A DENSITY HALES-JEWETT THEOREM FOR MATROIDS

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Abstract. We show that if \( \alpha \) is a positive real number, \( n \) and \( \ell \) are integers exceeding 1, and \( q \) is a prime power, then every simple matroid \( M \) of sufficiently large rank, with no \( U_{2,\ell} \)-minor, no rank-\( n \) projective geometry minor over a larger field than \( \text{GF}(q) \), and at least \( \alpha q^{r(M)} \) elements, has a rank-\( n \) affine geometry restriction over \( \text{GF}(q) \). This result can be viewed as an analogue of the multidimensional density Hales-Jewett theorem for matroids.

1. Introduction

For a matroid \( M \), let \(|M|\) denote the number of elements of \( M \). Furstenberg and Katznelson [3] proved the following result, implying that \( \text{GF}(q) \)-representable matroids of nonvanishing density and huge rank contain large affine geometries as restrictions:

**Theorem 1.1.** Let \( q \) be a prime power, \( \alpha \in \mathbb{R}^+ \) and \( n \in \mathbb{Z}^+ \). If \( M \) is a simple \( \text{GF}(q) \)-representable matroid of sufficiently large rank satisfying \(|M| \geq \alpha q^{r(M)}\), then \( M \) has an \( \text{AG}(n,q) \)-restriction.

Later, Furstenberg and Katznelson [4] proved a much more general result, namely the multidimensional density Hales-Jewett theorem, which gives a similar statement in the more abstract setting of words over an arbitrary finite alphabet. Considerably shorter proofs [1,13] have since been found. We will generalise Theorem 1.1 in a different direction:

**Theorem 1.2.** Let \( q \) be a prime power, \( \alpha \in \mathbb{R}^+ \) and \( n \in \mathbb{Z}^+ \). If \( M \) is a simple matroid of sufficiently large rank with no \( U_{2,q+2} \)-minor and with \(|M| \geq \alpha q^{r(M)}\), then \( M \) has an \( \text{AG}(n,q) \)-restriction.

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In fact, we prove more. The class of matroids with no $U_{2,q+2}$-minor is just one of many minor-closed classes whose extremal behaviour is qualitatively similar to that of the $GF(q)$-representable matroids. The following theorem, which summarises several papers [5, 6, 9], tells us that such classes occur naturally as one of three types:

**Theorem 1.3** (Growth Rate Theorem). Let $\mathcal{M}$ be a minor-closed class of matroids, not containing all simple rank-2 matroids. There exists a real number $c_\mathcal{M} > 0$ such that either:

1. $|M| \leq c_\mathcal{M} r(M)$ for every simple $M \in \mathcal{M}$,
2. $|M| \leq c_\mathcal{M} r(M)^2$ for every simple $M \in \mathcal{M}$, and $\mathcal{M}$ contains all graphic matroids, or
3. there is a prime power $q$ such that $|M| \leq c_\mathcal{M} q^{r(M)}$ for every simple $M \in \mathcal{M}$, and $\mathcal{M}$ contains all $GF(q)$-representable matroids.

We call a class $\mathcal{M}$ satisfying (3) base-$q$ exponentially dense. It is clear that these classes are the only ones that contain arbitrarily large affine geometries, and that the matroids with no $U_{2,q+2}$-minor form such a class. Our main result, which clearly implies Theorem 1.2, is the following:

**Theorem 1.4.** Let $\mathcal{M}$ be a base-$q$ exponentially dense minor-closed class of matroids, $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. If $M \in \mathcal{M}$ is simple, satisfies $|M| \geq \alpha q^{r(M)}$, and has sufficiently large rank, then $M$ has an $AG(n,q)$-restriction.

Finding such a highly structured restriction seems very surprising, given the apparent wildness of general exponentially dense classes. This will be proved using Theorem 1.3 and a slightly more technical statement, Theorem 6.1; the proof extensively uses machinery developed in [7], [8], [14] and [15].

We would like to prove a result corresponding to Theorem 1.4 for quadratically dense classes, those satisfying condition (2) of Theorem 1.3. The following is a corollary of the Erdős-Stone Theorem [2]:

**Theorem 1.5.** Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. If $G$ is a simple graph such that $|E(G)| \geq \alpha |V(G)|^2$ and $|V(G)|$ is sufficiently large, then $G$ has a $K_{n,n}$-subgraph.

In light of this, we expect that the unavoidable restrictions of dense matroids in a quadratically dense class are the cycle matroids of large complete bipartite graphs.

**Conjecture 1.6.** Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids, $\alpha > 0$ a real number, and $n$ a positive integer. If
$M \in \mathcal{M}$ is simple, satisfies $|M| \geq cr(M)^2$, and has sufficiently large rank, then $M$ has an $M(K_{n,n})$-restriction.

2. Preliminaries

We follow the notation of Oxley [16]. For a matroid $M$, we also write $\varepsilon(M)$ for $|\text{si}(M)|$, or the number of points or rank-1 flats in $M$. If $\ell \geq 2$ is an integer, we write $U(\ell)$ for the class of matroids with no $U_{2,\ell+2}$-minor.

The next theorem, a constituent of Theorem 1.3, follows easily from the two main results of [5].

**Theorem 2.1.** There is a function $\alpha_2 : \mathbb{Z} \times \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ so that, for all $\ell, n \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$ with $\ell, n \geq 2$ and $\gamma > 1$, if $M \in U(\ell)$ satisfies $\varepsilon(M) \geq \alpha_2(n, \gamma, \ell) r(M)$, then $M$ has a $PG(n - 1, q)$-minor for some $q > \gamma$.

The next theorem is due to Kung [11].

**Theorem 2.2.** If $\ell \geq 2$ and $M \in U(\ell)$, then $\varepsilon(M) \leq (\ell + 1) r(M) - 1$.

We will sometimes use the cruder estimate $\varepsilon(M) \leq (\ell + 1) r(M) - 1$ for ease of calculation, such as in the following simple corollary:

**Corollary 2.3.** If $\ell \geq 2$ is an integer, $M \in U(\ell)$, and $C \subseteq E(M)$ satisfies $r(M|C) < r(M)$, then $\varepsilon(M/C) \geq (\ell + 1) - r(M|C)\varepsilon(M)$.

*Proof.* Let $F$ be the collection of rank-$(r(M|C) + 1)$ flats of $M$ containing $C$. We have $\varepsilon(M|F) \leq \frac{r(M|C) + 1}{\ell - 1} \leq (\ell + 1) r(M|C)$ for each $F \in \mathcal{F}$. Moreover, $|\mathcal{F}| = \varepsilon(M/C)$, and $\varepsilon(M) \leq \sum_{F \in \mathcal{F}} \varepsilon(M|F)$; the result follows. \hfill \Box

We apply both Theorem 2.2 and Corollary 2.3 freely. The next result follows from [8, Lemma 3.1].

**Lemma 2.4.** Let $q$ be a prime power, $k \geq 0$ be an integer, and $M$ be a matroid with a $PG(r(M) - 1, q)$-restriction $R$. If $F$ is a rank-$k$ flat of $M$ that is disjoint from $E(R)$, then $\varepsilon(M/F) \geq \frac{q^{r(M/F) + k} - 1}{q - 1} - q^{2k - 1}.$

3. Connectivity

A matroid $M$ is weakly round if there is no pair of sets $A, B$ with union $E(M)$, such that $r_M(A) \leq r(M) - 1$ and $r_M(B) \leq r(M) - 2$. This is a variation on roundness, a notion equivalent to infinite vertical connectivity introduced by Kung in [12] under the name of non-splitting. Our tool for reducing Theorem 1.4 to the weakly round case is the following, proved in [14, Lemma 7.2].
Lemma 3.1. There is a function \( f_{3,1} : \mathbb{Z}^3 \to \mathbb{Z} \) so that, for all \( r,d,\ell \in \mathbb{Z} \) with \( \ell \geq 2 \) and \( r \geq d \geq 0 \), and every real-valued function \( g(n) \) satisfying \( g(d) \geq 1 \) and \( g(n) \geq 2g(n-1) \) for all \( n > d \), if \( M \in \mathcal{U}(\ell) \) satisfies \( r(M) \geq f_{3,1}(r,d,\ell) \) and \( \varepsilon(M) > g(r(M)) \), then \( M \) has a weakly round restriction \( N \) such that \( r(N) \geq r \) and \( \varepsilon(N) > g(r(N)) \).

Our next lemma, proved in [8, Lemma 8.1], allows us to exploit weak roundness by contracting an interesting low-rank restriction onto a projective geometry.

Lemma 3.2. There is a function \( f_{3,2} : \mathbb{Z}^4 \to \mathbb{Z} \) so that, for every prime power \( q \) and all \( n,\ell,t \in \mathbb{Z} \) with \( n \geq 1, \ell \geq 2 \) and \( t \geq 0 \), if \( M \in \mathcal{U}(\ell) \) is a weakly round matroid with a \( \mathrm{PG}(f_{3,2}(n,q,\ell,\ell)-1,q) \)-minor and \( T \) is a restriction of \( M \) with \( r(T) \leq t \), then there is a minor \( N \) of \( M \) of rank at least \( n \), such that \( T \) is a restriction of \( N \), and \( N \) has a \( \mathrm{PG}(r(N)-1,q) \)-restriction.

4. Stacks

We now define an obstruction to \( \mathrm{GF}(q) \)-representability. If \( q \) is a prime power, and \( h \) and \( t \) are nonnegative integers, then a matroid \( S \) is a \((q,h,t)\)-stack if there are pairwise disjoint subsets \( F_1, F_2, \ldots, F_h \) of \( E(S) \) such that the union of the \( F_i \) is spanning in \( S \), and for each \( i \in \{1, \ldots, h\} \), the matroid \( (S/(F_1 \cup \ldots \cup F_{i-1}))|F_i \) has rank at most \( t \) and is not \( \mathrm{GF}(q) \)-representable. We write \( F_i(S) \) for \( F_i \). Note that such a stack has rank at most \( ht \). When the value of \( t \) is unimportant, we refer simply to a \((q,h)\)-stack.

The next three results suggest that stacks are incompatible with large projective geometries. First we argue that a matroid obtained from a projective geometry by applying a small extension and contraction does not contain a large stack:

Lemma 4.1. Let \( q \) be a prime power and \( h \) be a nonnegative integer. If \( M \) is a matroid and \( X \subseteq E(M) \) satisfies \( r_M(X) \leq h \) and \( \mathrm{si}(M\setminus X) \cong \mathrm{PG}(r(M)-1,q) \), then \( M/X \) has no \((q,h+1)\)-stack restriction.

Proof. The result is clear if \( h = 0 \); suppose that \( h > 0 \) and that the result holds for smaller \( h \). Moreover, suppose that \( M/X \) has a \((q,h+1,t)\)-stack restriction \( S \). Let \( F = F_i(S) \). Since \( (M/X)|F \) is not \( \mathrm{GF}(q) \)-representable but \( M|F \) is, it follows that \( r_{M/F}(X) > 0 \). Therefore \( r_{M/F}(X) < r_M(X) \leq h \) and \( \mathrm{si}(M/F \setminus X) \cong \mathrm{PG}(r(M/F)-1,q) \), so by the inductive hypothesis \( M/(X \cup F) \) has no \((q,h)\)-stack restriction. Since \( M/(X \cup F)|(E(S)-F) \) is clearly such a stack, this is a contradiction. \( \square \)
Lemma 4.2. Let $q$ be a prime power and $h$ be a nonnegative integer. If $M$ is a matroid with a PG($r(M) - 1, q$)-restriction $R$ and a $(q, (h+1)/2)$-stack restriction, then $M$ has a rank-$h$ flat that is disjoint from $E(R)$.

Proof. If $h = 0$, then there is nothing to show; suppose that $h > 0$ and that the result holds for smaller $h$. Let $S$ be a $(q, (h+1)/2)$-stack restriction of $M$ and let $F_i = F_i(S)$ for each $i \in \{1, \ldots, (h+1)/2\}$. Let $S_1 = S\left(F_1 \cup \ldots \cup F_{(h+1)/2}\right)$. Clearly $S_1$ is a $(q, (h)/2)$-stack, so inductively there is a rank-$(h-1)$ flat $H$ of $M$ that is disjoint from $E(R)$.

Note that $(M/H)|E(R)$ has no loops. If $M/H$ has a nonloop $e$ that is not parallel to an element of $R$, then $cl_M(H \cup \{e\})$ is a rank-$h$ flat of $M$ disjoint from $E(R)$, and we are done. Therefore we may assume that $si(M/H) \cong si((M/H)|E(R))$, and so by Lemma 4.1 applied to the matroid $M|(E(R) \cup H)$, we know that $M/H$ has no $(q, h)$-stack restriction. However the sets $(E(S_1) - H) \cup F_{(h+1)/2 + 1}, F_{(h+2)/2}, \ldots, F_{(h+1)/2}$ clearly give rise to such a stack. This is a contradiction. □

Finally we show that a large stack restriction, together with a very large projective geometry minor, gives a projective geometry minor over a larger field:

Lemma 4.3. There are functions $f_{4.3} : \mathbb{Z}^4 \to \mathbb{Z}$ and $h_{4.3} : \mathbb{Z}^3 \to \mathbb{Z}$ so that, for every prime power $q$ and all $\ell, n, t \in \mathbb{Z}$ with $\ell, n \geq 2$ and $t \geq 0$, if $M \in \mathcal{U}(\ell)$ is weakly round and has a PG($f_{4.3}(n, q, t, \ell) - 1, q$)-minor and a $(q, h_{4.3}(n, q, t), t)$-stack restriction, then $M$ has a PG($n - 1, q'$)-minor for some $q' > q$.

Proof. Let $q$ be a prime power and $\ell \geq 2$, $n \geq 2$ and $t \geq 0$ be integers. Let $\alpha = \alpha_{2.1}(n, q, \ell)$, and let $h' > 0$ and $r \geq 0$ be integers so that $q^{r + h' - 1}/q - q^{2h' - 1}/q > \alpha q'$ for all $r' \geq r$. Set $h_{4.3}(n, q, \ell) = h = (h' + 1)/2$, and $f_{4.3}(n, q, t, \ell) = f_{3.2}(r + h', q, th, \ell)$.

Let $M \in \mathcal{U}(\ell)$ be weakly round with a PG($f_{4.3}(n, q, t, \ell) - 1, q$)-minor and a $(q, h, t)$-stack restriction $S$. We have $r(S) \leq th$; by Lemma 3.2 there is a minor $N$ of $M$, of rank at least $r + h'$, with a PG($r(N) - 1, q$)-restriction $R$, and $S$ as a restriction. By Lemma 4.2, there is a rank-$h'$ flat $F$ of $M$ that is disjoint from $E(R)$. Now $r(M/F) \geq r$; the lemma follows from Lemma 2.4, Theorem 2.1, and the definition of $h'$.

5. Lifting

The following is a restatement of Theorem 1.1:
Theorem 5.1. There is a function $f_{5.1} : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{Z}$ so that, for every prime power $q$ and all $n \in \mathbb{Z}^+$ and $\beta \in \mathbb{R}^+$, if $M$ is a GF($q$)-representable matroid satisfying $\varepsilon(M) \geq \beta q^{r(M)}$ and $r(M) \geq f_{5.1}(n, q, \beta)$, then $M$ has an AG($n - 1, q$)-restriction.

This next lemma uses the above to show that a bounded lift of a huge affine geometry itself contains a large affine geometry. The proof does not use the full strength of 5.1; the lemma would also follow from the much weaker ‘colouring’ Hales-Jewett Theorem [10].

Lemma 5.2. There is a function $f_{5.2} : \mathbb{N} \to \mathbb{N}$ so that, for every prime power $q$ and all $\ell, n, t \in \mathbb{N}$ so that $\ell, n \geq 2$ and $t \geq 0$, if $M \in \mathcal{U}(\ell)$ and $C \subseteq E(M)$ satisfy $r_M(C) \leq t$, and $M/C$ has an AG($f_{5.2}(n, q, \ell, t) - 1, q$)-restriction, then $M$ has an AG($n - 1, q$)-restriction.

Proof. Let $q$ be a prime power and $\ell \geq 2$, $n \geq 2$ and $t \geq 0$ be integers. Let $d$ be an integer large enough so that $(\ell + 1)^{-t} > \frac{q^{\ell-t} - d}{q-1}$, and let $m = f_{5.1}(n, q, (q^2(\ell + 1)^t)^{-1} + d$. Set $f_{5.2}(n, q, \ell, t) = m$.

Let $M \in \mathcal{U}(\ell)$ and let $C \subseteq E(M)$ be a set so that $r_M(C) \leq t$ and $M/C$ has an AG($m - 1, q$)-restriction $R$. We may assume that $C$ is independent and that $E(M) = E(R) \cup C$, so $M$ is simple and $r(M) = m + |C|$. Let $B$ be a basis for $M$ containing $C$, and let $e \in B - C$. Let $X = B - (C \cup \{e\})$. Now $\text{cl}_{M/C}(X)$ is a hyperplane of $R$, so $|\text{cl}_{M/C}(X)| = q^{m-2}$ and there are at least $q^{m-1} - q^{m-2} \geq q^{m-2}$ elements of $M$ not spanned by $X \cup C$. Each such element lies in a point of $M/X$ and is not spanned by $C$ in $M/X$. Moreover, $r(M/X) = t + 1$, so by Theorem 2.2, $M/X$ has at most $(\ell + 1)^t$ points; there is thus a point $P$ of $M/X$, not spanned by $C$, with $|P| \geq (\ell + 1)^t q^{m-2}$.

Now $P \subseteq E(R)$, so the matroid $(M/C)|P$ is GF($q$)-representable and has rank at most $m$, and $\varepsilon((M/C)|P) \geq (\ell + 1)^{-t} q^{m-2} > \frac{q^{m-d-1}}{q-1}$, so $r((M/C)|P) \geq m - d$. Furthermore, $\varepsilon((M/C)|P) \geq (q^2(\ell + 1)^t)^{-1} q^m \geq (q^2(\ell + 1)^t) - q^{r((M/C)|P)}$, so by Theorem 5.1 and the definition of $m$, the matroid $(M/C)|P$ has an AG($n - 1, q$)-restriction. However, $P$ is skew to $C$ in $M$ by construction, so $(M/C)|P = M|P$ and therefore $M$ also has an AG($n - 1, q$)-restriction, as required.

6. The Main Result

Since, for any base-$q$ exponentially dense minor-closed class $\mathcal{M}$, there is some $\ell \geq 2$ such that $\mathcal{M} \subseteq \mathcal{U}(\ell)$ and there is some $s$ such that $\text{PG}(s, q') \notin \mathcal{M}$ for all $q' > q$, the next theorem easily implies Theorem 1.4.
Theorem 6.1. There is a function \( f_{6.1} : \mathbb{Z}^3 \times \mathbb{R} \to \mathbb{Z} \) so that for every prime power \( q \) and all \( n, \ell \in \mathbb{Z} \) and \( \beta \in \mathbb{R}^+ \) with \( n, \ell \geq 2 \), if \( M \in \mathcal{U}(\ell) \) satisfies \( r(M) \geq f_{6.1}(n, q, \ell, \beta) \) and \( \varepsilon(M) \geq \beta q^{r(M)} \), then \( M \) has either an AG\((n - 1, q)\)-restriction or a PG\((n - 1, q')\)-minor for some \( q' > q \).

Proof. Let \( \beta > 0 \) be a real number, \( q \) be a prime power, and \( \ell, n \geq 2 \) be integers. Let \( \alpha = \alpha_{2.1}(n, q, \ell) \) and \( h = h_{4.3}(n, q, \ell) \). Set \( 0 = t_0, t_1, \ldots, t_h \) to be a nondecreasing sequence of integers such that

\[
t_{k+1} \geq f_{5.1}(f_{5.2}(n, q, \ell, k t_k), q, \beta((\ell + 1)^{k t_k} q \alpha)^{-1})
\]

for each \( k \in \{0, \ldots, h - 1\} \). Let \( m = \max(n, f_{4.3}(n, q, \ell, t_k)) \), and let \( r_1 \geq (h + 1)t_h \) be an integer large enough so that \( q^{(h+1)t_h-r_1-1} \leq \alpha \) and \( \beta q^{r} \geq \alpha_{2.1}(m, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^r \) for all \( r \geq r_1 \). Let \( d \) be an integer such that \( \beta q^d \geq 1 \), and let \( r_2 = f_{3.1}(r_1, d, \ell) \).

Let \( M_2 \in \mathcal{U}(\ell) \) satisfy \( r(M_2) \geq r_2 \) and \( \varepsilon(M_2) \geq \beta q^{r(M_2)} \); we will show that \( M_2 \) has either a PG\((n - 1, q')\)-minor for some \( q' > q \), or an AG\((n - 1, q)\)-restriction. The function \( g(r) = \beta q^r \) satisfies \( g(d) \geq 1 \) and \( g(r) \geq 2g(r-1) \) for all \( r > d \), so by Lemma 3.1 the matroid \( M_2 \) has a weakly rank restriction \( M_1 \) such that \( r(M_1) \geq r_1 \) and \( \varepsilon(M_1) \geq \beta q^{r(M_1)} \).

Let \( k \) be the maximal element of \( \{0, 1, \ldots, h\} \) such that \( M_1 \) has a \((q, k, t_k)\)-stack restriction; call this restriction \( S \). We split into cases depending on whether \( k = h \):

**Case 1**: \( k < h \).

Let \( M_0 = \text{si}(M_1/E(S)) \); note that \( r(M_0) \geq r(M_1) - k t_k \), and therefore that \( |M_0| \geq (\ell + 1) - k t_k |M_1| \geq (\ell + 1)^{k t_k} \beta q^{r(M_0)} \). Let \( F_0 \) be a rank-\((t_{k+1} - 1)\) flat of \( M_0 \), and consider the matroid \( M_0/F_0 \). If \( \varepsilon(M_0/F_0) \geq \alpha q^{r(M_0/F_0)} \), then we have the second outcome by Theorem 2.1, so we may assume that \( \varepsilon(M_0/F_0) \leq \alpha q^{r(M_0/F_0)} = \alpha q^{r(M_0) - t_{k+1} + 1} \). Let \( \mathcal{F} \) be the collection of rank-\(t_{k+1}\) flats of \( M_0 \) containing \( F_0 \). Since \( \cup \mathcal{F} = E(M_0) \), there is some \( F \in \mathcal{F} \) satisfying

\[
|F| \geq |\mathcal{F}|^{-1}|M_0|
\]

\[
\geq \varepsilon(M_0/F_0)(\ell + 1)^{-k t_k} \beta q^{r(M_0)}
\]

\[
\geq \alpha^{-1} q^{-r(M_0)+t_{k+1}-1}(\ell + 1)^{-k t_k} \beta q^{r(M_0)}
\]

\[
= \beta((\ell + 1)^{k t_k} q \alpha)^{-1} q^{r(M_0) - t_{k+1} + 1}.
\]

By the maximality of \( k \), we know that \( M_0/F \) is GF\((q)\)-representable, and \( r(M_0/F) = t_{k+1} \geq f_{5.1}(f_{5.2}(n, q, \ell, k t_k), q, \beta((\ell + 1)^{k t_k} q \alpha)^{-1}) \), so \( M_0/F \) has an AG\((f_{5.2}(n, q, \ell, k t_k) - 1, q)\)-restriction by Theorem 5.1. Now \( M_0 = \text{si}(M_1/E(S)) \) and \( r(S) \leq k t_k \), so by Lemma 5.2, \( M_1 \) has an AG\((n - 1, q)\)-restriction, and so does \( M_2 \).
Case 2: $k = h$.

Note that $\varepsilon(M_1) \geq \beta^{q^r(M_1)} \geq \alpha_{2,1}(m, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^{r(M_1)}$, so by Theorem 2.1 the matroid $M_1$ has a $\text{PG}(m - 1, q')$-minor for some prime power $q' > q - \frac{1}{2}$. If $q' > q$, then we have the second outcome, since $m \geq n$. Therefore we may assume that $M_1$ has a $\text{PG}(m - 1, q)$-minor. Since $M_1$ also has a $(q, h, t_h)$-stack restriction, the second outcome now follows from Lemma 4.3 and the definitions of $m$ and $h$. □

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References


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