathbb{N}}$, where the field K is either $\mathbb{Q}$ or $\mathbb{Q}(Z_j)_{j\in\mathbb{N}}$, for some sets of formal variables $\{X_i\}_{i\in\mathbb{N}}$ and $\{Z_j\}_{j\in\mathbb{N}}$.

Useful observations

Proposition (Samuel 1971)

If $\mathcal{R}$ is an integral domain and $A, T \in R$ are nonzero, then $\phi_{\mathcal{R}}(T) \leq \phi_{\mathcal{R}}(AT)$.

Proof.

The function $\phi(T) := \min_{0 \neq A \in R} \phi_{\mathcal{R}}(AT)$ satisfies

$$\phi_{\mathcal{R}}(T) \leq \phi(T) \leq \phi_{\mathcal{R}}(1T) = \phi_{\mathcal{R}}(T)$$

as a consequence of the minimality of $\phi_{\mathcal{R}}$ and taking 1 for A. It is clear that ϕ, and thus $\phi_{\mathcal{R}}$, has the desired property. $\square$

Proposition

[Folklore] If $\mathcal{R}$ is an integral domain, $A, T \in R$ are nonzero, and A is a nonunit, then $\phi_{\mathcal{R}}(T) < \phi_{\mathcal{R}}(AT)$.

Proof.

Since A is a nonunit, it follows AT does not divide T. Thus

$$\min_{Q \in R}\{\phi_{\mathcal{R}}(T + QAT)\} < \phi_{\mathcal{R}}(AT)$$

by virtue of the definition of R_α. By the proposition above (as $1 + QA \neq 0$ for all $Q \in R$), we have $\min_{Q\in R}\{\phi_{\mathcal{R}}(T + QAT)\} = \min_{Q\in R}\{\phi_{\mathcal{R}}(T(1 + QA))\} \geq \phi_{\mathcal{R}}(T)$. □

Killing transfinite computable functions

- Instead of killing the ordinal based functions work on computable suborderings of $\mathbb{Q}$. Ie (partial) computable relations

$$E_\phi(x, y) := \{(x, y) \in R \times R : \phi(x) \leq \phi(y).$$

  This is justified because E_ϕ is computable if ϕ is a computable transfinitely-valued Euclidean function.

- Idea Have a $\{E_i\}_{i \in \mathbb{N}}$ of partial computable binary relations. The idea is to determine whether $E_i(X_i, Y_i)$ or $E_i(Y_i, X_i)$ (if either computation converges) and assure this cannot be the case by making either X_i a power of Y_i or Y_i a positive power of X_i.
- At stage s, we introduce terms X_s and Y_s. For each $i \leq s$, we check whether $E_i(X_i, Y_i)\downarrow = 1$ or $E_i(Y_i, X_i)\downarrow = 1$. If either has newly converged, we put $X_i = Y_i^s$ if $E(X_i, Y_i)\downarrow = 1$ and $Y_i = X_i^s$ otherwise. Finally, at each stage s, we continue the enumeration of the ring, working towards $\mathbb{Q}\langle X_i, Y_i \rangle_{i \in \mathbb{N}}$

Making R_1 Π^0_2 complete

- Fix a Π^0_2-complete set S and a computable predicate $P(i, s)$ so that $i \in S$ if and only if $\exists^\infty s\,[P(i, s)]$.
- Begin with $\mathbb{Q}$ and expressions $\{Z_i\}_{i\in\mathbb{N}}$.
- Make $Z_i = X_{i,j} Y_{i,j}$ beginning with $j = 0$
- When the Π^0_2 predicate looks correct make $Y_{i,j}$ a unit, and move to fresh variables $Z_i = X_{i,j+1} Y_{i,j+1}$
- If $i \in S$, Z_i has rank 1, as every Y is turned into a unit,
- If $i \notin S$, Z_i has rank 2 as it gets stuck on some $Z_{i,j} Y_{i,j}$.

Killing Δ^0_2 rank functions

- Rely on the fact that if *any* rank function has something of rank n, then the *minimal* one has rank $\leq n$.
- Want to kill $\phi_e(x) = \lim_s \phi_e(x, s)$ i.e. show it has no limit or the limit is wrong. Might as well take $\phi_e(x, s)$ as primitive recursive.
- Wlog we assume that if $\phi_e(x, s) \neq \phi_e(x, s + 1)$, then one of the two is 0.
- At stage s we compute $\phi_e(X_e, s)$. If this is $\neq \phi_e(X_e, s + 1)$, introduce $\phi_e(X_e, s + 1) + 1$ many new variables $X_{e,s,0}, X_{e,s,1}, \ldots, X_{e,s,\phi_e(X_e,s)}$ to the ring R and declare their product equal to X_e.

No computable finitely valued, but computable transfinite

- Analyze Schrieber's construction.
- At each stage s, we create a term X_s. For each $i \leq s$ for which $\phi_i(X_i)$ newly converges, we create a new variable Y_i and set $X_i = Y_i^{\phi_i(X_i)+1}$.
- The X_i are mapped to $\mathbb{N}$ (or $\mathbb{N}+1$ depending on your notation system). They then drop to an assigned value if $\phi_i(X_i)\downarrow$.

Outline

Questions

Question

What's the correct answer? How to control even R_2?

Remark

- This is the limit of the techniques for this kind of ring. The point is that everything is defined by the rank 1 elements, and hence $\emptyset''$ can figure out the ranks of everything.
- Similar remarks seem to apply to any introduction of *algebraic* elements.
- We could not seem to control the introduction of $+$, so controlling $Z + QX$ to not be a unit for all Q in the ring.
- Maybe the answer is yes, meaning that all domians are actually controlled by low level ranked sets.
- Same questions apply to rings in place of domains.

Questions, ctd

Question

Is there a computable Euclidean domain (ring) having (classically) no Euclidean function with range strictly less than ω_1^{CK}? What about the reverse math? finite valued has upper bound ACA_0^+, and transfinite Π^1_1-CA.

Question

What more can be said [classically] about (transfinite) Euclidean domains? What more can be said about computable (transfinite) Euclidean domains?

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Thank you