

A Complete Development of Dedekind Cuts of \mathbb{Q}

Joshua Matthew Baker

March 3, 2008

Abstract

The field \mathbb{R} of real numbers is constructed from \mathbb{Q} , and the main properties are proven. The definitions are as in Chris Atkin's mathematical analysis course, except for multiplication, which I have defined myself.

1 Basic Definitions

Definition 1.1 A cut is a set $A \subseteq \mathbb{Q}$ with the following properties:

1. $A \neq \emptyset$ and $A \neq \mathbb{Q}$
2. If $\alpha \in A, \beta \in \mathbb{Q}$ and $\beta \geq \alpha$ then $\beta \in A$
3. If $\alpha \in A$ then there is a $\gamma \in A$ such that $\gamma < \alpha$

Given $\alpha \in \mathbb{Q}$, define $S(\alpha)$ to be the set of all rationals strictly greater than α . This is our first example of a cut.

Theorem 1.2 If $\alpha \in \mathbb{Q}$ then $S(\alpha)$ is a cut.

Proof We must check the three conditions in turn.

1. $S(\alpha) \neq \emptyset$ as $\alpha + 1 \in S(\alpha)$. Also, $S(\alpha) \neq \mathbb{Q}$ as $\alpha \notin S(\alpha)$.
2. If $\gamma \in S(\alpha)$, then $\gamma > \alpha$. Given any $\beta \geq \gamma$, naturally also $\beta > \alpha$, so $\beta \in S(\alpha)$.
3. If $\eta \in S(\alpha)$ then by definition $\alpha < \eta$, and letting $\xi = \frac{\alpha + \eta}{2}$ we see that $\xi < \eta$ and $\xi \in S(\alpha)$.

□

We define the cut 0 to be $S(0)$. Note that this is precisely the set of positive rationals. A cut A is said to be *nonnegative* if it does not contain the rational number 0. I will write $A \geq 0$ if A is nonnegative, and $A < 0$ if A is not nonnegative. Note that a nonnegative cut contains only positive numbers.

Definition 1.3 Given two cuts A, B , the sum $A + B$ is defined to be:

$$A + B = \{\alpha + \beta : \alpha \in A \wedge \beta \in B\}$$

Theorem 1.4 *If A and B are cuts, then $A + B$ is also a cut. If A and B are both nonnegative, then so is $A + B$.*

Proof Again, we check that the properties defining a cut hold for $A + B$:

1. By property (1) in definition 1.1, we can find $\alpha \in A$ and $\beta \in B$, so $\alpha + \beta \in A + B$ and $A + B \neq \emptyset$. Also, we can find $\alpha' \notin A$ and $\beta' \notin B$. If $\alpha' + \beta'$ were in $A + B$, then we would know $\eta \in A$ and $\xi \in B$ with $\alpha' + \beta' = \eta + \xi$. But necessarily $\alpha' < \eta$ and $\beta' < \xi$, so $\alpha' + \beta' < \eta + \xi$ and the above equality cannot be true. So $\alpha' + \beta' \notin A + B$ and $A + B \neq \mathbb{Q}$.
2. Given $\alpha + \beta \in A + B$ and $\gamma \geq \alpha + \beta$ we have $\gamma - \beta \geq \alpha \Rightarrow \gamma - \beta \in A$, so $\gamma = (\gamma - \beta) + \beta \in A + B$.
3. Given $\alpha + \beta \in A + B$, we know that there is an $\alpha' \in A$ such that $\alpha' < \alpha$ and a $\beta' \in B$ such that $\beta' < \beta$, so $\alpha' + \beta'$ is less than $\alpha + \beta$ and is in $A + B$.

To prove the second statement, we note that if $A, B \geq 0$ then every element of $A + B$ is the sum of two positive numbers, which cannot be zero. \square

2 Properties Of Addition

We continue to elucidate the algebraic properties of addition. The role of the identity is, as expected, played by 0. Additionally, an additive inverse is needed for a given cut. This is provided by the following definition.

Definition 2.1 *Let A be a cut. The cut $-A$ is defined by:*

$$-A = \{-\beta : (\exists \gamma \in \mathbb{Q} \setminus A). \gamma > \beta\}$$

Theorem 2.2 *Let A be a cut. $-A$ is also a cut.*

Proof We check the definition of a cut:

1. $A \neq \emptyset$, so let $\alpha \in A$. Then $-\alpha \notin -A$ so $-A \neq \mathbb{Q}$. Also, choose $\alpha' \in \mathbb{Q} \setminus A$. Then by definition, $-(\alpha' - 1) \in -A$ so $-A \neq \emptyset$.
2. Let $-\alpha \in -A$. Then by definition, there is a $\gamma \in \mathbb{Q} \setminus A$ that is greater than α . If $-\beta > -\alpha$, then $\alpha > \beta$ so $\gamma > \beta$ also and $\beta \notin A$, so $-\beta \in -A$.
3. Let $-\alpha \in -A$. Then by definition, there is a $\gamma \in \mathbb{Q} \setminus A$ that is greater than α . Then $\alpha < \frac{\gamma + \alpha}{2} < \gamma$ means that $-\alpha > -(\frac{\gamma + \alpha}{2}) \in -A$.

\square

Lemma 2.3 *If $A < 0$ is a cut, then $-A \geq 0$.*

Proof $0 \in A$ by definition. But by cut property (2) there is no $\gamma \notin A$ such that $-0 = 0 < \gamma$. So $0 \notin -A$. \square

Theorem 2.4 *Let A, B, C be cuts. The following properties are satisfied:*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A = 0 + A$
4. $A + (-A) = 0 = (-A) + A$

Proof

1. $A + B \ni \alpha + \beta = \beta + \alpha \in A + B$
2. $(A + B) + C \ni (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \in A + (B + C)$
3. Let $\alpha \in A, \xi \in A$. $\alpha + \xi > \alpha$, so $\alpha + \xi \in A$. Thus $A + 0 \subseteq A$.

Then again, if $\beta \in A$, there is a $\gamma \in A$ less than β , so write

$$\beta = \underset{\in A}{\gamma} + \underset{\in 0}{(\beta - \gamma)} \in A + 0$$

Thus $A \subseteq A + 0$.

4. Let $\alpha + \alpha' \in A + (-A)$. As $\alpha' \in -A$, $\alpha' = -\beta$ and there is a $\gamma \in A$ with $A \not\ni \gamma < \beta$. It must be shown that $\alpha - \beta \in 0$. But $\alpha > \gamma$, as if not, we'd have $\gamma \in \mathbb{Q}$, which is false. So $\alpha > \gamma > \beta$ and $\alpha - \beta \in 0$. Thus $A + (-A) \subseteq 0$.

Let $\pi \in 0$. Choose $\lambda \in \mathbb{Q} \setminus A$, $\mu \in A$. We can run the bisection method on these to find $\lambda_n \in \mathbb{Q} \setminus A$, $\mu_n \in A$ such that $\mu_n - \lambda_n < \pi$. By property (3) of cuts, $(\exists \mu' \in A) \mu' < \mu_n$. Then let $\lambda' = \lambda_n - (\mu_n - \mu')$ so that $\mu' - \lambda' = \mu_n - \lambda_n < \pi$, and $\lambda' < \lambda_n$ means $-\lambda' \in -A$. Then

$$[\underset{\in A}{\mu'} + (\pi - (\mu' - \lambda'))] + \underset{\in -A}{-\lambda'} = \pi$$

So $\pi \in A + (-A)$. Thus $0 \subseteq A + (-A)$. \square

3 Properties Of Multiplication

Note that now we have shown that the set of cuts is an abelian group under addition. With this we get useful theorems like uniqueness of inverses and $-(-A) = A$. We may take such facts for granted in the sequel.

Multiplication I shall define differently to the way Chris did, because I think his formula for AB looks horrible, and am not convinced it works as written. A sort of multiplication is first defined for nonnegative cuts, and then operation is then generalized with this as a basis.

Definition 3.1 If A, B are nonnegative cuts, then $A \times B$ is defined by:

$$A \times B = \{\alpha\beta : \alpha \in A \wedge \beta \in B\}$$

Theorem 3.2 If A, B are nonnegative cuts, then so is $A \times B$.

Proof It's back to checking the properties of a cut:

1. $A \times B \neq \emptyset$ as neither A nor B is empty. Also as all elements in A and B , every element in $A \times B$ is too. So $A \times B$ is nonnegative provided it is a cut, and also $0 \neq A \times B \Rightarrow A \times B \neq \mathbb{Q}$.
2. If $\alpha\beta \in A \times B$ and $\gamma > \alpha\beta$, then $\gamma/\beta > \alpha$ so $\gamma/\beta \in A$. Then $\gamma = (\gamma/\beta)\beta \in A \times B$.
3. If $\alpha\beta \in A \times B$, we know $(\exists \alpha' \in A). \alpha' < \alpha$, so $\alpha\beta > \alpha'\beta \in A \times B$.

□

Lemma 3.3 If A, B are nonnegative cuts, then the following hold:

1. $A \times B = B \times A$
2. $(A \times B) \times C = A \times (B \times C)$
3. $A \times (B + C) = A \times B + A \times C$

Proof

1. $A \times B \ni \alpha\beta = \beta\alpha \in B \times A$
2. $(A \times B) \times C \ni (\alpha\beta)\gamma = \alpha(\beta\gamma) \in A \times (B \times C)$
3. This is not as symmetric as the others and takes a little more care. Take $\alpha(\beta + \gamma) \in A \times (B + C)$. Then $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \in A \times B + A \times C$. So $A \times (B + C) \subseteq A \times B + A \times C$.

Now choose $\xi = \alpha\beta + \eta\gamma \in AB + AC$. If $\eta < \alpha$ then write

$$\xi = \eta(\alpha/\eta)\beta + \eta\gamma = \eta((\alpha/\eta)\beta + \gamma) \in A \times (B + C)$$

Otherwise, write

$$\xi = \alpha\beta + \alpha(\eta/\alpha)\gamma = \alpha(\beta + (\eta/\alpha)\gamma) \in A \times (B + C)$$

So $A \times B + A \times C \subseteq A \times (B + C)$.

□

Definition 3.4 Let A, B be cuts. Define AB by (\times -multiplication is to be performed before negation):

- If $A \geq 0$ and $B \geq 0$, $AB = A \times B$

- If $A \geq 0$ and $B < 0$, $AB = -A \times (-B)$
- If $A < 0$ and $B \geq 0$, $AB = -(-A) \times B$
- If $A < 0$ and $B < 0$, $AB = (-A) \times (-B)$

Theorem 3.5 *Cut multiplication is commutative.*

Proof Let A, B be cuts. We must check all four cases:

- If $A \geq 0$ and $B \geq 0$, $AB = A \times B = B \times A = BA$
- If $A \geq 0$ and $B > 0$, $AB = -A \times (-B) = -(-B) \times A = BA$
- If $A > 0$ and $B \geq 0$, $AB = -(-A) \times B = -B \times (-A) = BA$
- If $A > 0$ and $B > 0$, $AB = (-A) \times (-B) = (-B) \times (-A) = BA$

□

Theorem 3.6 *Cut multiplication is associative.*

Proof Let A, B, C be cuts. We must check all eight cases:

- $A \geq 0, B \geq 0, C \geq 0$
 $(AB)C = (A \times B) \times C = A \times (B \times C) = A(BC)$
- $A \geq 0, B \geq 0, C < 0$
 $(AB)C = -(AB) \times (-C) = -(A \times B) \times (-C)$
 $A(BC) = -A \times (-BC) = -A \times -(-B \times (-C)) = -A \times (B \times (-C))$
 By associativity of \times -multiplication, these are equal.
- $A < 0, B \geq 0, C \geq 0$
 $(AB)C = -(-AB) \times C = -(-(-A) \times B) \times C = -((-A) \times B) \times C$
 $A(BC) = -(-A) \times (BC) = -(-A) \times (B \times C)$
 By associativity of \times -multiplication, these are equal.
- $A \geq 0, B < 0, C \geq 0$
 $(AB)C = C(AB) = (CA)B = B(CA) = (BC)A = A(BC)$
- $A \geq 0, B < 0, C < 0$
 $(AB)C = (-AB) \times (-C) = (-(-A \times (-B))) \times (-C) = (A \times (-B)) \times (-C)$
 $A(BC) = A \times (BC) = A \times ((-B) \times (-C))$
 By associativity of \times -multiplication, these are equal.
- $A < 0, B < 0, C \geq 0$
 $(AB)C = (AB) \times C = ((-A) \times (-B)) \times C$
 $A(BC) = (-A) \times (-BC) = (-A) \times -(-(-B) \times C) = (-A) \times ((-B) \times C)$
 By associativity of \times -multiplication, these are equal.
- $A < 0, B \geq 0, C < 0$
 $(AB)C = C(AB) = (CA)B = B(CA) = (BC)A = A(BC)$

- $A < 0, B < 0, C < 0$
 $(AB)C = -(AB) \times (-C) = -((-A) \times (-B)) \times (-C)$
 $A(BC) = -(-A) \times (BC) = -(-A) \times ((-B) \times (-C))$
 By associativity of \times -multiplication, these are equal.

□ It was shown earlier that if A is not nonnegative, then $-A$ is. The opposite is not true, and we have to be careful about how we treat these cases. It is clear that $0 \geq 0$ as defined, but the equation $0 + 0 = 0$ shows that $-0 = 0$, which is still nonnegative.

Let us write $A > 0$ when $A \geq 0$ but $A \neq 0$. Then if $A > 0$, there is a positive rational number not in A , so $0 = -0 \in -A$ by definition. Thus $-A < 0$.

Also note that if $A \geq 0$, $0 + A \times 0 = A \times (0 + 0) = A \times (0 + 0) = A \times 0 + A \times 0$ so $A \times 0 = 0$. We can now prove:

Lemma 3.7 *Let A be a cut. Then $0A = A0 = 0$.*

Proof If $A \geq 0$, the result is proved. Otherwise $A0 = -(-A) \times 0 = -0 = 0$. □

Lemma 3.8 *Let A, B be cuts. Then:*

$$(-A)B = -AB = A(-B)$$

Proof The second equality is implied by the first and commutativity, but separate proofs are given for the sake of symmetry. We have to split cases again:

- $A = 0 \vee B = 0$:
By the preceding lemma, all three products are zero.
- $A > 0, B > 0$:
 $-AB = -A \times B = -A \times (- - B) = A(-B)$
 $-AB = -A \times B = -(- - A) \times B = (-A)B$
- $A > 0, B < 0$:
 $-AB = - - A \times (-B) = A(-B)$
 $-AB = - - A \times (-B) = (-(-A)) \times (-B) = (-A)B$
- $A < 0, B > 0$:
 $-AB = - - (-A) \times B = (-A)B$
 $-AB = - - (-A) \times B = (-A) \times (-(-B)) = A(-B)$
- $A < 0, B < 0$:
 $-AB = -(-A) \times (-B) = (-A)B$
 $-AB = -(-A) \times (-B) = A(-B)$

□

Given two cuts A, B , we define $A - B$ to be $A + (-B)$. We now take increasingly bigger steps toward proving the general distributive law. A nice little thing to use will be that if $A, B \geq 0$ $-(A + B)$

Lemma 3.9 *If A, D, E are nonnegative cuts, then*

$$A \times (D - E) = A \times D - A \times E$$

Proof

- If $D - E \geq 0$ then:

$$\begin{aligned} A(D - E) &= A \times (D - E) = A \times (D - E) + 0 = A \times (D - E) + A \times E - A \times E = \\ &= A \times (D - E + E) - A \times E = A \times D - A \times E = AD - AE \end{aligned}$$

- If $D - E < 0$ then:

$$\begin{aligned} A(D - E) &= -A \times -(D - E) = -A \times (E - D) = -(A \times E - A \times D) = \\ &= A \times D - A \times E = AD - AE \end{aligned}$$

□

Theorem 3.10 *If A, B, C are cuts, then*

$$A(B + C) = AB + AC$$

Proof First we treat the case when A is positive. We need cases again:

- $B \geq 0, C \geq 0$:

$$A(B + C) = A \times (B + C) = A \times B + A \times C = AB + AC$$

- $B \geq 0, C < 0$:

$$A(B + C) = A \times (B + C) = A \times (B - (-C)) = A \times B - A \times (-C) = AB + AC$$

- $B < 0, C \geq 0$:

$$A(B + C) = A(C + B) = AC + AB = AB + AC$$

- $B < 0, C < 0$:

$$\begin{aligned} A(B + C) &= -A((-B) + (-C)) = -(A(-B) + A(-C)) = -A(-B) + \\ &= -A(-C) = AB + AC \end{aligned}$$

Then if $A < 0$, we have $A(B + C) = -(-A)(B + C) = -((-A)B + (-A)C) = -(-A)B + -(-A)C = AB + AC$. □

4 Division

Now it has been shown that the set of cuts is a commutative ring. It also has identity, namely $S(1)$.

Theorem 4.1 *If A is a nonnegative cut, then $S(1)A = A$.*

Proof First take $\iota\alpha \in S(1)A$. As by definition $\iota > 1$ and α is positive, $\iota\alpha > \alpha$, so $\iota\alpha \in A$. Hence $S(1) \subseteq A$.

For the opposite containment, take $\alpha \in A$. Then there is an $\alpha' \in A$ with $\alpha' < \alpha$. Then $\alpha/\alpha' > 1$ so

$$\alpha = \underbrace{(\alpha/\alpha')}_{\in S(1)} \cdot \underbrace{\alpha'}_{\in A} \in S(1)A$$

If $A < 0$, then $A = -(-A) = -S(1)(-A) = S(1)A$. □

As using $S(1)$ all the time looks ugly, we shall write the cut $S(1)$ as just 1. The definition for the multiplicative inverse of an element is very much like that for the additive inverse:

Definition 4.2 Let A be a cut. If $A > 0$, define A^{-1} by:

$$A^{-1} = \{\beta^{-1} : (\exists \gamma \in 0 \setminus A). \gamma > \beta\}$$

If $A < 0$, define A^{-1} by:

$$A^{-1} = -(-A)^{-1}$$

Theorem 4.3 Let A be a nonzero cut. Then A^{-1} is also a cut. If $A > 0$ then also $A^{-1} > 0$.

Proof First assume that $A > 0$. We check the definition of a cut:

1. If $A > 0$ there is a positive rational number $\epsilon \notin A$. Then as $\epsilon/2 < \epsilon$, we have $2/\epsilon \in A^{-1}$. So $A^{-1} \neq \emptyset$. Also as everything in A^{-1} is positive by definition, $-1 \notin A^{-1}$ so $A^{-1} \neq \mathbb{Q}$.
2. Let $\alpha^{-1} \in A^{-1}$. Then by definition, there is a $\gamma \in 0 \setminus A$ that is greater than α . If $\beta^{-1} > \alpha^{-1}$, then $\alpha > \beta$ so $\gamma > \beta$ also and $\beta \notin A$, so $\beta^{-1} \in A^{-1}$.
3. Let $\alpha^{-1} \in A^{-1}$. Then by definition, there is a $\gamma \in 0 \setminus A$ that is greater than α . Then $\frac{\gamma+\alpha}{2} < \gamma$ means that $\alpha^{-1} > (\frac{\gamma+\alpha}{2})^{-1} \in A^{-1}$.

If $A < 0$, then $-A > 0$ so $(-A)^{-1}$ is a cut by the above. □

Theorem 4.4 Let A be a nonzero cut. Then:

$$AA^{-1} = 1 = A^{-1}A$$

Proof We first complete the proof for $A > 0$.

Let $\alpha\alpha' \in AA^{-1}$. As $\alpha' \in A^{-1}$, $\alpha' = \beta^{-1}$ and there is a $\gamma \in 0$ with $A \not\ni \gamma < \beta$. It must be shown that $\alpha/\beta \in 1$. But $\alpha > \gamma$, as if not, we'd have $\gamma \in A$, which is false. So $\alpha > \gamma > \beta$ and $\alpha/\beta > 1$. Thus $AA^{-1} \subseteq 1$.

Let $\pi \in 1$. Choose $\lambda \in 0 \setminus A, \mu \in A$. We can run the bisection method on these to find $\lambda_n \in 0 \setminus A, \mu_n \in A$ such that $\mu_n/\lambda_n < \pi$. By property (3) of cuts, $(\exists \mu' \in A). \mu' < \mu_n$. Then let $\lambda' = \lambda_n \mu' / \mu_n$ so that $\mu' / \lambda' = \mu_n / \lambda_n < \pi$, and $\lambda' < \lambda_n$ means $\lambda'^{-1} \in A^{-1}$. Then

$$[\underbrace{\mu'}_{\in A} (\underbrace{\pi \lambda'}_{\in A^{-1}} / \underbrace{\mu'}_{\in A^{-1}})]^{\lambda'^{-1}} = \pi$$

So $\pi \in AA^{-1}$. Thus $1 \subseteq AA^{-1}$.

If $A < 0$, we have $AA^{-1} = A(-(-A)^{-1}) = (-A)(-A)^{-1} = 1$. □

5 Order

So the set of cuts of \mathbb{Q} is a field! We define the field of real numbers \mathbb{R} to be this field, and we call a member of this field a real number. Now we define the canonical partial order on \mathbb{R} and deduce its properties.

Definition 5.1 *The binary relation \leq on \mathbb{R} is defined by:*

$$A \leq B \text{ iff } B \subseteq A$$

It will be useful to show that our use of \geq up to now is compatible with this definition. As if A is not nonnegative, then $0 \in A$ so as $0 \notin 0$ it follows that $A \not\subseteq 0$, that is, $0 \not\leq A$. Also if A is nonnegative, then all its elements are positive, so $A \subseteq 0$, that is, $0 \leq A$.

The relation \leq is a partial order on \mathbb{R} , which follows from the fact that \subseteq is a partial order on $\mathcal{P}(\mathbb{Q})$. It would be nice to show that it is actually a total order, and that the field of real numbers is "ordered by" \leq in the way we expect from order on the rationals. I can never remember the following definition:

Definition 5.2 *We say that a field \mathbb{F} is ordered by a partial order \leq if the following hold for all $A, B, C \in \mathbb{F}$:*

1. \leq is a total order on \mathbb{F}
2. $A \leq B \Rightarrow A + C \leq B + C$
3. $A, B \geq 0 \Rightarrow AB \geq 0$

Theorem 5.3 *\mathbb{R} is ordered by \leq .*

Proof We check against the definition:

1. Let $A, B \in \mathbb{R}$. Consider $B \setminus A$. If this is null then $B \subseteq A$ and $A \leq B$ and we're done. Otherwise choose $\beta \in B \setminus A$. Given any $\alpha \in A$ we must have $\beta < \alpha$, because if we had $\beta \geq \alpha$, then that would imply that $\beta \in A$. But then as B is a cut, $\beta < \alpha$ implies $\alpha \in B$. As α was arbitrary, $A \subseteq B$ and so $B \leq A$.
2. Say $A \leq B$. Then $B \subseteq A$. So take $\beta + \gamma \in B + C$. As $\beta \in A$, $\beta + \gamma$ is in $A + C$ also, so $B + C \subseteq A + C$ and $A + C \leq B + C$.
3. This we have already known for a while now.

□

6 The Subfield Isomorphic To \mathbb{Q}

When we defined $S(\alpha)$ for $\alpha \in \mathbb{Q}$ above not just to give us examples of cuts, but also because it is expected that we will have an easy way of refinding the rationals somewhere in this set of numbers we've just made. The function $S : \mathbb{Q} \rightarrow \mathbb{R}$ is actually a field isomorphism, which is shown by the following theorem. Also, you'll be relieved to hear that the ordering of the rationals is conserved.

Theorem 6.1 *Let $\alpha, \beta \in \mathbb{Q}$. Then:*

1. $S(\alpha + \beta) = S(\alpha) + S(\beta)$
2. $S(\alpha\beta) = S(\alpha)S(\beta)$
3. If $\alpha \leq \beta$ then $S(\alpha) \leq S(\beta)$

Proof

1. Let $\gamma \in S(\alpha + \beta)$. Then by definition $\gamma > \alpha + \beta$. $\gamma - \alpha > \beta$ so $\gamma - \alpha \in S(\beta)$. Also, as $S(\beta)$ is a cut, there is a $\eta \in S(\beta)$ with $\eta < \gamma - \alpha$. Then $\alpha < \gamma - \eta$ so $\gamma - \eta \in S(\alpha)$. Thus

$$\gamma = \underbrace{(\gamma - \eta)}_{\in S(\alpha)} + \underbrace{\eta}_{\in S(\beta)} \in S(\alpha) + S(\beta)$$

So $S(\alpha + \beta) \subseteq S(\alpha) + S(\beta)$.

Now let $\sigma + \tau \in S(\alpha) + S(\beta)$. Then by definition $\sigma > \alpha$ and $\tau > \beta$, so $\sigma + \tau > \alpha + \beta$, that is $\sigma + \tau \in S(\alpha + \beta)$. Thus $S(\alpha) + S(\beta) \subseteq S(\alpha + \beta)$.

2. This proof requires splitting into cases, because of how we defined multiplication. First, if α and β are both ≥ 0 , then picking $\alpha' \in S(\alpha)$ and $\beta' \in S(\beta)$ we have $\alpha'\beta' > \alpha\beta$ so $\alpha'\beta' \in S(\alpha\beta)$ and $S(\alpha)S(\beta) \subseteq S(\alpha\beta)$.

Now pick $\gamma \in S(\alpha\beta)$. Then $\gamma > \alpha\beta$, so $\gamma/\alpha > \beta$ and $\gamma/\alpha \in S(\beta)$. As $S(\beta)$ is a cut, we can pick $\eta < \gamma/\alpha$ in $S(\beta)$. Then $\alpha < \gamma/\eta$, so we can write

$$\gamma = \underbrace{(\gamma/\eta)}_{\in S(\alpha)} \underbrace{\eta}_{\in S(\beta)} \in S(\alpha)S(\beta)$$

Now suppose $\alpha < 0$. Then $S(\alpha\beta) = -S((-\alpha)\beta) = -S(-\alpha)S(\beta) = S(\alpha)S(\beta)$.

Finally, if $\beta < 0$, then $S(\alpha\beta) = -S(\alpha(-\beta)) = -S(\alpha)S(-\beta) = S(\alpha)S(\beta)$.

3. Let $\alpha \leq \beta$. Pick $\gamma \in S(\beta)$. $\gamma > \beta \geq \alpha$ so by definition, $\gamma \in S(\alpha)$. Thus $S(\beta) \subseteq S(\alpha)$ and $S(\alpha) \leq S(\beta)$.

□

Having proved that they act the same, we shall treat the cut $S(\alpha)$ as being the same as the rational α and treat $S[\mathbb{Q}]$ as being the same as \mathbb{Q} . In particular, it is thought that $\mathbb{Q} \subset \mathbb{R}$. To show how use is made of this lie, we offer the following theorem:

Theorem 6.2 *The rationals are dense in the reals. That is, if $A, B \in \mathbb{R}$ and $A < B$, then there is a rational γ with $A < \gamma < B$.*

Proof If $A < B$, then by definition, $B \subset A$, and $A \setminus B$ is nonempty. So pick $\alpha \in A \setminus B$. As A is a cut, we can find $\eta \in A$ such that $\eta < \alpha$, and naturally we have $\eta \notin B$. By property (2) of cuts, $B \subseteq S(\eta)$ and $S(\eta) \subseteq A$. Also, $B \subset S(\eta)$ as $\alpha \in S(\eta)$ but $\alpha \notin B$, and $S(\eta) \subset A$ as $\eta \in A$ but $\eta \notin S(\eta)$. Thus $A < S(\eta) < B$ and writing $\gamma = S(\eta)$ we have the result. \square