Resolute Sets and Initial Segment Complexity (Some Preliminary Results)

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Plan

- Introduce some new concepts relating to reals with low initial segment complexity.
- Relate them to known degree classes.
- Relate them to known notions of lowness for initial segment complexity.
- Our idea is that lowness concepts like K-trivialty, Schnorr triviality, (non-)complexity, have been so successful that perhaps we should seek relaxations of these concepts and if they are interesting.
- Here we relax K-triviality.
- Recall that we had no idea at the beginning that K-triviality would be so interesting.

Initial Segment Complexity

- Theme: $Q(A \upharpoonright n)$ for Kolmogorov complexities Q give insight into the relative randomness of Q, and interplay with computational power.
- For example Schnorr : A is random iff $K(A \upharpoonright n) \geq^{+} n$.
- Variations such as effective dimensions.
- Variations such as complex and autocomplex via orders and PA degrees etc.
- Recent Barmpalias, Hölzl, Lewis, Merkle "Computably enumerable sets that resemble Ω"
- Barmplais survey: "Measures of Randomness"

Initial Segment Lowness

- X is low for random means Y random iff Y is X-random.
- X is K-trivial iff for all n, $K(X \upharpoonright n) \leq K(n) + c$, for all n.

Theorem (Chaitin)

X is computable iff for all n, $C(X \upharpoonright n) \leq C(n) + c$ for all n. That is, C-trivial=computable.

A trivial story

- $A = \{\langle e, n\rangle : \exists s(W_{e,s} \cap A_s = \emptyset \wedge \langle e, n\rangle \in W_{e,s}$ and $\wedge \sum_{\langle e,n\rangle \leq j \leq s} 2^{-K(j)[s]} < 2^{-(e+2)})\}$.
- This A is K-trivial, non-computable.

Theorem (Downey, Hirschfeldt, Nies, Stephan)

If A is K-trivial then A solves Post's problem. That is, a one line description of a Turing incomplete set.

Theorem (Nies)

1. *A is K-trivial iff*
2. *A is low for randomness iff*
3. *$A \leq X$ for some A-random X (with Hirschfeldt)*
4. *A is low for K meaning $K^A = K$.*

Why do we feel they are important?

- K-trivials are ubiquitous with maybe 15 characterizations.
- Bring to the fore themes of traceing.
- Variations have been used to solve longstanding questions in computability theory and in logic.
- Newer variations with weaker initial segment properties.
- For example low for K up to g (Herbert, Hirschfeldt-Weber) $K(n) \leq K^A(n) + g(n)$ for (Δ_2^0) order g.
- First fundamentally enumerable property. No forcing.

Some old and new relatives

Definition

- We will say that A is weakly K-resolute iff for all (computable) orders h, $K(A \upharpoonright n) \geq^+ K(A \upharpoonright h(n))$.
- Similarly for C-.

Definition (Franklin, Greenberg, Stephan, Wu)

A is anti-complex iff for all orders h, $C(A \upharpoonright h(n)) \leq n$.

Definition (Lathrop and Lutz)

A is called ultracompressible iff for all orders g, $K(A \upharpoonright n) \leq^+ K(n) + g(n)$.

Definition

A is Kummer anti-complex iff for all orders g, $C(A \upharpoonright n) \leq C(n) + g(n)$.

- Note that in the first two definitions, think of h as fast growing, and the second two g as very slow growing.

- A natural way to construct sets with low initial segment complexity is to "thin them out".
- The f-shift of A, A_f, results by $f(n) \in A_f$ iff $n \in A$.

Definition

A is called Q-resolute ($Q \in \{C, K\}$) iff for all computable orders f with $f(n) \geq n$, and all m,

$$Q(A \upharpoonright m) =^{+} Q(A_f \upharpoonright m).$$

Lemma

1. *If A is Q-resolute then A is weakly Q-resolute.*
2. *If A is (weakly) Q-resolute and $B \equiv_Q A$ then B is (weakly) K-resolute.*
3. *If A is weakly Q-resolute and $B \leq_{wtt} A$, then $B \leq_Q A$.*

- Proof of 3. Let $B \leq_{wtt} A$, and hence $Q(B \upharpoonright n) \leq^+ Q(A \upharpoonright f(n))$ for some computable f. But A if A is weakly Q-resolute then $Q(A \upharpoonright n) \geq^+ Q(A \upharpoonright f(n))$.
- Proof of 2. Let f be the computable shift. Then $Q(B \upharpoonright n) \leq^+ Q(A \upharpoonright n) \leq^+ Q(A_f \upharpoonright n) \leq^+ Q(B_f \upharpoonright n)$. The weak case is similar.
- Proof of 1. Suppose that A is Q-resolute. Then $A \equiv_Q A_f$. Hence $Q(A \upharpoonright n) \geq^+ Q(A_f \upharpoonright f(n)) =^+ Q(A \upharpoonright f(n))$.

Anticomplex

Lemma

If A is weakly K-resolute then A is anticomplex.

Reason: $n \geq C(A \upharpoonright n) \geq^+ C(A \upharpoonright h(n))$.

Corollary

If A is weakly K resolute the $A \leq_{wtt} X$ for some Schnorr trivial set X.

REcall: A is Schnorr trivial iff for all computable measure machines M, there is a computable measure machine $\hat{M}$ such that for all n, $K_{\hat{M}}(A \upharpoonright n) \leq^+ K_M(n)$.
Recall: computable measure machine has $\mu(\mathrm{dom}(M)) = r$ for a computable real r.
Recall: X is Schnorr random iff for all such machines M, $K_M(X \upharpoonright n) \geq^+ n$. (Downey-Griffiths)

Basic Properties

Theorem

1. *Every K-resolute set X is ultracompressible.*
2. *Every C-resolute set is Kummer anti-complex.*
3. *The converse is not true.*

- 1 and 2. Let g be an order (slow growing) and choose and order f so that $f(g(n)) = n$. The $Q(X_f \upharpoonright n) \leq^+ Q(n) + Q(g(n))$. This is because X_f has at most $X_f \upharpoonright n$ has at most at most $g(n)$ nonzero bits. But X is Q-resolute. Hence $Q(X \upharpoonright n) \leq^+ Q(n) + Q(g(n))$. xiThis means $Q(X \upharpoonright n) \leq^+ Q(n) + g(n)$ and hence X is ultracompressible/Kummer-anticomplex.
- 3. Next frame

Array Computable Degrees

Definition (Downey, Jocksuch and Stob)

We say that $\mathbf{a}$ is array computable iff for some order h with $h(0) \geq 1$, all $g \leq_T \mathbf{a}$ is $h(n)$-c.e. (ie $g(n) = \lim_s g(n, s)$ with $|\{s \mid g(n, s+1) \neq g(n, s)\}| \leq h(n)$.

Strictly speaking, not a definition, but a theorem. It can be shown that any order will do, proof in Downey-Hirschfeldt or Nies for instance.

Theorem

If $\mathbf{a}$ is is array computable then it is completely ultracompressible and Kummer anti-complex. (and anticomplex)

Theorem (Kummer)

If $\mathbf{a}$ is array noncomputable then it contains a c.e. set which is neither.

- Suppose that X is array computable. let f be an order. Want to show $Q(A \upharpoonright n) \leq^+ Q(n) + f(n)$. Let $Q = K$. Let $g(n) = \log\log f(n)$, say. Then $X \upharpoonright n$ is g-traceable, as X is array computable, that is $X \upharpoonright n \in V_n$ with $|V_n| \leq g(n)$.
- Consuder then the algorithm M which on input σ simulates $U(\sigma)$, and if it halts we will define $M(\sigma\nu)$ potentially for all $\nu \in 2^{\log|V_{|U(\sigma)|}|}$.
- The idea is that when τ occurs in $V_{|U(\sigma)|}$, we use the next leaf ν of $2^{\log|V_{|U(\sigma)|}|}$ whn another string occurs in $V_{|U(\sigma)|}$. This works.
- The C case is easier.

- Kummer showed that if **a** is array noncomputable then it contains A such that $\exists^\infty n C(A \upharpoonright n) \geq^+ 2\log n$.

Theorem

There is an array computable X which is not weakly Q-resolute.

$R_e : \Phi_e^X$ total implies Φ_e^X is id-ce.
Placed as nodes τ on the tree. If $\tau\infty$ then we must build a witness.
We build h such that for all e, exists n, $K(X \upharpoonright n) \not\geq K(X \upharpoonright h(n)) - e$.
Nodes on the tree with guesses. Action: has a bit of measure, $2^{-e(\sigma)}$ which it lowers $K(A_s \upharpoonright n)$, freezes that and then challenges the opponent to match on $K_s(A_s \upharpoonright h(n))$ and we get to change $A_t[n+1, h(n)]$, etc. The point is to choose $n >> e$.

Shifts

- Note that if A is K-trivial then A is K-resolute.
- However, there are degrees which are completely K-resolute and not K-trivial.
- We begin with constructions of K-resolute sets.

Lemma

Let f be an order. Then every c.e. m-degree contains a set X such that $X \equiv_K X_f$.

- Given Y define $g(n) = f^n(0)$ and $X = \{g(n) \mid n \in Y\}$. $X \equiv_m Y$. Clearly $Q(X \upharpoonright n) \geq^+ X_f$. Conversely, given $X_f \upharpoonright n$, note that all the bits are 0 except perhaps the first $k+1$ members, $t_0, \ldots, t_k$, of $F = \{f^i(0) \mid i \in \omega\}$ below n. Note that $X(t_i) = X_f(t_{i+1})$ for $i < k$, thus to describe $X \upharpoonright n$ we only need $X_f \upharpoonright n$ and the value of $X(t_k)$. That is, $X \leq_{rK} X_f$.

Theorem

Every high c.e. degree contains a K-resolute set.

Theorem

Suppose that S is $CEA(\emptyset')$. Then there is a c.e. K-resolute set X with $X' \equiv_T S$.

- The idea for the high degree theorem is to use the m-degree results and use highness to guess totality for functions. That is, iterate the methodology making A "very sparse".
- The idea for the other one is similar, noting that enumeration is not actually necessary except for coding, but sparsification is. The "noise" is computable. This uses a tree.

Definition

- Given X, Y and g, let $X \otimes_g Y$ be the set obtained by replacing the $g(i)$-th bit of X by the i-th bit of Y.
- If g is increasing, and E is computable we say A is g-sparse if $A = E \otimes_f X$ for some X and computable f, with $g(f(i)) < f(i+1)$. A is sparse if it is g-sparse for all computable g.

Theorem

- *Every sparse set is Q-resolute.*
- *Every sparse hyperimmune set is high.*
- *Sparse sets form a meager class.*
- *There is a special Π^0_1 class of sparse (and hence resolute) sets.*

They are all pretty straightforward, the last one uses the expected construction. It also follows by the work of Ian Herbert and Lempp, Miller, Ng, Turetsky, Weber on low for dimension reals.

Completely K-resolute degrees

Definition (Ladner and Sasso)

X is called (strongly) contiguous if for all (not necessarily c.e.) $Y \equiv_T X$, $Y \equiv_{wtt} X$.

Theorem (Ng)

X is contiguous iff X is strongly contiguous.

Theorem

If X is strongly contiguous then X is completely Q-resolute, and for all c.e. $Y \equiv_T X$, $Y \equiv_Q X$.

Theorem

There are strongly contiguous degrees where the above is true removing the "c.e." before the Y.

Direct construction for the last one. Probably true for all contiguous.

Corollary

There are degrees $\mathbf{a}$ *containing only one* Q*-degree and are not* K*-trivial.*

That is because there are properly low_2, and superlow-non-K-trivial contiguous degrees, for instance. Note also the Herbert et al gives this, but there ones are all low, as low for information is GL_1.

Theorem

There exist degrees $\mathbf{a}$ *containing only one* Q*-degree and are neither contiguous nor* K*-trivial. They are completely* Q*-resolute.*

The proof is that Q allows for a small number of errors, but enough to make $A \not\leq_{wtt} B$, with $A \equiv_T B$. Think of $\Gamma^B = A$ where the use is allowed to change but only a small number of times, like a slowly growing order. This makes a $\leq_{rK}$ reduction.

An aside

Theorem

*If **a** is made so that all elements are sparse, then thay are all Schnorr trivial.*

Suppose that U is a computable measure machine. The proof is to think about the function $f(n)$ that computes where $\mu(\mathrm{dom}(U)) \leq 2^{-2^{n+1}}$, say. Use this and the sparseness to calculate M witnessing that $A \in \mathbf{a}$ is Schnorr trivial.

Notice it is enough to be narrow (Binns). Recall: A is narrow iff for all orders h, there is a Π^0_1 class P containing A with the width at level n $\leq h(n)$.

The construction of such degrees is essentially the contiguous one. Is that sufficient?

Back to the contiguous story

Theorem

Suppose that $\mathbf{a}$ *is wtt-bottomed. Then the bottom* A *is (weakly)* K*-resolute.*

Suppose that A is not h-resolute. Thus for all c there are infinitely many n with $K(A \upharpoonright n) \not\geq K(A \upharpoonright h(n))$. We build $B \equiv_T A$ to meet the requirements. The reduction $\Delta^B = A$ is by markers $\delta(n, s)$ The other one is simple permitting. $\delta(n, s)$ move as usual via some kind of kicking. We we move $\delta(n, s)$ only if $A \upharpoonright h(n)[s]$ changes, and *must* move it when $A \upharpoonright n$ changes.
Let Γ_e denote the e-wtt reduction. We meet

$$R_e : \Gamma^B \neq A.$$

When nothing else happens and n enters A_s we simply put $\delta(n, s)$ into B_s and kick $\delta(m, s)$ for $m \geq n$, to big numbers.

- R_e asserts control of n if $\ell(e, s) > h(n)$ at stage s for the first time. R_e asserts control of $A_s \upharpoonright h(n)$.
- We make a description of $A_s \upharpoonright h(n)$ matching $A_s \upharpoonright n$ with weighting $2^{-(e+2)}$. Such descriptions will be enumerated at e-expansionry stages.
- If $A_t \upharpoonright h(n)$ changes after this for the first time we will enumerate $\delta(n, s)$ into B_{s+1} (rather than perhaps some $\gamma(m, s)$ for $m \in [n+1, h(n)]$.)
- The key fact is that *if* $\gamma_e^B = A$ then after the nest expansion stage, if $A \upharpoonright h(n)$ changes, then it must be that some $i \leq n$ enters A also since else $B \upharpoonright \gamma(n, s)$ would not change.
- The result follows by finite injury, since if any R_e acted confinally, then it would demonstrate that $K(A \upharpoonright n) \geq^+ K(A \upharpoonright h(n))$.

Not every c.e. degree

Theorem

There exists a c.e. degree $\mathbf{a}$ *containing no ultracompressible set. They can be, e.g. high*$_2$ *or low.*

We build A to meet $R_e : \Phi_e^A = W_e \wedge \Gamma_e^{W_e} = A$ implies W_e is not weakly resolute at order h_e.
h_e is built at the mother node τ. Worker nodes σ try to demonstrate that $K(W \upharpoonright n) \not\leq K(W \upharpoonright n) + h(n) + e$. Again packets of measure at σ, and local control via links back at τ with enough layers to kill e. Put them in reverse order.

Thank You