Relative Enumerability vs Computability

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Programme

- Two basic concepts in recursion/computability theory are that of effective enumerability and computability.
- Much of classical computability theory, and particularly degree theory has focused upon classifying sets under relative *computability.*
- Here, computably enumerable sets have a special place to be sure, but a lot of insight can be gleaned from studying relative *enumerability.*

- I will speak on an eclectic group of papers based around studying relative enumerability. For example:
- A Δ_2 set that is barely Σ_2, (with G. LaForte and S. Lempp)
- Completing pseudojump operators, (with R. Coles, C. Jockusch and G. Laforte)
- Every set has a minimal jump enumeration, (with R. Coles)

Motivation

- Two papers mainly
- Lachlan, Uniform enumeration operations, JSL, 1975.
- Jockusch and Shore, Psuedo-jump operators, I, the r.e. case, TAMS, 1983.
- One of the ideas is to try to study possible natural solutions to Post's problem, via definable computable enumerable operators.
- This lead to questions about what kinds of things are compatible with general enumeration operators.

Lachlan's Paper

- Lachlan was concerned with the longstanding question of Sacks: *Is there a degree invariant solution to Post's Problem. That is there an r.e. operator W such that for all sets A, $A <_T W(A) <_T A'$ and for all sets $A \equiv_T B$, $W(A) \equiv_T W(B)$.*
- Various partial solutions. Strongest results in Slaman-Steel "Definable functions on the degrees". For instance, there is no solution with $A \leq_T B$ implies $W(A) \leq_T W(B)$.
- Lachlan made the first real result in this area. Call a solution uniformly

degree invariant if there is are function f, g such that for all e, i,

$$\{e\}^A = B \text{ and } \{i\}^B = A \text{ implies}$$

$$W(A) \equiv_T W(B) \text{ via } g(e,i), f(e,i).$$

Lachlan's Theorem

- There is no uniformly degree invariant solution to Post's Problem.

- Proof. *Suppose that* W *is UE. We claim that either* $W(\mathbf{0}) = \mathbf{0}'$, *or* $\exists \boldsymbol{d} W(\boldsymbol{d}) \leq \boldsymbol{d}$.

  (Actually Lachlan proves something stronger than this. He shows that if $\boldsymbol{b}$ is the degree of the unformity function f, then either $W(\boldsymbol{b}) \geq \boldsymbol{b}'$ or $W(W(\boldsymbol{b})) \leq W(\boldsymbol{b})$)

  We attempt to build a reduction $\Gamma^{W(\emptyset)} = K$. Failing that we argue that there is a D with $W(D) \leq_T D$.

The 0-*module* This module tries to define $W(D)(0)$ from D.

At stage s, suppose that all $m' < m$ *look bad.* (to be described) The we begin to work on m. This goes as follows. We define $A^0_m = \{\langle 0, m\rangle\}$. By the recursion theorem, we compute an index of procedures Φ_e, Φ_i witnessing $\emptyset \equiv_T A^0_m$.

- Now if we assume the strong uniformity of Lachlan, we can uniformly compute an index $n = n(i, j)$ witnessing $\Lambda_n^{W(\emptyset)} = W(A^0_m)$.

At the stage t that we see a

computation

$$\Lambda_n^{W(\emptyset)}(0) = W(A_m^0)(0)[t],$$

we enumerate a Γ_0 axiom for m, with use exceeding t. (saying $K_t(m) = K(m)$.) We then begin to work on $\langle 0, m+1 \rangle$. If we see $W(\emptyset)$ $^{-}$ $\lambda_n(0)$ change then we get a free clear on $\Gamma_0(k)$ for $k \geq m$.

We say that m *looks good* at stage v if $\Gamma_0^{W(\emptyset)}(m) \downarrow [v]$ and m enters K at stage v, so that the answer is wrong.

If m looks good at atage s we begin the 1-module. This module tries to define $W(D)(0)$ from D. That is, we would construct $A_k^{0,v,m}$ for $k > m$, as above, as $\{\langle 0, m \rangle, \langle v+1, k \rangle\}$, freezing the 0 module unless the situation changes there. (I.e. m is corrected by $W(\emptyset)$.) If the situation changes then the $A_k^{0,v,m}$ is thrown away forever, but that's okay since later we will work with $A_{k'}^{0,v',m'}$ for larger v'. We can ask that v' be so much larger that the algorithm below works.

Note that we could well ask for the 1-module that

$$\Gamma_{n(1,0,v,m)}^{W(\emptyset)}(z) = A_k^{0,v,m}(z) \text{ all } z \leq v.$$

Continue inductively.

There are 2 cases.

Case 1. For some i, there is no permanently good m. Then clearly $K \leq_T W(\emptyset)$ via some reduction

$\Gamma_{n(\text{lots of parameters})}$.

Case 2. Otherwise. Then construct D via the good m's inductively. Initially, D will contain the least $\langle 0, m\rangle$. It will then contain $\langle v, k\rangle$ for some v which is more or less the stage where m became good for 0, etc.

The key point to notice is that D can tell us the stage where longer and longer initial segments of $W(D)$ stop changing, since the elements are specifically chosen to be $\Lambda_{n(\text{parameters})}$-use large. Also we only need to list the principal function of D to then resurrect $W(D)$. Λ

- Various extensions of Lachlan's result and the like for the uniform case, with various determinacy considerations. Only Slaman-Steel and the result below for the general question without uniformity. (Namely Slaman-Steel prove that if f is degree invariant and $f(\boldsymbol{a}) < \boldsymbol{a}$ on a cone then f is constant up to degree on a cone. There is "press down" function.)

- with Shore, *Procedings of the American Mathematical Society.* Vol. 125 (1997), 3033-3037. This provides strong limitations on the possibilities for a degree invariant solution.

- Theorem (with Shore) If W is degree

invariant and at least the identity (i.e. $W(A) \leq_T A$) on a cone, then either $W(A)'' \equiv_T A''$ on a cone, or $W(A)''' \equiv_T A'''$ on a cone.

- The proof is basically an index set calculation plus an application of Steel (1982, JSL). One considers the set $V(A) = \{\langle i, j, m, n\rangle :$ $\{i\}^A$ total $\wedge \{j\}^{\{i\}^A} = A \wedge \{m\}^{W(A)} = W(\{i\}^A) \wedge \{n\}^{\{m\}^{W(A)}} = W(A)\}$.
- This encodes the uniformities of W.
- $V(A) \leq_T W(A)''$ by counting quantifiers.
- Coding shows that $W(A)'' \leq_T V(A)$.
- Hence, $A'' \leq_T V(A) \equiv_T$ $W(A)''$(uniformly) $\leq A'''$.

- Then one shows that $V \equiv_T W''$ is uniformly degree invariant, giving the result.

Basics

- A is called computably enumerable if A is the range of a computable function.
- $\{W_e : e \in \mathrm{N}\}$ the partial ordering of computably enumerable sets called $\mathcal{E}$.
- Reducibility: $A \leq B$ iff we can compute membership of A using B as "read only memory." Different access mechanisms give differing reducibi lities. Turing ($\leq_T$) and $m-$ where $A \leq_m B$ iff there is a computable function f such that $x \in A$ iff $f(x) \in B$.

- Reducibilities divide the world into equivalence classes called degrees. $\mathcal{R}$ denotes the computably enumerable degrees.

Question 1 How do $\mathcal{E}$ and $\mathcal{R}$ interact?

Post's Problem

- $\mathbf{o}$ degree of the computable sets; $\mathbf{o}'$ degree of the halting problem $K_0 = \{\langle x, y\rangle : x \in W_y\}$.
- All known degrees were either $\mathbf{o}$ or $\mathbf{o}'$. (cf. Rice's Theorem.)

Question 2 (Post's Problem) Any computably enumerable degrees $\boldsymbol{a}$ with $\mathbf{o} < \boldsymbol{a} < \mathbf{o}'$?

- Related. $\mathcal{E}^A$ denotes the sets computably enumerable relative to A.

Theorem 3 (Hammond, 1990) $\mathcal{E}^A \cong_{eff} \mathcal{E}^B$ *iff* $A' \equiv_T B'$.

(Proof uses automorphism machinery for "only if" and A simple calculation plus a representation trick for the other direction.)

Theorem 4 Hammond and Harrington, ta *If* $A'' \equiv_T B''$ *then* $\mathcal{E}^A \cong \mathcal{E}^B$.

- Recently with Coles and Laforte, we began to wonder about what exactly it means for a set and degree to be enumerable relative to some set. Some motivation provided by Jockusch who asked if the basic JS pesudo-jump result could be combined with e.g. cone avoidance. (The view would be that understand general properties of CE operators.)
- The basic Jockusch-Shore theorem : For all e, nontrivial, (that is, with $W_e^Q \not\leq_T Q$ for all c.e. Q)there is an incomplete A such that $W_e^A \oplus A \equiv_T A$.
- The proof is the following. There is a requirement R_e which says to code

$e \in K$ into $A \oplus W_e^A$. There is a negative one which is a Friedberg strategy saying $\{e\}^A \neq B$. One codes by coding markers. Call these $\gamma(e, s)$.

- One promises that if e enters K at stage s then $\gamma(e, s)$ enters $A \oplus W_e^A$.

- To meet the $\{e\}^A \neq B$ requirements, when we see the length of agreement exceed x and we also see W_e^A change on x yet A remains the same on the use of $\{e\}^A(y)[s]$, where $y = y(x, s)$ is the x-th follower, then we move the $\gamma(j, s)$ markers for all $j \geq x$. (We can do this for any $x \geq e$. (priority)) Then we put y into B.

- This succeeds unless some number

$x' < x$ enters A after this. If we actually fail then there is a reduction from A to W_e^A.

- The argument combines with most Friedberg strategies. For example,
- Theorem (Coles-Downey-Jockusch-Laforte) For any operator nontrivial on the d.r.e sets, there is an A of properly d.r.e. degree such that $A \oplus W_e^A \equiv_T A$.
- However Sacks strategies seem bad. The problem is to clear the use once the length os agreement is above x. But this is a move of the opponent, and he has no reason to synchronize with us.
- The obstacle can be turned around.
- Theorem (Coles-Downey-Jockusch-Laforte)

These is an operator V nontrivial on the ce sets and a ce set $C \not\equiv_T \emptyset$ such that for any ce set A with $A \oplus V^A \equiv_T K$, $C \leq_T A$.

Questions

- Open if the operator V can be made nontrivial for *all* sets. Conjecture it can be.
- Any natural operators? One possibility was the superhigh degrees where A is superhigh if $A' \equiv_{tt} \emptyset''$.
- (DCL) There are minimal pairs of superhighs.
- This lead to a complex of questions about classifying Δ_2^0 sets and strong jumps. Still work in progress
- Example. (DCL) There is a Σ_2^0 set A such that for all ce sets B

$B' \not\leq_{tt} A \oplus K$. (Friedberg's proof shows that this all works in the for sets below K, but this says that Sacks inversion fails.)

- Example (DCL) One can then try to study Δ^0_2 sets via "strong jumps" . For instance consider $\{e : [e]^A(e) \downarrow\}$ where $[e]$ denotes the e-th wtt reduction is the *wtt*-jump. One gets a hierarchy in $\mathbf{o}'$, and must refine the Kleene ordinal notations.

Barely Σ_2^0 sets

- All of this lead us to consider how hard is it to enumerate a set from another. The question considered by Lempp and Laforte and I is how hard is it to enumerate a set from a ce set.
- Theorem (DLL) There is a set of barely Σ_2^0 degree below K. Namely there is a 3-ce set A such that if W and V are ce sets and $W^V \equiv_T A$ then $W \equiv_T K$.

Jump Enumerations

- What can be said about $C(A) : \{\deg(B) : A \text{ is c.e. in } B\}$?
- This set seems very badly behaved. For instance,
- (CD) There is a non-ce set A ce in a minimal pair.
- However, when the jump is included, we get the following amazing (to me) result
- Theorem (CD) For any set A the set $C'(A) : \{\deg(B') : A \text{ is c.e. in } B\}$ *always* has a least member.

- Every set has a minimal *jump* enumeration.