Post’s Programme Revisited

Rod Downey
Victoria University
Wellington, New Zealand
Supported by the Marsden Fund and Alexander von Humboldt Stiftung

Joint work with Klaus Ambos-Spies and Martin Monath

Asian Logic Conference, June 2019
Post (1944) suggested “thinness” properties of complements of sets might solve his problem of finding a Turing incomplete c.e. set.

We know that it in its original form this proposal fails since

1. By e.g. Martin 1965, there are complete maximal c.e. sets (recall $M$ is maximal if it is a co-atom in $L^*$.)
2. Soare (1975) showed that all maximal sets are automorphic so no “extra” property will suffice to guarantee incompleteness.
3. By Cholak, Downey, Stob (1992) no property of $\overline{M}$ alone can guarantee incompleteness.

On the other hand, Harrington and Soare (1991) showed that there is a definable property $Q$ such that if $Q(A)$ then $A$ is incomplete.
On the other hand, there are fascinating interactions with strong reducibilities.

Simple sets solve Post’s problem for $m$-degrees. (Post, 1944)

($\eta$-) Maximal sets have minimal $m$-degrees. (Ershov 1971, Lachlan 1972).

Simple sets are not btt-cuppable. (Downey, 2000)

Dense simple sets are not tt-cuppable. (Kummer and Schaefer, 2007)

Hypersimple sets (recall $A$ is h-simple iff it meets all infinite strong arrays) are wtt-incomplete (Friedberg and Rogers, 1959) and indeed not wtt-cuppable (Downey and Jockusch, 1987).
Recall that $A$ is totally $\omega$-c.a. iff for all functions $f \leq_T A$, $f$ is $\omega$-c.a.. This means that for each $f$ there is a computable approximation $f(x) = \lim_s f(x, s)$ and a computable $h$, $|\{s \mid f(x, s + 1) \neq f(x, s)\}| < h(x)$.

**Theorem (Barmpalias, Downey and Greenberg 2010)**

A c.e. $a$ is is totally $\omega$-c.a. iff every (c.e.) set in $a$ is wtt-reducible to a $h$-simple c.e. set.

The totally $\omega$-c.a. degrees have turned out to be really interesting, with many characterizations, and systematizing the combinatorics of a number of constructions. They are definable in the c.e. degrees, etc. So they have natural independent interest.
Maximal sets

- A preliminary result.
- recall that $A$ is superlow if $A' \equiv_{tt} \emptyset'$.
- Equivalently, for c.e. $A$, there is a computable $h$ such that $J^A(e)$ is $h$-c.a. Here $J^A$ is the universal partial $A$-computable function.

**Theorem (Ambos-Spies, D, Monath)**

If $A$ is c.e. and superlow, then there is a maximal set $M$ with $A \leq_{wtt} M$. Indeed $A \leq_{ibT} M$. 
Proof

- **Requirements**: $A \leq_{wtt} M$.
  
  \( R_e : W_e \cap \overline{M} \text{ infinite implies } W_e \supseteq^* \overline{M}. \)
  
  \( N_e : \lim_s m_{e,s} = m_e \text{ exists where } m_{0,s} < m_{1,s} \ldots \text{ lists } \overline{M}. \)

- **Standard maximal set construction maximizes e-states**. The e-state of $z \in \overline{M}$ is \( \{j \leq e \mid z \in W_{j,s}\} \), a string.

- **Standard maximal set construction tries to put almost all of \( \overline{M} \)** into the same e-state.
If $\Gamma^M = A$ is the wtt-reduction, then if some $x \in A_{s+1} - A_s$, we need to change $M \upharpoonright \gamma(x)$.

This can only be done if there is some element in $\overline{M}_s$ which is below $\gamma(x)$ which can be put into $M - M_s$. We refer to this as coding.

The e-state machinery puts lots of elements from $\overline{M}_t$ into $M_{t+1}$, so we must be careful to leave enough elements to cope with this coding. For example, even for the 0-state, i.e. for a single requirement, all of the elements in $W_{0,s} \cap \overline{M}_s$ might be bigger than $\gamma(0)$ so we could not code 0 entering $A$.

We only do this e-state action when we can use the jump computation to tell us that we are safe, and few elements will enter $A$. 
One requirement $R_0$

- We have some part of the jump we control using the recursion theorem say $J^A(\langle g(0), j \rangle | j \in \mathbb{N})$. Jump computations on this have an approximation (known in advance) $J^A(\langle g(0), j \rangle)[s]$ changing at most $h(\langle g(0), j \rangle)$ many times.
- Anticipating things somewhat we write that as $f^A_0(j)[s]$ and the mind change number $n(0,j)$.
- For a single requirement, set aside a block of elements $B_1$ with at least $n(0,0) + 1$ many elements which we don’t raise the 0-state of.
- When we see at least $n(0,1) + 1$ many elements ($> \max B_1$) in the high state, the plan is to use these for $B_2$ and we define $f^A_0(0)[s] \downarrow$, with huge use $s$, and wait for the approximation to be confirmed. The interval $I_1 = [\max B_1, s]$. (We no longer code below $\max B_1$.)
- Now we declare that $B_1$ will code $I_1$, and dump all elements not in $B_1 \sqcup B_2$ below $s_0 = s$ into $M_{s+1} - M_s$.
- Each time some element enters $A$ between $\max B_1$ and $\max I_1$ we redefine the jump on argument 0 with use $s_0$ and on recovery we code all such this using an element in $B_1$. 
We repeat the process planning to use $B_2$ to code some interval $[\max l_1, s_1] = l_2$.

Now the coding is in the high state.

That is, we'll wait for at least $n(0, 2) + 1$ many elements in the high state (and these must be $> \max l_1$) for block $B_3$, etc.

So block $B_n$ looks after $I_n$.

For more than one states, this is done inductively. First note that $B_1$ might never be used as maybe the 0-state is not well-resided in $\overline{M}$. So there would need to be a version of “$B_1$ for $R_1$ guessing $R_0$ is inactive” and working in the same way as above for $R_0$, and getting re-stated each time the 0 state acts up.

There would be a version of $R_1$ guessing $R_0$ is infinitely often active. This demands that $B_2$ above would have a part of its block devoted to $B_1^{\infty \infty}$. It is only used when we see enough elements in state $\infty \infty \infty$ and these are verified by a part of the jump we build for this guess $f^{A \infty \infty}_\infty(1)$. 
Observations

- All of the definitions above are \textit{wtt-jump} computations, in that the use never changes once defined.
- That is, the proof only needs a new concept: We say that $A$ is \textit{wtt-sl} iff there is a uniformly computable approximation $g(x, s)$ of $\hat{J}^A(x)$ with $h(x)$ many mind changes where this is the partial wtt-jump. (i.e. $(\Phi_e, \varphi_e)$.) That is, the value of the the wtt-jump relative to $A$ is $\leq_{wtt} \emptyset'$.

\textbf{Theorem (ADM)}

$A \leq_{ibT} M$ for $M$ maximal if $A$ is c.e. and wtt-sl.

\textbf{Theorem (ADM)}

There are wtt-sl Turing complete c.e. sets.
A Characterization

- A modified version works for the following. $A$ is eventually uniformly $\text{wtt-array computable}$ iff there are computable functions $k, g$ and $h$, with $k(n, s) \leq k(n, s + 1) \leq 1$, $\lim_s k(n, s)$ exists for all $n$ such that
  1. $\lim_s g(x, s) = \hat{J}^A(x)$ for all $x$.
  2. $k(n, s) \leq k(n, s + 1) \leq 1$, $\lim_s k(n, s)$ exists for all $n$
  3. If $k(x, s) = 1$ then $g(x, t)$ has at most $h(x)$ further mind changes for $t > s$ (hence wlog $k(x, t) = 1$ for all $t > s$).
  4. If $\hat{J}^A(\langle e, y \rangle) \downarrow$ for all $y$, then for almost all $s$, $\lim_s k(\langle e, y \rangle, s) = 1$.

- More or less the same proof gives one direction of:

**Theorem (ADM)**

For a c.e. $A$, $A \leq_{ibT} M$ iff $A \leq_{wtt} M$ for $M$ maximal iff $A$ is eventually uniformly $\text{wtt-array computable.}$
The Other Direction

- Suppose that $\Gamma^M = A$, and $A$ not eventually uniformly wtt-ac. Choose $h(n) = 2^n$ for simplicity.
- Let $\ell(s) = \max\{z | \leq y\Gamma^M \upharpoonright z = A \upharpoonright z[s]\}$. Our assumptions about the enumerations of $A$, $M$ and the jump are that once $\ell(s) > n$, if $A_{s+1}(n) \neq A_s(n)$, $M_{s+1} \upharpoonright \gamma(n) \neq M_s \upharpoonright \gamma(n)$.
- We know that for any approximation for the wtt-jump $g(\langle e, x \rangle, s)$ there will be total $\hat{\Phi}^A_e(x)$ changing more than $h(\langle e, x \rangle)$ many times on infinitely many $x$.
- Initially have $k(\langle e, x \rangle, s) = 0$ for all $e, x$ and keep it like this unless told otherwise for $t > s$. The approximation $g(\langle e, x \rangle, s)$ is the natural one observing halting computations.
- For each $e$ carry out the following construction. When we see $\hat{\Phi}_e^A(0) \downarrow [s]$ let $I_0^e = [0, \gamma(\phi_e(0)))$. Wlog the $\hat{\Phi}_e^A$ are monotone, and we can continue $I_1^e = [\gamma(\phi_e(0)), \gamma(\phi_e(1)))$, defining a sequence of disjoint $e$-intervals \{ $I_n^e \mid n \in \mathbb{N}$ \}. If ever we see $|M_s \cap [0, \max I_n^e]| < 2^{\langle e, n \rangle}$, define $k(\langle e, n \rangle, s) = 1$. (Note that this ensures $k(\langle e, n \rangle, t) = 1$ for all $t > s$, also).
The assumption is that $\Gamma^M = A$, and $M$ is maximal.

Notice that if we define $k(\langle e, n \rangle, s) = 1$, $A \upharpoonright \phi_e(n)$ can change only $< 2^{\langle e, n \rangle} = h(\langle e, n \rangle)$ many times, since each change induces a change in $M \upharpoonright \gamma(\phi_e(n))$ and hence $M \upharpoonright \max I^n_e$. There are not enough elements to enter $M - M_s$ for this to happen more than $2^{\langle e, n \rangle} - 1$ many times. So 3 holds.

Now suppose that $\hat{\Phi}^A_e$ is total. Then for each $n$, we define $I^n_e$.

Moreover, since $M$ is maximal, we know that for almost all $n$, $|\overline{M} \cap I^n_e| \leq 1$.

Thus, for almost all $n$, there is an $s$ with $|\overline{M}_s \cap [0, \max I^n_e)| < 2^n \leq 2^{\langle e, n \rangle}$, and hence for all $e$ with $\hat{\Phi}^A_e$ total, and for almost all $n, s$, $k(\langle e, n \rangle, s) = 1$.

Therefore $A$ is eventually uniformly wtt-ac, a contradiction.

Same argument works for e.g. dense simple, hh-simple also.
Further work involves exploring approximations to wtt-functionals.

Thank you.