A DENSITY HALES-JEWETT THEOREM FOR MATROIDS

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Abstract. We show that if α is a positive real number, n and ℓ are integers exceeding 1, and q is a prime power, then every simple matroid M of sufficiently large rank, with no $U_{2,\ell}$-minor, no rank-n projective geometry minor over a larger field than $\mathrm{GF}(q)$, and at least $\alpha q^{r(M)}$ elements, has a rank-n affine geometry restriction over $\mathrm{GF}(q)$. This result can be viewed as an analogue of the multidimensional density Hales-Jewett theorem for matroids.

1. Introduction

For a matroid M, let $|M|$ denote the number of elements of M. Furstenberg and Katznelson [3] proved the following result, implying that $\mathrm{GF}(q)$-representable matroids of nonvanishing density and huge rank contain large affine geometries as restrictions:

Theorem 1.1. *Let q be a prime power, $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. If M is a simple $\mathrm{GF}(q)$-representable matroid of sufficiently large rank satisfying $|M| \geq \alpha q^{r(M)}$, then M has an $\mathrm{AG}(n,q)$-restriction.*

Later, Furstenberg and Katznelson [4] proved a much more general result, namely the multidimensional density Hales-Jewett theorem, which gives a similar statement in the more abstract setting of words over an arbitrary finite alphabet. Considerably shorter proofs [1,13] have since been found. We will generalise Theorem 1.1 in a different direction:

Theorem 1.2. *Let q be a prime power, $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. If M is a simple matroid of sufficiently large rank with no $U_{2,q+2}$-minor and with $|M| \geq \alpha q^{r(M)}$, then M has an $\mathrm{AG}(n,q)$-restriction.*

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In fact, we prove more. The class of matroids with no $U_{2,q+2}$-minor is just one of many minor-closed classes whose extremal behaviour is qualitatively similar to that of the $\mathrm{GF}(q)$-representable matroids. The following theorem, which summarises several papers [5,6,9], tells us that such classes occur naturally as one of three types:

Theorem 1.3 (Growth Rate Theorem)**.** *Let $\mathcal{M}$ be a minor-closed class of matroids, not containing all simple rank-2 matroids. There exists a real number $c_{\mathcal{M}} > 0$ such that either:*

(1) $|M| \le c_{\mathcal{M}} r(M)$ for every simple $M \in \mathcal{M}$,
(2) $|M| \le c_{\mathcal{M}} r(M)^2$ for every simple $M \in \mathcal{M}$, and $\mathcal{M}$ contains all graphic matroids, or
(3) there is a prime power q such that $|M| \le c_{\mathcal{M}} q^{r(M)}$ for every simple $M \in \mathcal{M}$, and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids.

We call a class $\mathcal{M}$ satisfying (3) *base-q exponentially dense.* It is clear that these classes are the only ones that contain arbitrarily large affine geometries, and that the matroids with no $U_{2,q+2}$-minor form such a class. Our main result, which clearly implies Theorem 1.2, is the following:

Theorem 1.4. *Let $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids, $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. If $M \in \mathcal{M}$ is simple, satisfies $|M| \ge \alpha q^{r(M)}$, and has sufficiently large rank, then M has an $\mathrm{AG}(n,q)$-restriction.*

Finding such a highly structured restriction seems very surprising, given the apparent wildness of general exponentially dense classes. This will be proved using Theorem 1.3 and a slightly more technical statement, Theorem 6.1; the proof extensively uses machinery developed in [7], [8], [14] and [15].

We would like to prove a result corresponding to Theorem 1.4 for *quadratically dense* classes, those satisfying condition (2) of Theorem 1.3. The following is a corollary of the Erdős-Stone Theorem [2]:

Theorem 1.5. *Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. If G is a simple graph such that $|E(G)| \ge \alpha |V(G)|^2$ and $|V(G)|$ is sufficiently large, then G has a $K_{n,n}$-subgraph.*

In light of this, we expect that the unavoidable restrictions of dense matroids in a quadratically dense class are the cycle matroids of large complete bipartite graphs.

Conjecture 1.6. *Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids, $\alpha > 0$ be a real number, and n be a positive integer. If*

$M \in \mathcal{M}$ is simple, satisfies $|M| \geq \alpha r(M)^2$, and has sufficiently large rank, then M has an $M(K_{n,n})$-restriction.

2. Preliminaries

We follow the notation of Oxley [16]. For a matroid M, we also write $\varepsilon(M)$ for $|\operatorname{si}(M)|$, or the number of points or rank-1 flats in M. If $\ell \geq 2$ is an integer, we write $\mathcal{U}(\ell)$ for the class of matroids with no $U_{2,\ell+2}$-minor.

The next theorem, a constituent of Theorem 1.3, follows easily from the two main results of [5].

Theorem 2.1. *There is a function $\alpha_{2.1} : \mathbb{Z} \times \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ so that, for all $\ell, n \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$ with $\ell, n \geq 2$ and $\gamma > 1$, if $M \in \mathcal{U}(\ell)$ satisfies $\varepsilon(M) \geq \alpha_{2.1}(n, \gamma, \ell)\gamma^{r(M)}$, then M has a $\mathrm{PG}(n-1, q)$-minor for some $q > \gamma$.*

The next theorem is due to Kung [11].

Theorem 2.2. *If $\ell \geq 2$ and $M \in \mathcal{U}(\ell)$, then $\varepsilon(M) \leq \frac{\ell^{r(M)}-1}{\ell-1}$.*

We will sometimes use the cruder estimate $\varepsilon(M) \leq (\ell+1)^{r(M)-1}$ for ease of calculation, such as in the following simple corollary:

Corollary 2.3. *If $\ell \geq 2$ is an integer, $M \in \mathcal{U}(\ell)$, and $C \subseteq E(M)$ satisfies $r_M(C) < r(M)$, then $\varepsilon(M/C) \geq (\ell+1)^{-r_M(C)}\varepsilon(M)$.*

Proof. Let $\mathcal{F}$ be the collection of rank-$(r_M(C)+1)$ flats of M containing C. We have $\varepsilon(M|F) \leq \frac{\ell^{r_M(C)+1}-1}{\ell-1} \leq (\ell+1)^{r_M(C)}$ for each $F \in \mathcal{F}$. Moreover, $|\mathcal{F}| = \varepsilon(M/C)$, and $\varepsilon(M) \leq \sum_{F \in \mathcal{F}} \varepsilon(M|F)$; the result follows. □

We apply both Theorem 2.2 and Corollary 2.3 freely. The next result follows from [8, Lemma 3.1].

Lemma 2.4. *Let q be a prime power, $k \geq 0$ be an integer, and M be a matroid with a $\mathrm{PG}(r(M)-1, q)$-restriction R. If F is a rank-k flat of M that is disjoint from $E(R)$, then $\varepsilon(M/F) \geq \frac{q^{r(M/F)+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}$.*

3. Connectivity

A matroid M is *weakly round* if there is no pair of sets A, B with union $E(M)$, such that $r_M(A) \leq r(M)-1$ and $r_M(B) \leq r(M)-2$. This is a variation on *roundness*, a notion equivalent to infinite vertical connectivity introduced by Kung in [12] under the name of *non-splitting*. Our tool for reducing Theorem 1.4 to the weakly round case is the following, proved in [14, Lemma 7.2].

Lemma 3.1. *There is a function $f_{3.1} : \mathbb{Z}^3 \to \mathbb{Z}$ so that, for all $r, d, \ell \in \mathbb{Z}$ with $\ell \geq 2$ and $r \geq d \geq 0$, and every real-valued function $g(n)$ satisfying $g(d) \geq 1$ and $g(n) \geq 2g(n-1)$ for all $n > d$, if $M \in \mathcal{U}(\ell)$ satisfies $r(M) \geq f_{3.1}(r, d, \ell)$ and $\varepsilon(M) > g(r(M))$, then M has a weakly round restriction N such that $r(N) \geq r$ and $\varepsilon(N) > g(r(N))$.*

Our next lemma, proved in [8, Lemma 8.1], allows us to exploit weak roundness by contracting an interesting low-rank restriction onto a projective geometry.

Lemma 3.2. *There is a function $f_{3.2} : \mathbb{Z}^4 \to \mathbb{Z}$ so that, for every prime power q and all $n, \ell, t \in \mathbb{Z}$ with $n \geq 1, \ell \geq 2$ and $t \geq 0$, if $M \in \mathcal{U}(\ell)$ is a weakly round matroid with a $\mathrm{PG}(f_{3.2}(n, q, t, \ell) - 1, q)$-minor and T is a restriction of M with $r(T) \leq t$, then there is a minor N of M of rank at least n, such that T is a restriction of N, and N has a $\mathrm{PG}(r(N) - 1, q)$-restriction.*

4. Stacks

We now define an obstruction to $\mathrm{GF}(q)$-representability. If q is a prime power, and h and t are nonnegative integers, then a matroid S is a (q, h, t)-*stack* if there are pairwise disjoint subsets $F_1, F_2, \ldots, F_h$ of $E(S)$ such that the union of the F_i is spanning in S, and for each $i \in \{1, \ldots, h\}$, the matroid $(S/(F_1 \cup \ldots \cup F_{i-1}))|F_i$ has rank at most t and is not $\mathrm{GF}(q)$-representable. We write $F_i(S)$ for F_i. Note that such a stack has rank at most ht. When the value of t is unimportant, we refer simply to a (q, h)-*stack*.

The next three results suggest that stacks are incompatible with large projective geometries. First we argue that a matroid obtained from a projective geometry by applying a small extension and contraction does not contain a large stack:

Lemma 4.1. *Let q be a prime power and h be a nonnegative integer. If M is a matroid and $X \subseteq E(M)$ satisfies $r_M(X) \leq h$ and $\mathrm{si}(M \backslash X) \cong \mathrm{PG}(r(M) - 1, q)$, then M/X has no $(q, h+1)$-stack restriction.*

Proof. The result is clear if $h = 0$; suppose that $h > 0$ and that the result holds for smaller h. Moreover, suppose that M/X has a $(q, h+1, t)$-stack restriction S. Let $F = F_1(S)$. Since $(M/X)|F$ is not $\mathrm{GF}(q)$-representable but $M|F$ is, it follows that $\sqcap_M(F, X) > 0$. Therefore $r_{M/F}(X) < r_M(X) \leq h$ and $\mathrm{si}(M/F \backslash X) \cong \mathrm{PG}(r(M/F) - 1, q)$, so by the inductive hypothesis $M/(X \cup F)$ has no (q, h)-stack restriction. Since $M/(X \cup F)|(E(S) - F)$ is clearly such a stack, this is a contradiction. $\square$

Now we show that a large stack on top of a projective geometry R allows us to find a large flat disjoint from R:

Lemma 4.2. *Let q be a prime power and h be a nonnegative integer. If M is a matroid with a $\mathrm{PG}(r(M)-1,q)$-restriction R and a $(q,\binom{h+1}{2})$-stack restriction, then M has a rank-h flat that is disjoint from $E(R)$.*

Proof. If $h=0$, then there is nothing to show; suppose that $h>0$ and that the result holds for smaller h. Let S be a $(q,\binom{h+1}{2})$-stack restriction of M and let $F_i = F_i(S)$ for each $i \in \{1,\ldots,\binom{h+1}{2}\}$. Let $S_1 = S\big|\left(F_1 \cup \ldots \cup F_{\binom{h}{2}}\right)$. Clearly S_1 is a $(q,\binom{h}{2})$-stack, so inductively there is a rank-$(h-1)$ flat H of M that is disjoint from $E(R)$.

Note that $(M/H)|E(R)$ has no loops. If M/H has a nonloop e that is not parallel to an element of R, then $\mathrm{cl}_M(H \cup \{e\})$ is a rank-h flat of M disjoint from $E(R)$, and we are done. Therefore we may assume that $\mathrm{si}(M/H) \cong \mathrm{si}((M/H)|E(R))$, and so by Lemma 4.1 applied to the matroid $M|(E(R)\cup H)$, we know that M/H has no (q,h)-stack restriction. However the sets $(E(S_1)-H)\cup F_{\binom{h}{2}+1}, F_{\binom{h}{2}+2},\ldots,F_{\binom{h+1}{2}}$ clearly give rise to such a stack. This is a contradiction. □

Finally we show that a large stack restriction, together with a very large projective geometry minor, gives a projective geometry minor over a larger field:

Lemma 4.3. *There are functions $f_{4.3}:\mathbb{Z}^4 \to \mathbb{Z}$ and $h_{4.3}:\mathbb{Z}^3\to\mathbb{Z}$ so that, for every prime power q and all $\ell,n,t\in\mathbb{Z}$ with $\ell,n\geq 2$ and $t \geq 0$, if $M \in \mathcal{U}(\ell)$ is weakly round and has a $\mathrm{PG}(f_{4.3}(n,q,t,\ell)-1,q)$-minor and a $(q,h_{4.3}(n,q,\ell),t)$-stack restriction, then M has a $\mathrm{PG}(n-1,q')$-minor for some $q'>q$.*

Proof. Let q be a prime power and $\ell \geq 2$, $n\geq 2$ and $t\geq 0$ be integers. Let $\alpha = \alpha_{2.1}(n,q,\ell)$, and let $h'>0$ and $r \geq 0$ be integers so that $\frac{q^{r'+h'}-1}{q-1} - q\frac{q^{2h'}-1}{q^2-1} > \alpha q^{r'}$ for all $r'\geq r$. Set $h_{4.3}(n,q,\ell) = h = \binom{h'+1}{2}$, and $f_{4.3}(n,q,t,\ell) = f_{3.2}(r+h',q,th,\ell)$.

Let $M \in \mathcal{U}(\ell)$ be weakly round with a $\mathrm{PG}(f_{4.3}(n,q,t,\ell)-1,q)$-minor and a (q,h,t)-stack restriction S. We have $r(S)\leq th$; by Lemma 3.2 there is a minor N of M, of rank at least $r+h'$, with a $\mathrm{PG}(r(N)-1,q)$-restriction R, and S as a restriction. By Lemma 4.2, there is a rank-h' flat F of M that is disjoint from $E(R)$. Now $r(M/F)\geq r$; the lemma follows from Lemma 2.4, Theorem 2.1, and the definition of h'. □

5. Lifting

The following is a restatement of Theorem 1.1:

Theorem 5.1. *There is a function $f_{5.1} : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{Z}$ so that, for every prime power q and all $n \in \mathbb{Z}^+$ and $\beta \in \mathbb{R}^+$, if M is a* $\mathrm{GF}(q)$*-representable matroid satisfying $\varepsilon(M) \geq \beta q^{r(M)}$ and $r(M) \geq f_{5.1}(n, q, \beta)$, then M has an $AG(n-1, q)$-restriction.*

This next lemma uses the above to show that a bounded lift of a huge affine geometry itself contains a large affine geometry. The proof does not use the full strength of 5.1; the lemma would also follow from the much weaker 'colouring' Hales-Jewett Theorem [10].

Lemma 5.2. *There is a function $f_{5.2} : \mathbb{Z}^4 \to \mathbb{Z}$ so that, for every prime power q and all $\ell, n, t \in \mathbb{Z}$ so that $\ell, n \geq 2$ and $t \geq 0$, if $M \in \mathcal{U}(\ell)$ and $C \subseteq E(M)$ satisfy $r_M(C) \leq t$, and M/C has an $\mathrm{AG}(f_{5.2}(n, q, \ell, t) - 1, q)$-restriction, then M has an $\mathrm{AG}(n-1, q)$-restriction.*

Proof. Let q be a prime power and $\ell \geq 2$, $n \geq 2$ and $t \geq 0$ be integers. Let d be an integer large enough so that $(\ell + 1)^{-t} > \frac{q^{2-d}}{q-1}$, and let $m = f_{5.1}(n, q, (q^2(\ell + 1)^t)^{-1}) + d$. Set $f_{5.2}(n, q, \ell, t) = m$.

Let $M \in \mathcal{U}(\ell)$ and let $C \subseteq E(M)$ be a set so that $r_M(C) \leq t$ and M/C has an $\mathrm{AG}(m-1, q)$-restriction R. We may assume that C is independent and that $E(M) = E(R) \cup C$, so M is simple and $r(M) = m + |C|$. Let B be a basis for M containing C, and let $e \in B - C$. Let $X = B - (C \cup \{e\})$. Now $\mathrm{cl}_{M/C}(X)$ is a hyperplane of R, so $|\mathrm{cl}_{M/C}(X)| = q^{m-2}$ and there are at least $q^{m-1} - q^{m-2} \geq q^{m-2}$ elements of M not spanned by $X \cup C$. Each such element lies in a point of M/X and is not spanned by C in M/X. Moreover, $r(M/X) = t+1$, so by Theorem 2.2, M/X has at most $(\ell + 1)^t$ points; there is thus a point P of M/X, not spanned by C, with $|P| \geq (\ell + 1)^{-t} q^{m-2}$.

Now $P \subseteq E(R)$, so the matroid $(M/C)|P$ is $\mathrm{GF}(q)$-representable and has rank at most m, and $\varepsilon((M/C)|P) \geq (\ell + 1)^{-t} q^{m-2} > \frac{q^{m-d}-1}{q-1}$, so $r((M/C)|P) \geq m - d$. Furthermore, $\varepsilon((M/C)|P) \geq (q^2(\ell+1)^t)^{-1} q^m \geq (q^2(\ell+1)^t)^{-1} q^{r((M/C)|P)}$, so by Theorem 5.1 and the definition of m, the matroid $(M/C)|P$ has an $\mathrm{AG}(n-1, q)$-restriction. However, P is skew to C in M by construction, so $(M/C)|P = M|P$ and therefore M also has an $\mathrm{AG}(n-1, q)$-restriction, as required. □

6. The Main Result

Since, for any base-q exponentially dense minor-closed class $\mathcal{M}$, there is some $\ell \geq 2$ such that $\mathcal{M} \subseteq \mathcal{U}(\ell)$ and there is some s such that $\mathrm{PG}(s, q') \notin \mathcal{M}$ for all $q' > q$, the next theorem easily implies Theorem 1.4.

Theorem 6.1. *There is a function* $f_{6.1} : \mathbb{Z}^3 \times \mathbb{R} \to \mathbb{Z}$ *so that for every prime power* q *and all* $n, \ell \in \mathbb{Z}$ *and* $\beta \in \mathbb{R}^+$ *with* $n, \ell \geq 2$*, if* $M \in \mathcal{U}(\ell)$ *satisfies* $r(M) \geq f_{6.1}(n, q, \ell, \beta)$ *and* $\varepsilon(M) \geq \beta q^{r(M)}$*, then* M *has either an* $\mathrm{AG}(n-1, q)$*-restriction or a* $\mathrm{PG}(n-1, q')$*-minor for some* $q' > q$*.*

Proof. Let $\beta > 0$ be a real number, q be a prime power, and $\ell, n \geq 2$ be integers. Let $\alpha = \alpha_{2.1}(n, q, \ell)$ and $h = h_{4.3}(n, q, \ell)$. Set $0 = t_0, t_1, \dots, t_h$ to be a nondecreasing sequence of integers such that

$$t_{k+1} \geq f_{5.1}(f_{5.2}(n, q, \ell, kt_k), q, \beta((\ell+1)^{kt_k} q\alpha)^{-1})$$

for each $k \in \{0, \dots, h-1\}$. Let $m = \max(n, f_{4.3}(n, q, \ell, t_h))$, and let $r_1 \geq (h+1)t_h$ be an integer large enough so that $q^{(h+1)t_h - r_1 - 1} \leq \alpha$ and $\beta q^r \geq \alpha_{2.1}(m, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^r$ for all $r \geq r_1$. Let d be an integer such that $\beta q^d \geq 1$, and let $r_2 = f_{3.1}(r_1, d, \ell)$.

Let $M_2 \in \mathcal{U}(\ell)$ satisfy $r(M_2) \geq r_2$ and $\varepsilon(M_2) \geq \beta q^{r(M_2)}$; we will show that M_2 has either a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or an $\mathrm{AG}(n-1, q)$-restriction. The function $g(r) = \beta q^r$ satisfies $g(d) \geq 1$ and $g(r) \geq 2g(r-1)$ for all $r > d$, so by Lemma 3.1 the matroid M_2 has a weakly round restriction M_1 such that $r(M_1) \geq r_1$ and $\varepsilon(M_1) \geq \beta q^{r(M_1)}$.

Let k be the maximal element of $\{0, 1, \dots, h\}$ such that M_1 has a (q, k, t_k)-stack restriction; call this restriction S. We split into cases depending on whether $k = h$:

Case 1: $k < h$.

Let $M_0 = \mathrm{si}(M_1/E(S))$; note that $r(M_0) \geq r(M_1) - kt_k$, and therefore that $|M_0| \geq (\ell+1)^{-kt_k}|M_1| \geq (\ell+1)^{-kt_k}\beta q^{r(M_0)}$. Let F_0 be a rank-$(t_{k+1} - 1)$ flat of M_0, and consider the matroid M_0/F_0. If $\varepsilon(M_0/F_0) \geq \alpha q^{r(M_0/F_0)}$, then we have the second outcome by Theorem 2.1, so we may assume that $\varepsilon(M_0/F_0) \leq \alpha q^{r(M_0/F_0)} = \alpha q^{r(M_0) - t_{k+1} + 1}$. Let $\mathcal{F}$ be the collection of rank-t_{k+1} flats of M_0 containing F_0. Since $\cup\mathcal{F} = E(M_0)$, there is some $F \in \mathcal{F}$ satisfying

$$\begin{aligned} |F| &\geq |\mathcal{F}|^{-1}|M_0| \\ &\geq \varepsilon(M_0/F_0)(\ell+1)^{-kt_k}\beta q^{r(M_0)} \\ &\geq \alpha^{-1}q^{-r(M_0)+t_{k+1}-1}(\ell+1)^{-kt_k}\beta q^{r(M_0)} \\ &= \beta((\ell+1)^{kt_k}q\alpha)^{-1}q^{r(M_0|F)}. \end{aligned}$$

By the maximality of k, we know that $M_0|F$ is $\mathrm{GF}(q)$-representable, and $r(M_0|F) = t_{k+1} \geq f_{5.1}(f_{5.2}(n, q, \ell, kt_k), q, \beta((\ell+1)^{kt_k}q\alpha)^{-1})$, so $M_0|F$ has an $\mathrm{AG}(f_{5.2}(n, q, \ell, kt_k) - 1, q)$-restriction by Theorem 5.1. Now $M_0 = \mathrm{si}(M_1/E(S))$ and $r(S) \leq kt_k$, so by Lemma 5.2, M_1 has an $\mathrm{AG}(n-1, q)$-restriction, and so does M_2.

Case 2: $k = h$.

Note that $\varepsilon(M_1) \geq \beta q^{r(M_1)} \geq \alpha_{2.1}(m, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^{r(M_1)}$, so by Theorem 2.1 the matroid M_1 has a $\mathrm{PG}(m-1, q')$-minor for some prime power $q' > q - \frac{1}{2}$. If $q' > q$, then we have the second outcome, since $m \geq n$. Therefore we may assume that M_1 has a $\mathrm{PG}(m-1, q)$-minor. Sine M_1 also has a (q, h, t_h)-stack restriction, the second outcome now follows from Lemma 4.3 and the definitions of m and h. □

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