INDIFFERENT SETS FOR GENERICITY

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Abstract. This paper investigates indifferent sets for comeager classes in Cantor space focusing on the class of all 1-generic sets and the class of all weakly 1-generic sets. Jockusch and Posner showed that there exist 1-generic sets that have indifferent sets [10]. Figueira, Miller and Nies have studied indifferent sets for randomness and other notions [7]. We show that any comeager class in Cantor space contains a comeager class with a universal indifferent set. A forcing construction is used to show that any 1-generic set, or weakly 1-generic set, has an indifferent set. Such an indifferent set can be computed by any set in \( \mathcal{GL}_2 \) which bounds the (weakly) 1-generic. We show by approximation arguments that some, but not all, \( \Delta^0_2 \) 1-generic sets can compute an indifferent set for themselves. We show that all \( \Delta^0_2 \) weakly 1-generic sets can compute an indifferent set for themselves. Additional results on indifferent sets, including one of Miller, and two of Fitzgerald, are presented.

1. Introduction

The 1-generic sets play a central role in computability theory. Jockusch and Posner showed that there exists a 1-generic set \( G \), and an infinite set of locations \( I \), such that no matter how \( G \) is changed on the locations specified by \( I \), the resulting set is also 1-generic [10]. We call such an \( I \), an indifferent set for \( G \) with respect to 1-genericity. The term indifferent set was introduced by Figueira, Miller and Nies [7]. Figueira, Miller and Nies primarily investigated indifferent sets for random sequences. For example, they showed that any Martin-Löf random sequence has an indifferent set with respect to Martin-Löf randomness.

This paper investigates indifferent sets for comeager classes in Cantor space focusing on the comeager class of all 1-generic sets and the comeager class of all weakly 1-generic sets. In Section 3, we will consider universal indifferent sets for comeager classes in Cantor space. If \( \mathcal{A} \) is a comeager class in Cantor space, then a universal indifferent set for \( \mathcal{A} \), is a set \( I \) such that for all \( A \in \mathcal{A} \), no matter how \( A \) is changed on the bits specified by \( I \), the resulting sequence remains in \( \mathcal{A} \). We will show in Theorem 3.3 that any comeager class in Cantor space contains a comeager class with a universal indifferent set. The classes of weakly \( n \)-generic sets are natural examples of comeager classes in computability theory. We will establish in Theorem 3.4, that for any \( n \), the class of weakly \( n \)-generic sets has a universal indifferent set.

The class of all 1-generic sets is arguably the most important example of a comeager class in computability theory. Figueira, Miller and Nies were aware that all 1-generic sets have indifferent sets, though this result was

\[1\] We will define a 1-generic set and other key concepts in Section 2.
Fitzgerald established a few preliminary results on indifferent sets for 1-generic sets while he was a student at Victoria University of Wellington. These results were never published and we will present two of them here. We cannot investigate the question of indifferent sets for 1-generic sets using the approach of Section 3, because Miller has established that there is no universal indifferent set for the class of all 1-generic sets. We present his result in Theorem 3.6. Instead we will look for an indifferent set for a given 1-generic set. In Theorem 4.5, we establish a strong existence result for indifferent sets for 1-generic sets. All 1-generic sets $G$ have an indifferent set $I$ that is also 1-generic. Further, such a set $I$ can be found below any set $A \in \overline{GL}_2$ that bounds $G$. An easy corollary to Theorem 4.5 is that given any countable class of 1-generic sets, there is a set which is indifferent for all elements of this class. Fitzgerald established that any $\Delta^0_2$ 1-generic set has a co-c.e. indifferent set. We present his result in Theorem 4.8.

One use of indifferent sets is as coding locations. In Corollary 4.7, we use encoding with indifferent sets to show that if $X \in \overline{GL}_2$, then for every 1-generic set $G$ such that $X \geq_T G$, there is another 1-generic set $\hat{G}$ such that $X \equiv_T G \oplus \hat{G}$.

In Section 5 we examine the relationship between sparseness and indifferent sets. Fitzgerald showed that any indifferent set for a 1-generic set must be hyperimmune. In Theorem 5.3 we establish a sparseness condition that is sufficient for a set to be the indifferent set for some 1-generic set.

An implication of Theorem 4.5 is that any 1-generic set has a $GL_1$ indifferent set. This contrasts strongly with indifferent sets for Martin-Löf randomness which must be complete [7]. The fact that indifferent sets for 1-generic sets can be computationally weak raises the possibility that a 1-generic set might be able to compute its own indifferent set. We investigate this question in Section 6 for $\Delta^0_2$ 1-generic sets. We establish that some but not all $\Delta^0_2$ 1-generic sets have this property. We consider which c.e. sets bound a 1-generic set with this property. We show that any c.e. set which is not of totally $\omega$-c.a. degree bounds such a 1-generic set. On the other hand no c.e. set of totally $\omega$-c.a. degree bounds such a 1-generic set. These results are presented in Theorems 6.8 and 6.16 respectively.

In Section 7, we consider similar questions for weakly 1-generic sets. These results offer interesting contrasts with those for 1-generic sets. First in Theorem 7.1, we show that any hyperimmune set $I$ computes a weakly 1-generic set $G$ that $I$ is an indifferent set for. This tells us that $I$ is an indifferent set for some weakly 1-generic set if and only if $I$ is hyperimmune (Corollary 7.2). In Theorem 7.4 we show that if a set $G$ is weakly 1-generic and $I$ is a set whose principal function escapes domination by any $G$-computable function then $I$ is an indifferent set for $G$. Further, we show in Theorem 7.5, just as is the case for 1-generic sets, that if $A \in \overline{GL}_2$ then $A$ computes an indifferent set for any weakly 1-generic set it bounds. A difference to the case of 1-generic sets is provided in Theorem 7.6. In this theorem we show that any $\Delta^0_2$ weakly 1-generic set computes a set it is indifferent to.

We conclude in Section 8 with some open questions.

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2This was observed by the anonymous referee of their paper.
Given $A$ and $X \subseteq \omega$, we will write both $A_{\downarrow X}$ and $A \Delta X$ to denote the symmetric difference of $A$ and $X$, i.e. the set $(A \setminus X) \cup (X \setminus A)$. This is the set which differs from $A$ at precisely the elements of $X$. If $\sigma, \tau \in 2^{<\omega}$, then by $\sigma \Delta \tau$ we mean $\sigma 0^\omega \Delta \tau 0^\omega$. Our central definition is the following.

**Definition 2.1.** Let $A \subseteq 2^\omega$ and $I \subseteq \omega$.

1. Take $A \in I$. If for all $X \subseteq I$ we have that $A_{\downarrow X} \in A$ then we call $I$ an indifferent set for $A$ with respect to $A$.
2. We call $I$ a universal indifferent set for $A$ if $I$ is an indifferent set for all $A \in A$ with respect to $A$.

We will be interested in the case in which $A$ is comeager. Meager and comeager sets were introduced by Baire and while they have a wider definition, we will restrict our attention to Cantor space. A subset of Cantor space is comeager if it contains the intersection of a countable family of open dense sets. A subset is meager if its complement is comeager.

We denote the $e$th c.e. set of finite strings by $S_e$.

Let $A \subseteq \omega$ and $S \subseteq 2^{<\omega}$. We will say that $A$ meets $S$ if $\exists \sigma \prec A (\sigma \in S)$, and that $A$ avoids $S$ if $\forall \sigma \prec A (\forall \tau \in S) (\sigma \not\prec \tau)$. We will also say that a string $\sigma$ meets $S$ if for some $\tau \preceq \sigma$, $\tau \in S$, and $\sigma$ avoids $S$ if no string comparable with $\sigma$ is in $S$. We call a set of strings $S$ dense if for all $\sigma \in 2^{< \omega}$ there exists $\tau \in S$ such that $\tau \succeq \sigma$.

**Definition 2.2.** If $G \subseteq \omega$ meets or avoids all sets of finite strings computably enumerable in $0^{n-1}$, then $G$ is $n$-generic. If $G$ meets all dense sets computably enumerable in $0^{n-1}$, then $G$ is weakly $n$-generic.

For an introduction to $n$-generic sets see survey papers by Jockusch and Kumabe [9, 12] and Kumabe’s thesis [11]. Weakly $n$-generic sets were introduced by Kurtz [13]. In this paper we will focus our study on $1$-generic and weakly $1$-generic sets.

A function $f$ is $\omega$-c.a. if $f(x) = \lim_s g(x, s)$ for some computable $g$ and there is some computable $h$ such that for all $x$,

$$|\{s : g(x, s + 1) \neq g(x, s)\}| \leq h(x).$$

A Turing degree is a is array non-computable, or ANC if for any $\omega$-c.a. function $g$, there is a function $f \leq_T a$ such that $f$ escapes domination by $g$. This class of degrees was introduced by Downey, Jockusch and Stob [3, 6]. A degree $a$ is in $\text{GL}_n$ if $a^n = (a \lor \emptyset)^{n-1}$. We write $\overline{\text{GL}_n}$ for the complement of $\text{GL}_n$. A well-known fact is that the degree of any $1$-generic set is $\text{GL}_1$.

The use of $\overline{\text{GL}_2}$ degrees and ANC degrees to perform Cohen forcing constructions was noted by Jockusch and Posner [10], and by Downey, Jockusch and Stob [6] respectively. Forcing using $\overline{\text{GL}_2}$ degrees makes use of the following characterisation of Martin: a $\in \overline{\text{GL}_2}$ if and only if for any function $g \leq_T a \lor \emptyset$ there is a function $f \leq_T a$ such that $f$ escapes domination by $g$ [14]. The following theorem of Cai and Shore extends these ideas and helps us understand the computational power required to undertake the forcing constructions used in this paper [2]. An $A$-computable notion of forcing $P$, is a set $P$ of forcing conditions with a partial order $\leq_P$ on $P$ which contains
a greatest element 1, such that \( P \) is computable in \( A \). Let \( C \) be a sequence of dense subsets of \( P \). A sequence \( \langle p_i \rangle \) is \( C \)-generic if it meets each element of \( C \) and for all \( i, p_i \geq p_{i+1} \).

**Theorem 2.3** (Cai, Shore). Suppose that \( P \) is an \( A \)-computable notion of forcing, that \( C = \langle D_n \rangle \) is a sequence of sets dense in \( P \), and that there is a function \( d(x,y) = \Phi(A \oplus \emptyset; x,y) \) witnessing their density, i.e. \( \forall p \in P \forall n (d(p,n) \leq p \land d(p,n) \in D_n) \).

1. If \( A \in \text{GL}_2 \) then there is a \( C \)-generic sequence computable in \( A \).
2. If \( A \in \text{ANC} \) and the use from \( \emptyset \) in the computation of \( \Phi(A \oplus \emptyset; x,y) \) is bounded by a function computable in \( A \), then there is also a \( C \)-generic sequence computable in \( A \).

A function computable in \( \emptyset \) with use bounded by a computable function is called \textit{wtt-reducible} to \( \emptyset' \). A c.e. degree \( a \) is of \textit{totally \( \omega \)-c.a.} degree if for all \( f \leq_T a \), \( f \) is \( \omega \)-c.a. The class of totally \( \omega \)-c.a. sets was introduced by Downey, Greenberg and Weber \([5]\) and also studied by Balcázar, Downey and Greenberg \([1]\). A forthcoming monograph of Downey and Greenberg generalises this concept \([4]\). The terminology below follows that monograph.

Let \( R = (R, \leq_R) \) be a computable well-ordering of a computable set \( R \). An \( R \)-computable approximation of a function \( f \) is a computable approximation \( \langle f_s \rangle_{s<\omega} \) of \( f \), equipped with a uniformly computable sequence \( \langle o_s \rangle_{s<\omega} \) of functions from \( \omega \) to \( R \) such that for all \( x \) and \( s \):

- \( o_{s+1}(x) \leq_R o_s(x) \).
- If \( f_{s+1}(x) \neq f_s(x) \), then \( o_{s+1}(x) <_R o_s(x) \).

The sequence \( \langle o_s \rangle_{s<\omega} \), together with the well-foundedness of \( R \), witnesses the fact that the approximation \( \langle f_s \rangle_{s<\omega} \) indeed reaches a limit.

**Definition 2.4.** A function \( f : \omega \rightarrow \omega \) is \( R \)-computably approximable (or \( R \)-c.a.) if it has an \( R \)-computable approximation.

It is possible to use this definition to establish a hierarchy in the Turing degrees by restricting ourselves to certain well-orderings. Every ordinal \( \alpha \) has a unique expression as the sum

\[
\omega^{\alpha_1}n_1 + \omega^{\alpha_2}n_2 + \cdots + \omega^{\alpha_k}n_k
\]

where \( n_i < \omega \) are nonzero and \( \alpha_1 > \alpha_2 > \cdots > \alpha_k \) are ordinals. This is called the \textit{Cantor normal form} of \( \alpha \). Further,

\[
\varepsilon_0 = \sup \left\{ \omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots \right\}
\]

is the least ordinal \( \gamma \) such that \( \omega^\gamma = \gamma \), so for all \( \alpha < \varepsilon_0 \), every ordinal appearing in the Cantor normal form of \( \alpha \) is strictly smaller than \( \alpha \).

Let \( R = (R, <_R) \) be a computable well-ordering, and let \( | \cdot | : R \rightarrow \text{otp}(R) \) be the unique isomorphism between \( R \) and its order-type. The pullback to \( R \) of the Cantor normal form function is the function \( \text{nf}_R \) whose domain is \( R \) and is defined by letting

\[
\text{nf}_R(x) = \langle (z_1, n_1), (z_2, n_2), \ldots, (z_k, n_k) \rangle
\]

where \( n_i < \omega \) are nonzero, \( z_1 \in R, z_1 >_R z_2 >_R \cdots >_R z_k \), and

\[
|z| = \omega^{|z_1|}n_1 + \omega^{|z_2|}n_2 + \cdots + \omega^{|z_k|}n_k.
\]
Definition 2.5. A computable well-ordering $R$ is canonical if its associated Cantor normal form function $nf_R$ is also computable.

Downey and Greenberg have established that for every ordinal $\alpha \leq \varepsilon_0$ there is a canonical well-ordering of order-type $\alpha$ and further that any two canonical well-orderings of order-type $\alpha$ are computably isomorphic. Hence we can define a function $f$ as being $\alpha$-c.a. if it is $R$-c.a. for some canonical well-ordering of order-type $\alpha$.

Definition 2.6. If $\alpha \leq \varepsilon_0$, then a Turing degree $a$ is totally $\alpha$-c.a. if every function $f \in a$ is $\alpha$-c.a. It is not difficult to show that $a$ is totally $\alpha$-c.a. if and only if every function $f \leq_T a$ is $\alpha$-c.a. The following theorem establishes that the $\alpha$-c.a. degrees do indeed form a hierarchy.

Theorem 2.7 (Downey, Greenberg [4]). Let $\alpha \leq \varepsilon_0$. There is a totally $\alpha$-c.a. degree which is not totally $\gamma$-c.a. for any $\gamma < \alpha$ if and only if $\alpha$ is a power of $\omega$. If $\alpha$ is a power of $\omega$ then in fact there is a c.e. degree which is totally $\alpha$-c.a. but not totally $\gamma$-c.a. for any $\gamma < \alpha$.

3. Universal indifferent sets

We will begin by looking at indifferent sets for comeager subsets of Cantor space. Not all comeager subsets of Cantor space have a universal indifferent set. A trivial example is the set comprised of all 1-generic sets and the empty set. If this comeager set had a universal indifferent set $I$, then we could add the least element of $I$ to the empty set and obtain a finite 1-generic set.

However, we will show that any comeager subset of Cantor space contains a comeager subset with a universal indifferent set. In order to prove this, we will establish a property of a countable class of dense sets of strings $S$, that will ensure that the class of sets that meet all elements of $S$ has a universal indifferent set.

Let $S$ be a dense set of strings. A typical approach to constructing a sequence $A$ that meets $S$ is to find some string $\sigma \in S$ and then fix $\sigma$ to be an initial segment of $A$. However, as the following lemma shows, the segment that we fix does not need to be an initial segment of $A$.

Lemma 3.1. If $S$ is a dense set of strings, then for all $n \in \omega$, there some $\tau_n \in 2^{\omega}$ such that for all $X \in 2^\omega$, if $(X \upharpoonright n) \tau_n \prec X$ then $X$ meets $S$.

Proof. Fix some enumeration of $S$. To find $\tau_n$, let $\sigma_1, \ldots, \sigma_k$ be a list of all finite strings of length $n$. Define $\rho_1$ such that $\sigma_1 \rho_1 \in S$ and $\rho_1$ is the first string observed with this property. Once $\rho_i$ is defined, if $i < k$, define $\rho_{i+1}$ such that $\sigma_{i+1} \rho_1 \rho_2 \ldots \rho_i \rho_{i+1} \in S$ and $\rho_{i+1}$ is the first string observed with this property. Define $\tau_n = \rho_1 \ldots \rho_k$.

Take any $X \subseteq \omega$ such that $(X \upharpoonright n) \tau_n \prec X$. Fix the $i$ such $(X \upharpoonright n) = \sigma_i$. Thus $\sigma_1 \rho_1 \ldots \rho_i \prec X$ and hence $X \in S$. □

Given a dense set of strings $S$, let $\tau_i$ witness the truth of Lemma 3.1 for each $i \in \omega$. We can inductively define a pair of functions $t : \omega \rightarrow 2^{<\omega}$ and $l : \omega \rightarrow \omega$ by

$$l(0) = 0, \quad t(0) = \tau_0, \quad l(n + 1) = l(n) + |t(n)|, \quad t(n + 1) = \tau_{l(n+1)}.$$
If for any \( n \) we have that \((X \upharpoonright l(n))t(n) \prec X\), then \( X \) meets \( S \). This section will make use of pairs of function of this type so we will introduce the following term. We call a pair of functions \((t, l) \in (2^{<\omega})^\omega \times \omega^\omega \) suitable for \( S \) if for all \( n \):

1. \( l(n + 1) = l(n) + |t(n)|. \)
2. For all \( X \in 2^\omega \) if \((X \upharpoonright l(n))t(n) \prec X\), then \( X \) meets \( S \).

Let \( S \) be a class of dense sets of strings. We say that \( S \) has property \((\ast)\) if for all \( S \subseteq S \) there is a pair of functions \((t, l) \) suitable for \( S \) such that for all \( n \), the following set is also in \( S \):

\[
\{ \sigma \tau \rho : \exists m > n[|\sigma| = l(2m) \land \tau = t(2m) \land \rho = t(2m + 1)] \}. \tag{3.1}
\]

**Lemma 3.2.** If \( S = \{S_i\}_{i \in \omega} \) is a countable sequence of dense sets of strings with property \((\ast)\), then there is a universal indifferent set for \( A = \{X \subseteq \omega : (\forall i)X \text{ meets } S_i\} \).

**Proof.** For each \( i \in \omega \), we can take \((t_i, l_i)\) to be a pair of suitable functions for \( S_i \) that witness property \((\ast)\). Let \( I \) be an infinite set such that:

\[
(\forall i)(\forall \omega) \exists m > n[|\sigma| = l_1(2m) \land \tau = t_1(2m) \land \rho = t_1(2m + 1)] \}
\]

We claim that \( I \) is a universal indifferent set for \( A \). Let \( A \subseteq A \) and \( X \subseteq I \). Consider any dense set \( S_i \). Now because of property \((\ast)\) there are infinitely many \( n \) such that \((X \upharpoonright l_1(2n))t_1(2n)\rho(2n + 1) \prec A\). Now for almost all of these \( n \) we have that \([I \cap [l_1(2n), l_1(2n + 2) - 1]] \leq 1\). 

For such \( n \) either

1. \( A_{[X]} \succ (A_{[X]} \upharpoonright l_1(2n))t_1(2n)\); or
2. \( A_{[X]} \succ (A_{[X]} \upharpoonright l_1(2n + 1))t_1(2n + 1)\).

In either case we have that \( A_{[X]} \) meets \( S_i \). Thus \( A_{[X]} \in \mathcal{A} \). \( \square \)

**Theorem 3.3.** Let \( A \subseteq 2^\omega \) be comeager. There exists \( B \subseteq A \) and \( I \subseteq \omega \) such that \( B \) is comeager and \( I \) is a universal indifferent set for \( B \).

**Proof.** Let \( S = \{S_i\}_{i \in \omega} \) be a countable sequence of dense sets of strings such that \( \{X \subseteq \omega : (\forall i)X \text{ meets } S_i\} \subseteq \mathcal{A} \).

We extend \( S \) as follows. Let \( S_0 = S \). We build \( S_{n+1} \) from \( S_n \) as follows. For each \( S \in S_n \) we choose a pair of functions \((t, l) \) suitable for \( S \) and add \( S \) along with the countably many dense sets of strings defined by (3.1) to \( S_{n+1} \). Then \( S_\omega = \bigcup_n S_n \) has property \((\ast)\) and is a countable set of dense sets of strings. Thus there is a universal indifferent set for \( B = \{X \subseteq \omega : (\forall S \in S_\omega)X \text{ meets } S\} \). \( \square \)

If a set \( X \) can enumerate an open dense subset \( S \), i.e. \( S = W_\epsilon(X) \) for some \( \epsilon \), then using the procedure of Lemma 3.1, \( X \) can uniformly in \( \epsilon \) compute a pair of functions \((t, l) \) that are suitable for \( S \).

**Theorem 3.4.** Fix \( n \geq 1 \). If \( B \geq_T \emptyset^{n-1} \) and \( B' \geq_T \emptyset^{n+2} \) then \( B \) computes a universal indifferent set for the class of all weakly \( n \)-generic sets.

**Proof.** Fix \( n \geq 1 \) and let \( A = \emptyset^{n-1} \), let \( \{S_i\}_{i \in \omega} \) be an \( A \) computable enumeration of all c.e. sets of strings enumerable in \( A \). Using \( A \) as an oracle, we can in a uniform manner define \((t_i, l_i)\) such that if \( S_i \) is a dense set of strings, then

1. \((t_i, l_i) \) is suitable for \( S_i \).
(2) All sets defined from \((t_1, l_1)\) as in (3.1) are also enumerable in \(A\).

Hence the collection of open dense sets enumerable in \(A\) has property \((\ast)\).

The set \(D = \{i : S_i(A) \text{ is dense}\}\) is \(\Delta^0_2(B)\) because \(B'\overset{T}{\geq} B''\). Let \(D_s\) be a \(B\)-computable approximation to \(D\).

From \(B\) we define \(x_0 = 0\). Once \(x_s\) is defined, we find some \(t \geq s\) such that for all \(e \leq s\), one of the following is true:

1. \(D_t(e) = 0\); or
2. For some \(n_i\), we have \(l_i(2n_i)[t] \downarrow > x_s\).

We let \(x_{s+1}\) be the least element of \(\omega\) greater than both \(x_s\) and \(\max\{l_i(2n_i) : l_i(2n_i)[t] \downarrow \text{ and } e \leq s\}\). We let \(I = \{x_n : n \in \omega\}\). Now because our approximation to \(D\) must converge, \(I\) is well defined and has the desired property.

**Corollary 3.5.** If \(A, B \in 2^{\omega}\) with \(A \leq_T B\) and \(A'' \leq_T B''\) then \(B\) computes a universal indifferent set for the class of open dense sets enumerable in \(A\).

**Proof.** The restriction of \(A\) to \(0^n\) for some \(n\) is not necessary. □

Theorem 3.3 tells us that the class of 1-generic sets contains a comeager subset with a universal indifferent set. Theorem 3.4 provides a specific example, namely the class of all weakly 2-generic sets. However, is there a subset with a universal indifferent set. Theorem 3.4 provides a specific example, namely the class of all weakly 2-generic sets. However, is there a universal indifferent set for class of 1-generic sets itself? Miller, in unpublished work, has proved that there is not.

**Theorem 3.6** (Miller). Let \(I \subseteq \omega\) be infinite. There are sets \(G, A \subseteq \omega\) such that \(G\) is 1-generic, \(A\) is not 1-generic, and \(G \Delta A \subseteq I\). In other words, \(I\) is not universally indifferent for the class of 1-generic sets.

**Proof.** We define the characteristic function of \(G\) in stages. At stage 0 we pick some element \(i_0 \in I\) greater than 0. We define \(\sigma_0 = 0^n10^n\emptyset(0)\) where \(n\) is chosen so that \(|0^n1| = i_0\) i.e. \(i_0\) is the location of the 0 in \(\sigma_0\) before the coding location of \(\emptyset'(0)\).

At stage \(s + 1\), we define \(\sigma_{s+1}\) extending \(\sigma_s\) as follows. If \(\sigma_s\) avoids \(S_s\), then define \(\tau_s = \lambda\). If \(\sigma_s\) does not avoid \(S_s\) then let \(\tau_s\) be the first string enumerated into \(S_s\) that extends \(\sigma_s\). We take \(i_{s+1} \in I\) such that \(i_{s+1} > |\tau_s|\) and we now define \(\sigma_{s+1} = \tau_s0^n10^n\emptyset'(s + 1)\) where \(n\) is chosen so that \(|\tau_s0^n1| = i_{s+1}\). Let \(G = \bigcup_n \sigma_n\). The construction ensures that the set \(G\) is 1-generic. Define \(X = \{i_n : \sigma_n\text{ avoids }S_n\}\). Define \(A = G_{\upharpoonright X}\). From \(A\) and \(\sigma_n\), we can determine whether or not \(\sigma_n\) meets or avoids \(S_n\). Thus we can determine \(\sigma_n\) and so we can determine \(\emptyset'(n + 1)\). This shows that \(A \geq_T \emptyset'\) and consequently \(A\) cannot be 1-generic. □

4. INDIFFERENCE AND 1-GENERICITY

We know that there is no universal indifferent set for the class of all 1-generic sets. However we will establish that all 1-generic sets have an indifferent set with respect to the class of all 1-generic sets. It is known that there exists 1-generic sets with indifferent sets.

**Theorem 4.1** (Jockusch and Posner [10]). If \(A \in \mathcal{GL}_2\), then there exists \(G, I\) Turing below \(A\) such that \(G\) is 1-generic and \(I\) is an indifferent set for \(G\).
The terminology used by Jockusch and Posner is different. They constructed a function $f : \omega \to \{0, 1, 2\}$ such that $f$ is 1-generic and any characteristic function obtained from $f$ by replacing 2’s with 0’s and 1’s is also 1-generic. Given such an $f$, let $G = \{x : f(x) = 1\}$ and $I = \{x : f(x) = 2\}$. Clearly $I$ is an indifferent set for $G$. This result can be strengthened by the work of Cai and Shore.

**Theorem 4.2** (Jockusch and Posner; Cai and Shore [2, 10]). If $A \in \ANC$ then there exists $G, I$ Turing below $A$ such that $G$ is 1-generic and $I$ is an indifferent set for $G$.

We will strengthen Jockusch and Posner’s result in a different direction. Theorem 4.5 establishes that any $X \in \GL_2$ computes an indifferent set for any 1-generic set it bounds. We make use of the following well-known lemma.

**Lemma 4.3.** If $G$ is a 1-generic set and $X \subseteq \omega$ is finite, then $G \upharpoonright X$ is 1-generic.

Given a 1-generic set $G$ and $e, n \in \omega$ we can find some point $m$ such that no matter how we change $G$ on the first $n$ bits, the resulting set meets or avoids $S_e$ after $m$ bits.

**Lemma 4.4.** Let $G$ be a 1-generic set. There is a function $f_G : \omega^2 \to \omega$ with $f_G \leq_T G \oplus \emptyset'$ such that for all $n, e \in \omega$, for all $X \subseteq \{0, 1, \ldots, n - 1\}$, we have that $G \upharpoonright X \upharpoonright m$ meets or avoids $S_e$.

Proof. Take any $n$ and $e$ and let $X_1, X_2, \ldots, X_{2^n}$ be a list of all subsets of the set $\{0, 1, \ldots, n - 1\}$. For all $i, 1 \leq i \leq 2^n$, we have that $G \upharpoonright X_i$ is 1-generic as $X_i$ is finite. Hence there is some $m_i$ such that $G \upharpoonright X_i \upharpoonright m_i$ meets or avoids $S_e$. This $m_i$ is computable in $G \oplus \emptyset'$ because we can query whether successive initial segments of $G \upharpoonright X_i$ meet or avoid $S_e$ until we find one that does. Define $f_G(n, e) = \max\{m_i : 1 \leq i \leq 2^n\}$. □

We are now ready to give our basic existence result for indifferent sets for 1-generic sets.

**Theorem 4.5.** Given $A \geq_T G$ where $A \in \GL_2$ and $G$ is a 1-generic set, there exists $I \leq_T A$ such that $I$ is an indifferent set for $G$ with respect to 1-genericity and $I$ is 1-generic.

Proof. Let $G$ be 1-generic. We can make $I$ both 1-generic and an indifferent set for $G$ by satisfying the following requirements for all $e \in \omega$.

$Q_e$: $I$ meets or avoids $S_e$.

$R_e$: For all $X \subseteq I$ we have that $G \upharpoonright X$ meets or avoids $S_e$.

These requirements can be met by constructing an $I$ that meets the following dense sets:

$C_e = \{\sigma \in 2^{<\omega} : \sigma$ meets or avoids $S_e\}$, and

$D_e = \{\sigma \in 2^{<\omega} : \forall X \subseteq \sigma, G \upharpoonright |\sigma|$ meets or avoids $S_e\}.$

We say $X \subseteq \sigma$ if $X \subseteq \{i : i < |\sigma| \land \sigma(i) = 1\}.$
In this proof the notion of forcing that we use is just Cohen forcing so 
\( P = 2^{<\omega} \) and \( \sigma \leq_P \tau \) if \( \sigma \geq \tau \). There is a function computable in \( \emptyset' \) that uniformly witnesses the density of the sequence of sets \( \langle C_n \rangle \): let \( c(\sigma, e) \) be the first extension of \( \sigma \) to enter \( S \), if such an extension exists, or \( \sigma \) otherwise. There is also a function computable in \( G \oplus \emptyset' \) that witnesses the density of the sequence of sets \( \langle D_n \rangle \). We define \( d(\sigma, e) = \sigma \theta_G(\langle |\sigma|, e \rangle - |\sigma|) \).

If \( I \supset d(\sigma, e) \) and \( X \subseteq I \), then \( G_{|X|} \upharpoonright |d(\sigma, e)| \) can only differ from \( G \) on the first \( |\sigma| \) bits. As \( f_G(|\sigma|, e) = |d(\sigma, e)| \) by Lemma 4.4, \( G_{|X|} \upharpoonright |d(\sigma, e)| \) meets or avoids \( S_e \).

By applying Theorem 2.3, there is an A-computable \( \preceq_P \) decreasing sequence \( \langle \sigma_i \rangle \) that meets all sets in the sequences \( \langle C_n \rangle \) and \( \langle D_n \rangle \). Hence taking \( I = \lim \sigma_i \), we have that \( I \) is 1-generic and \( I \) is an indifferent set for \( G \).

**Corollary 4.6.** If \( \{G_i\}_{i \in \omega} \) is a countable family of 1-generic sets and \( A \supseteq T \)
\( \oplus_{i \in \omega} G_i \) with \( A \in \text{GL}_2 \), then there is an \( I \leq_T A \) such that for all \( i \in \omega \), \( I \) is an indifferent set for \( G_i \).

**Proof.** We simply replace the sequence of sets \( \langle D_n \rangle \) with
\[ D_{e,i} = \{ \sigma \in 2^{<\omega} : \forall X \subseteq \sigma, G_{i,\langle|\sigma|\rangle} \upharpoonright |\sigma| \text{ meets or avoids } S_e \} \].

Now because the sequence \( \{G_i\}_{i \in \omega} \) is uniformly computable in \( A \oplus \emptyset' \) that witnesses the density of this sequence.

One possible use of indifferent sets for 1-generic sets is as coding locations. We can take a 1-generic \( G \), and then form another 1-generic \( \tilde{G} \) by changing \( G \) on some bits of an indifferent set. We can recover these changes from the join of \( G \) and \( \tilde{G} \). The following theorem is an application of this idea.

**Corollary 4.7.** Given \( A \geq_T G \) where \( A \in \text{GL}_2 \) and \( G \) is a 1-generic set, there exists a 1-generic \( \tilde{G} \) such that \( G \oplus \tilde{G} \equiv_T A \).

**Proof.** \( G \) has an \( A \)-computable indifferent set \( I \) that is also 1-generic. As \( I \) is 1-generic there are infinitely many even numbers in \( I \) and infinitely many odd numbers in \( I \). Define the following \( A \) computable subset of \( I \). If \( 0 \in A \), then let \( x_0 \) be the first even element of \( I \), otherwise let \( x_0 \) be the first odd element of \( I \). We inductively define \( x_{i+1} \) to be the first even element of \( I \) greater than \( x_i \) if \( i + 1 \in A \) and the first odd element of \( I \) greater than \( x_i \) otherwise. Let \( X = \{ x_i : i \in \omega \} \). As \( G, X \leq_T A \) we have that \( G_{|X|} \leq_T A \). Further, \( A \leq_T G \oplus G_{|X|} \) because \( x \in A \) if and only if the \( i \)th position where \( G \) and \( G_{|X|} \) differ is even. Thus \( A \equiv_T G \oplus G_{|X|} \).

In our final result for this section, we will show that any \( \Delta^0_2 \) 1-generic set has a co-c.e. indifferent set. This was originally shown by Fitzgerald using a full approximation argument. We give an alternative proof.

**Theorem 4.8 (Fitzgerald).** If \( G \) is a \( \Delta^0_2 \) 1-generic set, then there exists a co-c.e. set \( I \) such that \( I \) is an indifferent set for \( G \).

**Proof.** Let \( G \in \Delta^0_2 \) be a 1-generic set. We will show that there is a function \( f \leq_T \emptyset' \) such that for any set \( I \) such that \( p_I \), the principal function of \( I \), majorizes \( f \) we have that \( I \) is an indifferent set for \( G \). Because for any \( \Delta^0_2 \) function \( f \), there is a co-c.e. set whose principal function majorizes \( f \) we are done.
Define $f(e)$ as follows. Set $n_{e,0} = 0$ and then $n_{e,i+1} = f_{G}(n_{e,i}, e) + 1$. Now set $f(e) = n_{e,e+1}$. The function $f$ is computable in $\emptyset'$ because $f_{G} \leq_T G \oplus \emptyset' = \emptyset'$. Let $I$ be a co-c.e. set such that for all $x$, $p_{I}(x) \geq f(x)$. Take any $e$. Now consider the pairwise disjoint sets $[n_{e,i}, n_{e,i+1} - 1]$ for $i$ such that $0 \leq i < e + 1$. Because $n_{e,e+1} = f(e) \leq p_{I}(e)$, there must be some $i$ such that $[n_{e,i}, n_{e,i+1} - 1] \cap I = \emptyset$. Now because $n_{e,i+1} - 1 = f_{G}(n_{e,i})$, for any $X \subseteq I$, $G_{[X]} \upharpoonright f_{G}(n_{e,i}, e)$ only differs from $G$ on the first $n_{e,i}$ many bits. Thus $G_{[X]}$ meets or avoids $S_{e}$ and so $I$ is a co-c.e. indifferent set for $G$. □

5. Sparsity of indifferent sets

In the previous section, we used a forcing construction to build an indifferent set. That construction succeeded by placing large intervals of zeros into the indifferent set. This indicates that indifferent sets for 1-generic sets may need to be sparse. The following theorem of Fitzgerald confirms this intuition by showing that they must be hyperimmune.

**Theorem 5.1** (Fitzgerald). If $I$ is an indifferent set for some 1-generic set $G$, then $I$ is hyperimmune.

**Proof.** Let $G$ be 1-generic and $X \subseteq \omega$ be a set that is not hyperimmune. Let $f$ be a computable and strictly increasing function such that for all $j$, $[f(j), f(j+1) - 1] \cap X \neq \emptyset$. Then there exists some $X \subseteq \omega$ such that for all $j$ there is a unique $n \in [f(j), f(j+1) - 1] \cap X$ if and only if $|[f(j), f(j+1) - 1]| \cap G$ is even.

Consider the set $S \subseteq 2^{<\omega}$, defined by $\sigma \tau \in S$ if for some $j$, $|\sigma| = f(j)$, $|\tau| = f(j+1) - 1 - f(j)$ and $\tau$ has an even number of bits. The set $S$ is a c.e. dense set of strings. Now $G_{[X]}$ cannot meet $S$ because for all $j$, $|[f(j), f(j+1) - 1] \cap G_{[X]}|$ must be odd. Thus $G_{[X]}$ is not 1-generic and so $X$ is not an indifferent set for $G$. □

Thus an indifferent set needs to be sparse. In fact if a set is sufficiently sparse, then it must be the indifferent set for some 1-generic set. The following lemma is a version of Lemma 3.1 for sets that are not necessarily dense.

**Lemma 5.2.** There exists an $\omega$-c.a. function $g : \omega^{2} \rightarrow 2^{<\omega}$ such that for all $n, e \in \omega$, for all $X \subseteq \omega$, if $(X \upharpoonright n)g(n, e) \prec X$ then $X$ meets or avoids $S_{e}$.

**Proof.** Fix some enumeration of the sets $S_{e}$ such that at most one string enters $S_{e}$ at any stage $s$. Given $n$ and $e$ we use the following process to determine $g(n, e)$. We let $\tau_{0} = \lambda$. Now we inductively define $\tau_{s+1}$ as follows. If we have some string $\sigma \rho$ enumerated into $S_{e}$ at stage $s$ with $|\sigma| = n$, $\rho \supseteq \tau_{s}$ and such that $\sigma$ has not been satisfied, then we define $\tau_{s+1} = \rho$ and we regard $\sigma$ as being satisfied. Otherwise, we define $\tau_{s+1} = \tau_{s}$. We let $g(n) = \lim_{s} \tau_{s}$. As $\tau_{s}$ can change at most $2^{n}$ times we have that $g$ is $\omega$-c.a.

Assume that for some $n, e \in \omega$ and $X \subseteq \omega$ we have that $(X \upharpoonright n)g(n, e) \prec X$. If during the construction of $g(n, e)$ the string $X \upharpoonright n$ was satisfied then for some $\tau \supseteq g(n, e)$ we have that $(X \upharpoonright n)\tau \in S_{e}$ and so $X$ meets $S_{e}$. If not, then there is no $\tau \supseteq g(n, e)$ such that $(X \upharpoonright n)\tau \in S_{e}$. Thus $(X \upharpoonright n)g(n)$ avoids $S_{e}$. □
Theorem 5.3. Take I to be an infinite subset of ω. There is an ω-c.a. function f, such that if

$$\exists^\infty n [n, f(n)] \cap I = \emptyset,$$

then I is an indifferent set for some 1-generic G.

Proof. Take f(n) = n + \max\{|g(n, e)| : e \leq n\} where g is the function defined in Lemma 5.2. As g is ω-c.a., so is f. Now given I such that $$\exists^\infty n [n, f(n)] \cap I = \emptyset$$ we can take a sequence n₀, n₁, ... such that for all i, $$[n_i, f(n_i)] \cap I = \emptyset,$$ and $$f(n_i) < n_{i+1}.$$ We define G as $$\lim_\sigma \sigma_i$$ where $$\sigma_0 = 0^\omega,$$ and with inductive assumption that $$|\sigma_i| = n_i,$$ we define $$\sigma_{i+1} = \sigma_i f(n_i, i) 0^{k_i}$$ where $$k_i = n_{i+1} - |\sigma_i f(n_i, i)|$$ (k_i ensures that $$|\sigma_{i+1}| = n_{i+1}$$ and $$k_i > 0$$ because $$n_{i+1} > f(n_i) \geq n_i + g(n_i, i))$$.

Now let X ⊆ I and take any e ∈ ω. We have that $$(G[X] \upharpoonright n_e) g(n_e, e) \prec G[X]$$ because I ∩ [n_e, f(n_e)] = ∅ and so by Lemma 5.2, G[X] meets or avoids S_e.

6. 1-generic sets that compute their own indifferent sets

We know that every 1-generic set has as indifferent set which is GL₁. This indicates that it may be possible for a 1-generic set to compute an indifferent set for itself. Such a 1-generic set would have the following interesting property. For all A ⊆ T G there exists a 1-generic $\bar{G}$ such that A is the join of G and $\bar{G}$. To show this, let G compute an indifferent set for itself with principal function $p_l$. If A ⊆ T G, then let X = \{p_l(x) : x ∈ A\}, and so A ⊆ T G ⊕ G[X]. In this section we will show that such 1-generic sets exist. In fact we will show that any c.e. set that is not of totally ω^l-c.a. degree computes a 1-generic set with this property. First, by appealing to known results, we will show that not all 1-generic sets have this property.

Proposition 6.1. Any 1-generic set bounded by a set with a strong minimal cover does not compute an indifferent set for itself.

Proof. If A ⊆ T B ⊆ T G ⊆ T I, where G is 1-generic and I is an indifferent set for G, then by the discussion above, there is a 1-generic $\bar{G}$ such that A = G ⊕ $\bar{G}$. If $\bar{G} \prec T A$, then as $\bar{G} \preceq T B$ we have that A is not a strong minimal cover of B. If $\bar{G} \succeq T A$, then A splits into two incomparable degrees one of which cannot be below B and so again A cannot be a strong minimal cover of B.

Corollary 6.2. Any 1-generic set bounded by an array computable c.e. set does not compute an indifferent set for itself.

Proof. This corollary follows from Ishmukhametov’s theorem that any array computable c.e. set has a strong minimal cover [8].

Corollary 6.3. There exists a $\Delta^0_2$ 1-generic set G that fails to compute an indifferent set for itself.

Proof. Any c.e. set bounds a 1-generic set and there exist array computable c.e. sets.
We will now work towards proving that every c.e. set that is not of totally \( \omega^\omega \)-c.a. degree computes a \( 1 \)-generic set which computes an indifferent set for itself. We first start by proving that \( \emptyset' \) computes such a \( 1 \)-generic set. Following this, we will examine the proof and turn it into a permissions argument.

**Theorem 6.4.** There exists a \( \Delta_2^0 \) \( 1 \)-generic set \( G \) that computes an indifferent set for itself.

**Proof.** For this construction we will build a set \( G \) and a reduction \( \Gamma \) such that \( \Gamma(G) \) is the principal function of \( I \), an indifferent set for \( G \). We have the following requirements:

- \( I_e \): \( \Gamma(G; e) \downarrow \) and if \( e > 0 \), \( \Gamma(G; e) > \Gamma(G; e - 1) \).
- \( R_e \): \( \exists w \) such that \( \forall X \subseteq \text{rng} \Gamma(G), G\mid_X \upharpoonright w \) meets or avoids \( S_e \).

If we meet all these requirements, then \( G \) is \( 1 \)-generic and \( \text{rng} \Gamma(G) \) is an infinite indifferent set for \( G \). Our requirements will be prioritised as follows:

- \( R_0 > I_0 > R_1 > I_1 > \ldots \). The construction of \( G \) is a finite injury argument.

Requirement \( I_e \) needs attention at stage \( s \) if \( \Gamma(G; e)\mid_s \uparrow \). Our action is to choose a new large number \( n \) such that \( G_s(n) = 0 \) (this will always be possible because there will only be finitely many places where \( G_s \) is \( 1 \)). We then set \( \Gamma(G; e) = n \) with use \( n \), define \( G_{s+1} = G_s \) and restrain \( G_{s+1} \upharpoonright (n + 1) \) with priority \( e \).

Each requirement \( R_e \) will have a restraint, \( r \), imposed on it by higher priority requirements. There will also be a value \( w(e, s) = 0 \) (this is our candidate for the witness \( w \)). We start with \( w(e, 0) = 0 \) for all \( e \). A requirement \( R_e \) needs attention at stage \( s \) if there exists \( \sigma \in S_{e,s} \) with \( |\sigma| > w(e, s) \), such that:

1. \( (\sigma \upharpoonright w(e, s)) \Delta(G_s \upharpoonright w(e, s)) \subseteq \text{rng} \Gamma(G)\mid_s \).
2. No initial segment of \( \sigma \upharpoonright w(e, s) \) is in \( S_{e,s} \).

The reason we need to pay attention to \( R_e \) at this stage is because \( \sigma \upharpoonright w(e, s) \) differs from \( G_s \) only on elements of \( \text{rng} \Gamma(G)\mid_s \) but \( \sigma \upharpoonright w(e, s) \) does not meet or avoid \( S_e \). This means that \( w(e, s) \) is not a correct witness for requirement \( R_e \).

Our action is like that of a greedy algorithm. We change \( G_s \) to look like \( \sigma \) anywhere we can while maintaining our constraints. Specifically we set:

\[
G_{s+1}(x) = \begin{cases} 
G_s(x) & x \leq r \\
\sigma(x) & r < x < |\sigma| \\
0 & |\sigma| \leq r.
\end{cases}
\]

We set \( w(e, s + 1) = |\sigma| \), and we restrain \( G_{s+1} \upharpoonright w(e, s + 1) \) with priority \( e \). For all \( e' < e \) we set \( w(e', s + 1) = w(e', s) \). For all \( e' > e \) we set \( w(e', s + 1) = w(e, s) \). This ends the construction.
Verification. The key step to verifying this construction is to show that each requirement needs attention finitely often. Take any requirement, and assume that at stage $s_0$, all higher priority requirements have stopped acting and a final restraint $r$ is placed on this requirement. First consider a requirement of the form $f_e$. If $\Gamma(G; e)[s_0]$ is not defined, then it will be defined at the next stage and its use preserved.

Now consider a requirement of the form $R_e$. Let $n = 2^r$. We want to show that $R_e$ acts finitely often. To prove this, for each stage $s \geq s_0$, we will choose an element of $c_s$ of the linear order $(\omega + 1)^n$ (if $a, b \in (\omega + 1)^n$ then $a < b$ if on the first coordinate that $a$ and $b$ differ the $a$ value on this coordinate is smaller than the $b$ value). We will ensure that for all $s \geq s_0$, $c_{s+1} \leq c_s$ and if $R_e$ acts at stage $s + 1$, then $c_{s+1} < c_s$. Hence as $(\omega + 1)^n$ is a well-order, we have that $R_e$ can only act finitely often.

To determine $c_s$ we first let $T \subseteq S_e$ be the set of all strings that cause $R_e$ to act after stage $s_0$. Because any element of $T$ causes $R_e$ to change, the next element of $T$ to enter $S_e$ must be of strictly greater length. Let $\sigma_{0,s}, \sigma_{1,s}, \ldots, \sigma_{m_s,s}$ be a list of all strings in $T_s$ that agree with $G_s$ after $r$. We assume that the strings are numbered so that $|\sigma_{0,s}| < |\sigma_{1,s}| < \ldots < |\sigma_{m_s,s}|$. For any $s \geq s_0$, we have that $m_s < n$. This is because if $m_s \geq n$, then because there are only $n$ strings of length $r$, there would be $i < j \leq m_s$ with $\sigma_{i,s} \upharpoonright r = \sigma_{j,s} \upharpoonright r$. As both strings agree with $G_s$ after $r$, this implies that $\sigma_{i,s} < \sigma_{j,s}$. However, it is not possible for two such strings to cause requirement $R_e$ to act (by condition (ii) for $R_e$ to need attention and the fact that strings enter $T$ in increasing order).

Fix some $i \leq m_s$. Provided that for all $t \geq s$ and all $j < i$ we have that $\sigma_{j,t+1} = \sigma_{j,t}$, our objective is to bound the size of the set $\{t : \sigma_{i,t+1} \neq \sigma_{i,t}\}$. First we will find a set of strings that includes all possible values that $G_t \upharpoonright |\sigma_{i,t}|$ could take for $t \geq s$. We define a string $\rho$ as being an $(i, s)$-candidate if:

1. $\rho \neq G_s \upharpoonright |\sigma_{i,s}|$.
2. $\rho \geq_{\text{lex}} G_s \upharpoonright |\rho|$.

If $\sigma_{i,s} \neq \sigma_{i,s+1}$ then $\sigma_{i,s+1}$ must differ from $\sigma_{i,s}$ at some point in $\text{rng}\Gamma(G_s)[s]$. We define:

$$C_s(i) = \{ x \in \omega : x \in \text{rng}\Gamma(\rho)[s] \text{ and } \rho \text{ is an } (i, s)\text{-candidate} \}.$$

We will show that the size of this set bounds the possible number of times that $\sigma_{i,s}$ changes. Thus we define:

$$c_s(i) = \begin{cases} |C_s(i)| & \text{if } i \leq m_s \\ \omega & \text{otherwise.} \end{cases}$$

This ends our definition of $c_s$.

Lemma 6.5. If $i \leq \min\{m_s, m_{s+1}\}$, and $\sigma_{i,s+1} = \sigma_{i,s}$ then $c_{s+1}(i) = c_s(i)$.

Proof. We have that $\sigma_{i,s}$ agrees with $G_s \upharpoonright |\sigma_{i,s}|$ after $r$, and $\sigma_{i,s+1}$ agrees with $G_{s+1} \upharpoonright |\sigma_{i,s+1}|$ after $r$, hence $G_s \upharpoonright |\sigma_{i,s}| = G_{s+1} \upharpoonright |\sigma_{i,s+1}|$. Thus if $\rho$ is an $(i, s)$-candidate, then it is also an $(i, s+1)$-candidate.

Now at stage $s + 1$, if a new axiom is enumerated for $\Gamma$, it must be on some extension of $G_s \upharpoonright w(e, s)$. As $w(e, s) \geq |\sigma_{i,s+1}|$, any new axiom must
be on a string which is not an \((i,s+1)\)-candidate. Hence we have that \(C_{s+1}(i) = C_s(i)\) and consequently \(c_{s+1}(i) = c_s(i)\).

\[\square\]

**Lemma 6.6.** If \(R_e\) does not act at stage \(s + 1\), then \(c_{s+1} = c_s\).

**Proof.** If \(R_e\) does not act at stage \(s + 1\), then \(T_{s+1} = T_s\) and \(G_{s + 1} \upharpoonright w(e, s + 1) = G_{s+1} \upharpoonright w(e, s)\) so \(m_{s+1} = m_s\). Thus for all \(i \leq m_s\), \(\sigma_i = \sigma_{i,s+1}\) and so be the previous lemma \(c_s(i) = c_{s+1}(i)\). For \(i\) with \(m_s < i < n\) we have that \(c_{s+1}(i) = \omega = c_s(i)\).

\[\square\]

**Lemma 6.7.** If \(R_e\) acts at stage \(s + 1\), then \(c_{s+1} < c_s\).

**Proof.** If \(R_e\) acts at stage \(s + 1\), then some new string enters \(T_{s+1}\). This means that either there is some \(i \leq \min\{m_s, m_{s+1}\}\) with \(\sigma_{i,s+1} \neq \sigma_{i,s}\), or \(m_{s+1} = m_s + 1\). For the second case we have that \(c_{s+1}(i) = c_s(i)\) for all \(i \leq m_s\) and \(c_{s+1}(m_{s+1}) < \omega = c_s(m_{s+1})\). Thus \(c_{s+1} < c_s\).

For the first case, let \(i\) be least such that \(\sigma_{i,s+1} \neq \sigma_{i,s}\). We have already established that \(c_{s+1}(j) = c_s(j)\) for all \(j < i\).

Our first claim is that \(G_{s+1} \upharpoonright [\sigma_{i,s}] >_{lex} G_s \upharpoonright [\sigma_{i,s}]\). This claim holds because first \(G_{s+1} \upharpoonright [\sigma_{i,s+1}]\) cannot be an extension of \(G_s \upharpoonright [\sigma_{i,s}]\) (if so it would not be the \(i\)th string in order of size in \(T_{s+1}\) that agrees with \(G_{s+1}\) after \(r\)). Hence \(\sigma_{i,s+1}\) does not agree with \(G_s\) at some point between \(|\sigma_{i-1,s}|\) (or \(r\) if \(i = 0\)) and \(|\sigma_{i,s}|\). The only places that \(\sigma_{i,s+1}\) can disagree with \(G_s\) are those elements of \(\text{rng}(G)[s]\). This is a subset of \(C_s(i)\). Now if \(\Gamma(G; m)[s] \downarrow = n\) then \(G_s(n) = 0\). This implies that we must form \(G_{s+1} \upharpoonright [\sigma_{s}]\) from \(G_s \upharpoonright [\sigma_s]\) by changing some elements to 1 and so \(G_{s+1} \upharpoonright [\sigma_{i,s}] >_{lex} G_s \upharpoonright [\sigma_{i,s}]\).

Consequently the \((i,s+1)\)-suitable strings are a strict subset of the \((i,s)\)-suitable strings. Further there must have been some element of \(C_s(i)\) that is now an element of \(G_{s+1}\) and hence \(C_{s+1}(i) \subset C_s(i)\) and so \(c_{s+1}(i) < c_s(i)\).

\[\square\]

Hence the stages at which \(R_e\) acts can be mapped to a decreasing sequence of a well-order and so requirement \(R_e\) needs attention finitely often. As each requirement needs attention finitely often, we have that the use of \(\Gamma(G; e)\) is bounded in the construction and so requirement \(I_e\) is met. For all \(e, \lim_s w(e, s)\) exists and witnesses the fact that requirement \(R_e\) is met.

Once we know that there exist 1-generic sets that compute their own indifferent sets, we can investigate where in the \(\Delta^0_2\) degrees such 1-generic sets exist. The verification of Theorem 6.4 showed that each requirement was met by constructing a descending sequence in \((\omega + 1)^n\) for some \(n\). In the following theorem, we will show how this can be turned into a permission argument. We can build a 1-generic set \(G\) that computes an indifferent set for itself below some c.e. set \(A\), if \(A\) will provide us with enough permissions. We will show that if \(A\) is not of totally \(\omega^\omega\)-c.a. degree then sufficient permissions must be given.

**Theorem 6.8.** If a c.e. set \(A\) is not of totally \(\omega^\omega\)-c.a. degree then there exists \(G, I\) with \(A \succ_T G \succ_T I\) such that \(G\) is 1-generic, and \(I\) is an indifferent set for \(G\).

**Proof.** Let \(A\) be such a set and let \(f = \Psi(A)\) be a function that witnesses that \(A\) is not of totally \(\omega^\omega\)-c.a. degree. Let \(f(x, s) = \Psi(A; x)[s]\) be a computable
approximation to \( f \). We will build \( G \) via a reduction \( \Phi(A) \), and set \( I = \text{rng}\Gamma(G) \) where \( \Gamma(G) \) is a strictly increasing function. To construct \( G \) we will need to obtain \( A \)-permissions. We will do this by attempting to show that \( f \) is \( \omega^\omega \)-c.a. In fact to simplify the exposition of the proof we will attempt to show that \( f \) is \( (\omega + 1)^\omega \)-c.a. The result will hold because our well-order \( (\omega + 1)^\omega \) can just be regarded as another canonical well-ordering of \( \omega^\omega \).

The failure of this approach will ensure that we get the permissions we need to build \( G \) and \( I \). We can assume that our approximation to \( A \) has the property that for all stages \( s, f(x,s) \) is defined for all \( x \leq s \). We let \( u(x,s) \) be the use of \( A_s \) in the computation of \( f(x,s) \). We can assume that \( u \) is non-decreasing in both arguments.

Our first requirement is that \( \Phi(A) \) is total. Then taking \( G = \Phi(A) \) we have for all \( e \in \omega \):

\[
I_e: \quad \Gamma(G; e) \downarrow \text{ and if } e > 0, \, \Gamma(G; e) > \Gamma(G; e - 1).
\]

\[
R_e: \quad \exists w \text{ such that } \forall X \subseteq \text{rng}\Gamma(G), \, G[X] \uparrow w \text{ meets or avoids } S_e.
\]

The approach we use to meet a requirement \( R_e \) is as follows. Each time the requirement is injured, we start afresh with a new function \( g_e \) that we use to approximate \( f \). We pick some number \( q_0 \) greater than the restraint imposed on \( R_e \), and we attempt to run the same strategy of Theorem 6.4 using \( q_0 \) instead of \( r \). We call \( q_0 \) a sub-strategy of \( R_e \). To run this sub-strategy we need to be able to change our approximation to \( G \) after \( q_0 \). However, we cannot do this unless we get permission from \( A \) to do so. To get this permission we will link the \( A \)-use of the computation of \( G \) to the \( A \)-use of the computation of \( f \). Specifically, if we act on sub-strategy \( q_0 \) at stage \( s \), then will ensure that for all \( t \geq s \), we have that \( \phi(A; q_0)[t] \geq u(s,t) \).

Suppose that at some later stage \( t \), we see some string \( r \) enter \( S_{e,t} \) and we want to change \( G \) after \( q_0 \) to agree with \( r \). We ask \( A \) for permission by defining \( g_e(x,t+1) = f(x,t+1) \) for all \( x \in [r+1, s] \). Now if \( f \) changes on any of these values, then we know that \( A \) has changed below \( u(s,t) \leq \phi(A; q_0)[t] \) and thus we can change \( G \) as desired. If on the other hand, \( f \) fails to change, then we have made some progress in building an approximation to \( f \). We will show this approximation is a \( (\omega + 1)^\omega \) computable approximation by defining a function \( o_e(x,s) \) that tracks the changes of \( g_e \). While we are waiting for permission to change \( G \) at \( q_0 \) we initiate another sub-strategy \( q_1 \). We will keep initiating new strategies at stages at which it appears that all earlier sub-strategies are successfully approximating \( f \). We will show in the verification that one sub-strategy must succeed because otherwise we will establish that \( f \) is \( (\omega + 1)^\omega \)-c.a. We will assign all active sub-strategies one of two states: **good or waiting**. The **good** state indicates that a sub-strategy currently believes it has established the existence of a \( w \) witnessing that we have met \( R_e \). The witness for a strategy \( q \) at stage \( s \) will be \( \phi(A; q)[s] \). The **waiting** state indicates that the sub-strategy is seeking permission to change the current approximation to \( G \).

**Construction.** Our requirements will be prioritised as follows: \( R_0 > I_0 > R_1 > I_1 > \ldots \). A requirement \( I_e \) needs attention at stage \( s + 1 \) if \( \Gamma(G; e)[s] \uparrow. \)
If a requirement $R_e$ is injured then all of its sub-strategies are halted. We say that a sub-strategy $q$ in state \textit{waiting} has permission to act if $\Phi(A; q)[s] \uparrow$.

A sub-strategy $q$ of requirement $R_e$ needs attention if:

1. It is in state \textit{waiting} and has permission to act; or
2. It is in state \textit{good} and there exists $\sigma \in S_{e,s}$ with $|\sigma| > \phi(A; q)[s]$, such that:
   
   a. $(\sigma \uparrow \phi(A; q)[s]) \Delta(G_s \uparrow \phi(A; q)[s]) \subseteq \text{rng} \Gamma(G)[s]$, and
   
   b. No initial segment of $\sigma \uparrow \phi(A; q)[s]$ is in $S_{e,s}$.

A requirement $R_e$ needs attention at stage $s$ if:

1. It has a sub-strategy that needs attention; or
2. No sub-strategy of $R_e$ needs attention, and all sub-strategies (if any) are in state \textit{waiting}.

At stage $0$, set $G_s = 0^\omega$ and for all $e, x$ set $g_e(x, 0) \uparrow$ and $o_e(x, 0) \uparrow$.

At stage $s + 1$, find the highest priority requirement that needs attention. If there are no requirements that need attention, set $G_{s+1} = G_s$.

If the requirement that needs attention is of the form $I_e$, take some large value $n$ such that $G_s(n) = 0$ and set $G_{s+1} = G_s$ and $\Gamma(G; e)[s + 1] = n$. Injure all lower priority requirements and restrain $G_{s+1} \uparrow n$ with priority $e$.

If the requirement is of the form $R_e$ and $R_e$ has some sub-strategy that needs attention, then let $q$ be the smallest such sub-strategy. If $q$ is in state \textit{good}, then change $q$ to state \textit{waiting} and set $G_{s+1} = G_s$. Halt any sub-strategy $q'$ of $R_e$ with $q' > q$. If $q$ is in state \textit{waiting}, then let $\sigma$ be the string in $S_{e,s}$ that caused $q$ to change its state to \textit{waiting}. We set:

$$G_{s+1}(x) = \begin{cases} 
G_s(x) & x \leq q \\
\sigma(x) & q < x < |\sigma| \\
0 & |\sigma| \leq x.
\end{cases}$$

We will commit to keeping $\phi(A; x)[t] \geq u(s + 1, t)$ for all $t \geq s + 1$ for any $x$ such that $x \geq q$. Halt any sub-strategy $q'$ of $R_e$ with $q' > q$.

If no sub-strategy of $R_e$ needs attention, then we start a new sub-strategy on $s + 1$ and we will commit to keeping $\phi(A; s + 1)[t] \geq u(s + 1, t)$ for all $t \geq s + 1$.

Now that we have updated $G_s$, we update the reduction $\Phi$. For all $x \leq s + 1$ we set $\Phi(A; x)[s + 1] = G_{s+1}(x)$. As we only change $G_{s+1}$ when we get permission from our approximation to $A$, $\Phi$ is a valid Turing functional (we will later verify that $\Phi(A)$ is total). We will set the use of $\Phi(A; x)[s + 1]$ for all $x \leq s + 1$ to be as small as possible while still honouring the commitments made.

We will now define the functions $g_e$ and $o_e$. For all $e \leq s + 1$, we do the following. Take any $x \leq s + 1$. If $u(x, s + 1) \leq \phi(A; q)[s + 1]$ for some sub-strategy $q$ of $R_e$ that is in state \textit{waiting}, we define $g_e(x, s + 1) = f(x, s + 1)$. Otherwise we set $g_e(x, s + 1)$ to be undefined. Let $r_e$ be the restraint currently imposed on requirement $R_e$. Let $n_x = \sum_{i=r_e+1}^{s+1} 2^i$. For all $x$ such that $g_e(x, s + 1)$ is defined, we will define $o_e(x, s + 1)$ to be an element of the well order $(\omega+1)^{n_x}$. First, for all $x$ we define $c_{s+1,x}$ to be an element of $(\omega+1)^{(2^x)}$. We let $c_{s+1,x} = 0^{(2^x)}$ if $x$ is not an active sub-strategy of $R_e$. Otherwise, we define $c_{s+1,x}$ to be the element of $(\omega+1)^{(2^x)}$ as in Theorem 6.4 (we take $x$
instead of \( r \), and \( T \subseteq S_{e,s} \) to be the strings that cause this sub-strategy to change from state \textit{good} to \textit{waiting}). We define \( o_e(r+1,s) = c_{s+1,x} \) and then for \( x > r + 1 \) we define \( o_e \) inductively by \( o_e(x+1,s) = o_e(x,s) c_{s+1,x+1} \).

**Verification.** Take any requirement and assume that by stage \( s_0 \) all higher priority requirements have finished acting. If the requirement is of the form \( I_e \) then by stage \( s_0 + 1 \) we have defined \( \Gamma(G;e)[s+1] \) and restrained \( G \) on the use of this computation.

Assume the requirement is of the form \( R_e \). The first observation is that any sub-strategy can only be acted on a finite number of times. This follows from the verification of Theorem 6.4. Given any \( x \), let \( s \) be the last stage at which we act on any sub-strategy (for any requirement \( R_e \)) less than or equal to \( x \). Now \( \phi(A;x) = u(s) \) and so we have that \( \Phi(A) \) is total.

**Lemma 6.9.** For all \( x,s,t \) with \( s < t \), if \( o_e(x,s) \downarrow \) and \( o_e(x,t) \downarrow \) then \( o_e(x,s) \geq o_e(x,t) \).

**Proof.** Assume that this is false and that there is some least \( x \) for which there exist \( s \) and \( t \) with \( s < t \) such that \( o_e(x,s) < o_e(x,t) \). Now because \( o_e(x,s) = c_{s+r+1}c_{s+r+2} \ldots c_{s,x} \), there must be some least \( y \) with \( r+1 \leq y \leq x \) such that \( c_{s,y} < c_{t,y} \) and \( y \) must be an active sub-strategy of \( R_e \) at stage \( t \) (because otherwise \( c_{t,y} = 0(2^y) \)). This means that \( y \) was also an active sub-strategy of \( R_e \) at stage \( s \) because \( y \leq x \leq s \) and the sub-strategy \( y \) must have been started at stage \( y \).

Now no higher priority requirements have acted since stage \( s \). Additionally no \( R_e \) sub-strategies less than \( y \) have acted because if they did, then sub-strategy \( y \) would be halted. Hence if for any \( s' \) with \( s \leq s' < t \) we have that \( G_{s'} \upharpoonright \phi(A;y)[s'] \neq G_{s'+1} \upharpoonright \phi(A;y)[s'] \) this must have been the result of sub-strategy \( y \) acting and so by the verification of Theorem 6.4 we know that \( c_{s,y} \geq c_{t,y} \). This contradicts our assumption. \( \square \)

**Lemma 6.10.** For all \( x,s,t \) with \( s < t \) if \( g_e(x,s) \downarrow \) and \( g_e(x,t) \downarrow \) with \( g_e(x,s) \neq g_e(x,t) \) then \( o_e(x,s) > o_e(x,t) \).

**Proof.** If \( g_e(x,s) \downarrow \), then \( g_e(x,s) = f(x,s) \) and there exists some \( R_e \) sub-strategy \( q \leq x \) such that \( q \) is in state \textit{waiting} at stage \( s \) and \( \phi(A;q)[s] \geq u(x,s) \). Now if \( q \) is in state \textit{waiting} at stage \( s \) then all sub-strategies \( q' < q \) of \( R_e \) are also in state \textit{waiting} at stage \( s \) (when strategy \( q \) started \( q' \) was in state \textit{waiting} and if \( q' \) ever changed to state \textit{good} then \( q \) would be halted).

If \( g_e(x,t) \downarrow \neq g_e(x,s) \), then \( f(x,s) \neq f(x,t) \) and so \( A_s \upharpoonright u(x,s) \neq A_t \upharpoonright u(x,s) \). This means that some smallest sub-strategy \( q' \leq q \) will have been given permission to act and have acted. As this is the smallest sub-strategy to act, we know that \( c_{s,q'} \geq c_{t,q'} \) by the verification of Theorem 6.4. If \( q' = r + 1 \) this gives us that \( o_e(q',s) > o_e(q',t) \). If \( q' > r + 1 \) then as \( o_e(q'-1,t) \geq o_e(q'-1,t) \) by the previous lemma we again have that \( o_e(q',s) = o_e(q'-1,s) c_{s,q'} > o_e(q'-1,t) c_{t,q'} = o_e(q',t) \). As \( x \geq q' \), we have that \( o_e(x,s) > o_e(x,t) \). \( \square \)

**Lemma 6.11.** If \( R_e \) acts infinitely often, then for infinitely many \( x \) there are infinitely many \( s \) such that \( g_e(x,s) \downarrow \).

**Proof.** As observed, any sub-strategy only needs to act a finite number of times to meet requirement \( R_e \). Hence if \( R_e \) acts infinitely often then \( R_e \)
must have an infinite number of sub-strategies and all of the sub-strategies must have some final state of waiting. Let \( r_e \) be the final restraint imposed upon \( R_e \) by higher priority requirements. Pick any \( x > r_e \) and take the least sub-strategy \( q \) such that \( q > x \) and some stage \( s \) such that \( q \) is in state waiting for all stages \( t \geq s \). In this case \( g_e(x, t) \) is defined for all \( t \geq s \).

If \( g_e(x, s) \downarrow \), then \( g_e(x, s) = f(x, s) \). Thus if \( R_e \) acts infinitely often, let \( \hat{g}_e(x, s) = g_e(x, \min\{t \geq s : g_e(x, t) \downarrow\}) \), and let \( \hat{o}_e(x, s) = o_e(x, \min\{t \geq s : o_e(x, t) \downarrow\}) \). For all \( x > r_e \), \( \lim_s \hat{g}_e(x, s) = \lim_s g_e(x, s) = f(x) \) and so \( \hat{g}_e \) and \( \hat{o}_e \) witness that \( f \) is \( (\omega + 1)^\omega \)-c.a. This contradicts our initial assumption and so \( R_e \) acts finitely often. Hence all requirements act finitely often and so our construction is successful.

**Corollary 6.12.** If \( A \) is a c.e. set and \( A'' >_T \emptyset'' \) then \( A \) bounds a 1-generic set that computes an indifferent set for itself.

**Proof.** If a c.e. set \( A \) is of totally \( \omega^\omega \)-c.a. degree then \( A'' \equiv_T \emptyset'' \) [4].

We will now prove an extension of Corollary 6.2. We will show in Theorem 6.16 that no 1-generic set bounded by a c.e. set that is of totally \( \omega^\omega \)-c.a. degree computes a set to which it is indifferent. To establish this result, we will show that any approximation to a 1-generic set that computes an indifferent set to itself changes frequently. Given \( G \) and \( I \) with \( p_I = \Gamma(G) \), we define the number of times \( G \) changes on the use of the \( x \)-th point of \( I \) as follows.

\[
\#G(I, x) = |\{s : \Gamma(G; x)[s] \downarrow \wedge G_s \uparrow \gamma(G; x)[s] \neq G_{s+1} \uparrow \gamma(G, x)[s]\}|
\]

**Proposition 6.13.** Assume that \( I \preceq_T G \preceq_T \emptyset' \) with \( p_I = \Gamma(G) \) and the reduction \( \Gamma \) has the property that \( \forall x, \Gamma(G; x + 1) > \gamma(G; x) \). If for some computable function \( q \), and some computable approximation to \( G \), \( G_s \) we have that:

\[
\forall x q(x + 1) - q(x) \geq (\#G(I, q(x + 1)) + 1)^2
\]

then \( I \) is not an indifferent set for \( G \).

**Proof.** We will construct a set \( A \) and a c.e. set of strings \( V \) such that \( A\Delta G \subseteq I \) and \( A \) does not meet or avoid \( V \). The set \( A \) will be defined as the limit of a sequence \( \alpha_0 < \alpha_1 < \ldots \). We have the following requirements:

\[
R_e: \quad \alpha_{e+1} > \alpha_e, \quad \alpha_e \text{ does not avoid } V, \quad \alpha_{e+1} \text{ does not meet } V \quad \text{and} \quad \alpha_{e+1} \Delta(G \uparrow |\alpha_{e+1}|) \subseteq \text{rng}(\Gamma(G)).
\]

We will build an approximation \( \alpha(e, s) \) such that for all \( e, \alpha_e = \lim_s \alpha(e, s) \). A requirement \( R_e \) needs attention at stage \( s \), if for all \( x \leq q(e + 1) \) we have that \( \Gamma(G; x)[s] \downarrow \); if \( x > 0 \) then \( \Gamma(G; x)[s] \geq \gamma(G; x - 1)[s] \) and either:

1. There is no extension of \( \alpha(e, s) \) in \( V_s \), or
2. \( \alpha(e + 1, s) \) is undefined, or
3. The requirement last acted at stage \( t \) and

\[
G_t \uparrow |\alpha(e + 1, t)| \neq G_s \uparrow |\alpha(e + 1, t)|.
\]

At stage \( s = 0 \), we set \( \alpha(0, 0) = \lambda \) and we declare \( \alpha(e, 0) \) to be undefined for all \( e > 0 \). At stage \( s + 1 \), we find the highest priority requirement that
needs attention \( R_e \). If no requirement needs attention then for all \( e \) such that \( \alpha(e, s) \downarrow \) we set \( \alpha(e, s + 1) = \alpha(e, s) \) and otherwise we set \( \alpha(e, s + 1) \uparrow \).

If \( R_e \) is the highest priority requirement that needs attention, then take \( n_e \) to be the largest integer such that \( n_e^2 \leq q(e + 1) - q(e) \). We partition \( \{ \Gamma(G; x) : q(e + 1) - n_e^2 < x \leq q(e + 1) \} \) into sets \( I_{s,1}, I_{s,2}, \ldots, I_{s,n_e} \), each of size \( n_e \), and such that \( \max I_{s,j} < \min I_{s,j+1} \). Define \( \rho = G_s \mid |\alpha(e, s)| \) and \( u = \gamma(G_s, q(e + 1))[s] \). Let \( m \) be the number of times that requirement \( R_e \) has acted previously in the construction. We will take as our first construction assumption that \( m < n_e - 1 \) and verify this assumption later.

If there is no extension of \( \alpha(e, s) \) in \( V_s \), then we take \( i = \min I_{s,n_e} \), and we add the string \( \alpha(e, s)\sigma \) to \( V_{s+1} \), where \( \sigma \) is chosen so that \( |\rho\sigma| = u \) and \( \rho\sigma\Delta(G_s \upharpoonright u) = \{i\} \).

We define the following set of candidates for \( \alpha(e + 1, s + 1) \):

\[ U_{s+1} = \{ \alpha(e, s)\sigma : |\alpha(e, s)\sigma| = u \wedge \rho\sigma\Delta(G_s \upharpoonright u) = \{i\} \wedge i \in I_{s,n_e-m-1} \}. \]

We will assume for now that there is some string \( \alpha(e, s)\sigma \in U_{s+1} \) that is incomparable with all \( \alpha(e + 1, t) \) for all \( t \leq s \) (where defined) and also incomparable with any extension of \( \alpha(e, s) \) added to \( V \) by this requirement. This is the second construction assumption. We take such a string \( \alpha(e, s)\sigma \) (e.g. the lexicographically least) and set \( \alpha(e + 1, s + 1) = \alpha(e, s)\sigma \). For all \( d \leq e \) we set \( \alpha(d, s + 1) = \alpha(d, s) \) and for all \( f > e + 1 \), we set \( \alpha(f, s + 1) \) to be undefined.

**Verification.** If a requirement acts at a stage \( s \), we will say that this action is successful if both of the construction assumptions made are met. We will inductively show that:

1. Whenever requirement \( R_e \) acts, it does so successfully.
2. \( R_e \) acts finitely often.

**Lemma 6.14.** If \( R_e \) acts at stage \( s + 1 \), and it has acted less than \( n_e - 1 \) times before, then \( R_e \) acts successfully.

**Proof.** As our first construction assumption is met by hypothesis we only need to check that \( U_{s+1} \) contains a suitable string. Any strings enumerated into \( V_{s+1} \) by this requirement extend \( \alpha(e, t) \), for some \( t \leq s \). By our induction hypothesis (that the second construction assumption is met for \( R_{e-1} \)) this set of strings forms an anti-chain (for the case \( c = 0 \), we have that \( a(0, t) = \lambda \) for all \( t \)). Thus there is only one string enumerated in \( V_{s+1} \) by this requirement that we need to avoid. In addition, there are at most \( n_e - 2 \) strings that are equal to \( \alpha(e + 1, t) \) for some \( t \leq s \) as this is the maximum number of times this requirement has acted before.

Let \( m \) be the number of times this requirement has acted before. Let \( T \) be the set of all strings we want to avoid. Note that \( |T| \leq n_e - 1 \). By induction on the stages at which \( R_e \) acts, we can assume that \( T \) is an anti-chain (i.e. the second construction assumption has held every previous time \( R_e \) has acted). The size of the set \( U_{s+1} \) is \( n_e \). We will show that for all \( \tau \in T \), there is at most one string \( \alpha(e, s)\sigma \in U_{s+1} \) such that \( \tau \preceq \alpha(e, s)\sigma \), or \( \alpha(e, s)\sigma \preceq \tau \). Hence as \( |U_{s+1}| > |T| \), we can choose some string \( \sigma \in U_{s+1} \) such that \( T \cup \{ \alpha(e, s)\sigma \} \) is an anti-chain and set \( \alpha(e + 1, s + 1) = \alpha(e, s)\sigma \).

Take any \( \tau \in T \). If \( \tau \) is not an extension of \( \alpha(e, s) \) then \( \tau \) must extend some \( \alpha(e, t) \) for some \( t < s \) with \( \alpha(e, t) \) defined and \( \alpha(e, t) \neq \alpha(e, s) \).
Because the second construction assumption holds for all higher priority requirements, we know that \( \alpha(e, s) \) and \( \alpha(e, t) \) are incomparable and hence \( \tau \) is incomparable with all elements of \( U_{s+1} \). So we can assume that \( \tau \geq \alpha(e, s) \) and we can write \( \tau = \alpha(e, s)\hat{\tau} \). Let \( \rho = G_s \upharpoonright |a(e, s)| \). Now because \( U_{s+1} \) is an anti-chain the only way that \( \alpha(e, s)\hat{\tau} \) can be comparable with two elements of \( U_{s+1} \) is if \( \alpha(e, s)\hat{\tau} \preceq \alpha(e, s)\sigma_j \) and \( \alpha(e, s)\hat{\tau} \preceq \alpha(e, s)\sigma_k \) for distinct \( \alpha(e, s)\sigma_j, \alpha(e, s)\sigma_k \in U_{s+1} \). In this case, it must be that \( \rho\hat{\tau} \preceq G_s \) because any common initial segment of two elements of \( U_{s+1} \) must agree with \( G_s \) after \( |a(e, s)| \).

Let \( t \) be the stage when either \( \tau \) was added to \( V \) or the least stage when \( \tau \) was equal to \( \alpha(e+1, t) \). Let \( m' \) be the number of times that the requirement has acted before stage \( t \). Hence \( m' < m \). As no higher priority requirement has acted we have that \( \rho \preceq G_t \). By construction, it must be that \( \rho\hat{\tau} \Delta(G_t \upharpoonright |\tau|) = \{a\} \) for some \( a \in I_{t, n_e} \) if \( \tau \) was added to \( V \), and \( a \in I_{t, n_e, m'-1} \) otherwise. Now any element of the range of \( \Gamma(G)[t] \) less than \( \alpha \) is also an element of the range of \( \Gamma(G)[s] \) because any element of the range of \( \Gamma(G)[t] \) less than \( \alpha \) has use less than \( a \) and \( \rho\hat{\tau} \) agrees with both \( G_s \) and \( G_t \) before \( a \).

We can draw a contradiction from this fact because \( \alpha(e, s)\hat{\tau} \preceq \alpha(e, s)\sigma_j \) implies that the first element of the range of \( \Gamma(G)[s] \) greater than or equal to \( a \) cannot be in \( I_{t, n_e, m'-k} \) if \( k \leq m \). This in turn implies that \( a \) cannot be in \( I_{t, n_e, m'-k} \) if \( k \leq m \) which contradicts our assumption that \( a \in I_{t, n_e, m'-1} \) or \( a \in I_{t, n_e} \). Hence there is some suitable string in \( U_{s+1} \) and requirement \( R_e \) acts successfully.

\begin{lemma}
If requirement \( R_e \) acts \( n_e - 1 \) times then it never needs attention again.
\end{lemma}

\begin{proof}
If \( R_e \) acts at stage \( t \), and then again at some \( s > t \), it must be that for some \( d \leq e \), we have that for some stage \( s' \) with \( t < s' \leq s \):

\[ G_{s'} \upharpoonright |\alpha(d + 1, t)| \neq G_{s'+1} \upharpoonright |\alpha(d + 1, t)|. \]

Now \( |\alpha(d + 1, t)| = \gamma(G; q(d + 1)) |t| \leq \gamma(G; q(e + 1)) |t| \). Thus \( G_t \upharpoonright \gamma(G; q(e + 1)) |t| \neq G_{s'} \upharpoonright \gamma(G; q(e + 1)) |t| \).

This means that the number of times that \( R_e \) acts is at most \( \#G(I, q(e + 1)) \). But as \( \#G(I, q(e + 1)) + 1 \) \( \leq q(e + 1) - q(e) \), by our choice of \( n_e \) we have that \( \#G(I, q(e + 1)) \leq n_e - 1 \).
\end{proof}

We have that each requirement acts finitely often. This means that for all \( e, \alpha_e = \lim_s \alpha(e, s) \) is defined. Further \( \alpha_{e+1} \succeq \alpha_e \) because if \( \alpha(e+1, s) \downarrow \) then \( \alpha(e+1, s) \succeq \alpha(e, s) \). Let \( A = \bigcup_e \alpha_e \). Now \( \alpha_{e+1} \Delta(G \upharpoonright |\alpha_{e+1}|) \subseteq \text{rng} \Gamma(G) \) because if \( s \) is a stage when requirement \( R_e \) acts then \( \alpha(e+1, s)\Delta(G_s \upharpoonright |\alpha(e+1, s)|) \subseteq \text{rng} \Gamma(G)[s] \) and the use of any indifferent point in \( \text{rng} \Gamma(G)[s] \cap |\alpha(e+1, s)| \) is at most \( |\alpha(e+1, s)| \). Now if the approximation to \( G \) changes on the first \( |\alpha(e+1, s)| \) bits, then \( \alpha(e+1, s) \neq \alpha_{e+1} \) because the requirement will need attention again.

As each requirement is met we have that \( A \) does not avoid \( V \) (for all \( e, \alpha_e \) has some extension in \( V \)). Let \( \sigma \) be an element of \( V \). There is some requirement \( R_e \) that enumerated \( \sigma \) into \( V \) and some stage \( s \) such that \( \sigma \succeq \alpha(e, s) \). If \( \alpha_e \neq \alpha(e, s) \) then these two strings are incomparable and so \( \alpha \) does not meet \( \sigma \). If \( \alpha_e = \alpha(e, s) \), then by construction \( \alpha_{e+1} \) is incomparable with \( \sigma \). Thus \( A \) is not 1-generic and so \( I \) is not an indifferent set for \( G \).
\end{proof}
Theorem 6.16. If a c.e. set $A$ is of totally $\omega$-c.a. degree then $A$ does not bound a 1-generic set $G$ that computes an indifferent set for itself.

Proof. Assume that $G = \Phi(A)$ and that $\Gamma(G)$ is equal to the principal function of some set $I$. We can assume that for all $x > 0$, $\Gamma(G; x + 1) > \gamma(G; x)$ because if $G$ can compute an indifferent set for itself then it can compute one with this property.

We will show that $I$ is not an indifferent set for $G$. Our approach is to build a function $f \leq_T A$, where $f(x) = \Psi(A; x)$. Because $A$ is of totally $\omega$-c.a. degree we know that there is a pair of computable functions $(g, h)$ such that $f(x) = \lim_s g(x, s)$ and if we define $\#g(x) = |\{s + 1 : g(x, s) \neq g(x, s + 1)\}|$, then $\#g(x) \leq h(x)$. However, we do not know which computable functions these are. We will take an enumeration of all pairs of partial computable functions: $(g_e, h_e)_{e < \omega}$. Our objective in the construction of $f$ is to find the pair of functions $g$ and $h$ that witness that $f$ is $\omega$-c.a., and then use this pair to slow the approximation of $A$ sufficiently so that $I$ is not an indifferent set for $G$.

We call $(g_e, h_e)$ a valid $\omega$-c.a. approximation for $f$ if:

1. $g_e$ and $h_e$ are total.
2. $\forall x, \#g_e(x) \leq h_e(x)$.
3. $\forall x, \lim_s g_e(x, s) = f(x)$.

We cannot determine whether $(g_e, h_e)$ is a valid $\omega$-c.a. approximation for $f$. But given $x$ and $s$, we can determine the following. We call $(g_e, h_e)$ a valid $\omega$-c.a. approximation for $f$ until $x$ at stage $s$ if for all $y \leq x$:

1. $h_e(y)[s] \downarrow$.
2. $g_e(y, \max\{j : g_e(y, j)[s] \downarrow\}) = \Psi(A; x)[s]$.
3. $|\{j \in \omega : g_e(y, j)[s] \downarrow \neq g_e(y, j + 1)[s] \downarrow\}| \leq h_e(y)[s]$.

A pair $(g_e, h_e)$ is a valid $\omega$-c.a. approximation for $f$ if and only if for all $x$ there is an $s$ such that $(g_e, h_e)$ is a valid $\omega$-c.a. approximation for $f$ until $x$ at stage $s$.

Our requirements for the construction are:

1. For all $x$, $\Psi(A; x) \downarrow$.
2. Either $A$ is computable, or there some approximation to $G$, and some strictly increasing computable function $q$, such that:

$$\forall x (q(x + 1) - q(x) > (\#G(I, q(x)) + 1)^2).$$

We will assume that our enumeration of $A$ is sufficiently fast so that for all $x \leq s$, $\Gamma(\Phi(A); x)[s] \downarrow$. During the construction we will build a computable function $r(x, s)$. The function $r(x, s)$ determines a point in the set $I$. This point in $I$ is computable from $A$ by composing $\Gamma$ and $\Phi$. Hence it has an $A$-use. We will denote this use as $\gamma \circ \Phi(A; r(x, s))$. When possible, we will set the use of $\Psi(A; x)[s]$ to be this value.

Let $e, s, x \in \omega$ with $e, x \leq s$. At any stage $s$, we say that $e$ can take control of $x$ if:

1. $x$ is not controlled by any $d < e$.
2. $(g_e, h_e)$ is a valid $\omega$-c.a. approximation for $f$ until $x$ at stage $s$.
Construction. At each stage in the construction we will define \( r(x, s) \) for all \( x \leq s \). We will also ensure that \( \Psi(A; x)[s] \) is defined for \( x \leq s \).

At stage 0 we define \( \Psi(A; 0)[0] = 0 \), \( r(0, 0) = 0 \) and we define the use of \( \Psi(A; 0)[0] \) to be \( \gamma \circ \Phi(A; r(0, 0))[r(0, 0)] \).

At stage \( s + 1 \), we ask whether there is any \( e, x \leq s \) such that \( e \) can take control of \( x \). If so we take the least such \( e \), and give it control of the least \( x \) it can take control of. Define \( r(0, s + 1) = 0 \). Now define inductively

\[
r(x + 1, s + 1) = \begin{cases} 
  r(x, s + 1) + (h_e(x + 1) + 1)^2 & \text{if } e \text{ controls } x + 1, \\
  r(x, s + 1) + 1 & \text{if no } e \text{ controls } x + 1.
\end{cases}
\]

Now for all \( x \leq s + 1 \), such that \( \Psi(A_{s+1}; x)[s] \uparrow \) we define

\[
\Psi(A_{s+1}; x)[s + 1] = \begin{cases} 
  \max\{g_e(x, j)[s] : j \leq s\} + 1 & \text{if } e \text{ controls } x, \\
  0 & \text{if no } e \text{ controls } x.
\end{cases}
\]

For these \( x \), we set the \( A \) use of the computation of \( \Psi(A_{s+1}; x)[s + 1] \) to be \( \gamma \circ \Phi(A; r(x, s + 1))[r(x, s + 1)] \).

Verification. First we show that \( f = \Psi(A) \) is total. This can be done by showing that the use of any computation does not tend to infinity. This is true if for all \( x \), \( \lim_n r(x, s) \) exists. Now \( \lim_n r(0, s) = 0 \) and if the \( \lim_n r(x, s) \) exists, then as \( x + 1 \) has finitely many owners so \( \lim_n r(x + 1, s) \) exists.

By assumption \( A \) is \( \omega \text{-c.a.} \), so it follows that there is some least \( e \) such that \( (g_e, h_e) \) is a valid \( \omega \text{-c.a.} \) approximation to \( f \). We claim that \( e \) takes control of almost all \( x \). If \( d < e \), then there is some \( x \) such that for all \( s \), \( (g_d, h_d) \) is never a valid approximation for \( f \) until \( x \) at stage \( s \). Then \( d \) never takes control of any element greater than or equal to \( x \). As only a finite subset of \( \omega \) is controlled any \( d < e \), \( e \) will take control of the remaining elements of \( \omega \). This makes \( r : \omega \to \omega \), defined by \( r(x) = \lim_n r(x, s) \), computable, because for almost all \( x \) the limit occurs at the stage \( e \) takes ownership of \( x \).

If for almost all \( x \), \( \Psi(A; x) = 0 \), then \( A \) is computable. We just wait until a stage \( s \) at which \( e \) takes ownership of \( x \). At this point, \( A \) cannot change on the use of \( \Psi(A; x)[s] \); if it did, then \( \Psi(A; x) \) would be set to some non-zero value. As the use of \( \Psi(A) \) is unbounded we have that \( A \) is computable.

Otherwise, there exists infinitely many \( x \) where \( \Psi(A; x) \) is defined after \( e \) takes ownership of \( x \). The set of these \( x \) is c.e. and so has some infinite computable subset. We can take this subset to be the range of a strictly increasing computable function \( p \).

Now we can speed-up the enumeration of \( G \) so that stage \( t \) in the new enumeration corresponds to the first stage \( s \) when \( e \) has control of all \( x \leq t \) (that it will take control of) and \( (g_e, h_e) \) is a valid approximation for \( f \) until \( t \). Now the computable function \( r(p(x)) \) has the property that for all \( x \),

\[
 r(p(x + 1)) - r(p(x)) \geq r(p(x + 1)) - r(p(x + 1) - 1) \geq (h_e(p(x + 1)) + 1)^2.
\]

We claim that for all \( x \), \#\(G(I; r(x)) \leq h_e(x) \). This is because if \( G_{t+1} | \gamma(G; r(x))[t] \neq G_t \ | \gamma(G; r(x))[t] \), then \( A_{t+1} \neq A_t \ | \gamma \circ \Phi(A; (r(x))) \). This means that \( \Psi(A; x)[t + 1] > \Psi(A; x)[t] \) which in turn implies that \( g_e(x; t + 1) > g_e(x; t) \). We know these changes are bounded by \( h_e(x) \). Thus for all \( x \),

\[
r(p(x + 1)) - r(p(x)) \geq (\#G(I, r(p(x + 1))) + 1)^2.
\]
We now apply Proposition 6.13 to establish that $I$ is not an indifferent set for $G$ with $q = r \circ p$. \hfill \Box

### 7. Indifferent Sets for Weakly 1-generic Sets

In this section we will further examine indifferent sets for weakly 1-generic sets. We start with an observation about the function $g(n, e)$ defined in Lemma 5.2. Fix $e \in \omega$. If $S_e$ is dense, then $g(n, e)$, as a function of the first variable, is computable, because we do not need to ask $\emptyset'$ whether some string $\sigma$ has an extension in $S_e$. The answer is always “yes.” We can use this observation to prove that any hyperimmune set computes a weakly 1-generic set that is the indifferent set for. For this section we will take $g(n, e)$ to be a partial computable function such that if $S_e$ is dense, then $g(n, e)$ agrees with the function of Lemma 5.2 and if $S_e$ is not dense then for some $n$, for all $m \geq n$, $g(m, e) \uparrow$. This is the result of assuming that the answer to each $\emptyset'$ query is “yes”; at some point $g(m, e)$ will fail to halt if $S_e$ is not dense.

**Theorem 7.1.** If $I$ is hyperimmune, then $I$ is the indifferent set for some weakly 1-generic set $G$ with $G \leq_T I$.

**Proof.** We will construct $G \leq_T I$ and meet the following requirements to ensure that $G$ is weakly 1-generic.

$$R_e: \begin{cases} 
\text{If } S_e \text{ is dense, then } \exists n \ ((G \upharpoonright n)g(n, e) \prec G \text{ and} \newline
[n, n + g(n, e)] \cap I = \emptyset). 
\end{cases}$$

We order our requirements by priority as follows: $R_0 > R_1 > \ldots$. We say that requirement $R_e$ needs attention at stage $s + 1$ if:

1. $e \leq s$.
2. $R_e$ is not currently satisfied.
3. $g(s, e)[pf(s)] \downarrow$.
4. $I \cap [s, s + |g(s, e)|] = \emptyset$.
5. $s + 1$ is not restrained by any $R_d$ with $d < e$.

At stage 0, we set $G_0 = 0^\omega$. At stage $s + 1$, if no requirement needs attention then we set $G_{s+1} = G_s$. Otherwise let $R_e$ be the highest priority requirement that needs attention. We define $G_{s+1} = (G_s \upharpoonright s)(g(s, e))0^\omega$ and if $x \leq s + |g(s, e)|$, we restrain $x$ with priority $e$. We declare $R_e$ satisfied and $R_f$ unsatisfied for all $f > e$.

**Verification.** Because for all $x$, $G_x \upharpoonright x = G \upharpoonright x$, we have that $G \leq_T I$. To show that all requirements are met, consider any requirement $R_e$ and let $s_0 \geq e$ be a stage such that for all $d \leq e$, $R_d$ no longer requires attention and further that no $x \geq s_0$ is restrained by a higher priority requirement. Now if $S_e$ is not dense then $R_e$ is met trivially but further for some $x$, for all $y \geq x$, $g(y, e) \uparrow$ and so $R_e$ only requires attention finitely often.

If $S_e$ is dense then we claim that there exists some $t \geq s_0$ such that both $g(t, e)[pf(t)] \downarrow$, and $I \cap [t, t + |g(t, e)|] = \emptyset$. If this claim holds then we have that $(G \upharpoonright t)g(t, e) \prec G$ and so requirement $R_e$ is met. It needs attention finitely often because once it is acted on it will be declared satisfied and never again require attention.

To establish the claim, define the following computable function $h : \omega \rightarrow \omega$. To determine $h(x)$, let $n_{x,0} = x$ and $n_{x,i+1} = n_{x,i} + |g(n_{x,i}, e)| + 1$. Now
define \( h(x) \) to be the least stage such that for all \( i \) with \( 0 \leq i \leq x + 1 \), \( g(n_{x,i}, e) \) has halted. We also require \( h(x) \) to be greater than \( n_{x,x+1} \). If we take some \( x \geq s_0 \) such \( p_I(x) > h(x) \), then there must be some \( i \leq x \) such that \([n_{x,i}, n_{x,i+1}] \cap I = \emptyset \). Further we have that \( g(n_{x,i}, e)[p_I(x)] \downarrow \) so \( g(n_{x,i}, e)[p_I(x)] \downarrow \). Now take \( t = n_{x,i} \).

\[ \square \]

**Corollary 7.2.** A set \( I \subseteq \omega \) is the indifferent set for some weakly 1-generic set if and only if \( I \) is hyperimmune.

**Proof.** The other direction, the fact that if \( I \) is an indifferent set for some weakly 1-generic set \( G \) then \( I \) is hyperimmune, follows from the proof of Theorem 5.1. The proof of Theorem 5.1 builds a dense c.e. set of strings so this proof applies to the case of weakly 1-generic sets as well.

We will now consider which degrees can compute an indifferent set for a given weakly 1-generic set. In Lemma 5.2, we showed that we could make a set \( G \) meet or avoid \( S_e \) by ensuring that \((G \upharpoonright n)g(n,e) \prec G \). Now if \( S_e \) is dense, then as the following lemma indicates, this is also a necessary condition for \( G \) to meet \( S_e \).

**Lemma 7.3.** If \( G \) is a weakly 1-generic set and \( S_e \) is dense, then there exist infinitely many \( n \) such that \((G \upharpoonright n)g(n,e) \prec G \).

**Proof.** Given a dense set of strings \( S_e \) we will define another dense set of strings \( V_e \) as follows. At stage 0 we add \( g(0, e) \) to \( V_e \) and set \( l_0 = |g(0, e)| \).

At stage \( s + 1 \), for all \( \sigma \) of length \( l_s \), we add \( \sigma g(l_s, e) \) to \( V_e \) and set \( l_{s+1} = l_s + |g(l_s, e)| \). As \( V_e \) is dense, it follows that \( G \) meets \( V_e \) infinitely often.

Hence for infinitely many \( i \), \((G \upharpoonright l_i)g(l_i, e) \prec G \).

\[ \square \]

**Theorem 7.4.** If \( G \) is a weakly 1-generic set, and \( I \) is a set such that \( p_I \) escapes domination by all \( f \leq_T G \), then \( I \) is an indifferent set for \( G \).

**Proof.** If \( G \) is a weakly 1-generic set, and \( S_e \) is dense, there is a sequence \( n_0, n_1, \ldots \) computable in \( G \) such that \((G \upharpoonright n_i)g(n_i,e) \prec G \). Further we can require that \( n_{i+1} > n_i + |g(n_{i+1}, e)| \). Now as \( p_I \) escapes domination by any \( G \)-computable function, for some \( x \) we have that \( p_I(x) > n_x \). Thus there is some \( i \leq x \) such that such that \( I \cap [n_i, n_{i+1}] = \emptyset \). Thus we have that for all \( X \subseteq I \), \( G_{[X]} \upharpoonright n_x g(n_x, e) \prec G_{[X]} \) and so \( G_{[X]} \) meets \( S_e \). This is true for any \( e \) so \( I \) is an indifferent set for \( G \) with respect to weak 1-genericity.

\[ \square \]

**Theorem 7.5.** If \( G \) is a weakly 1-generic set, \( A \geq_T G \) and \( A \in \mathcal{GL}_2 \), then \( A \) computes an indifferent set for \( G \).

**Proof.** For the previous proof to hold, we needed \( p_I \) to escape domination by a set of functions \( f_e \) for each \( e \) such that \( S_e \) is dense. From \( G \oplus \emptyset' \) we can define a function that majorizes \( f(x) = \max\{f_e(x) : e \leq x \wedge S_e \text{ is dense}\} \). If a set \( S_e \) is not dense, then the corresponding \( f_e \) may fail to be total. However, we can concurrently attempt to compute \( f_e(x) \), for any \( x \), using \( G \) and search for a witness that shows \( S_e \) is not dense using \( \emptyset' \). Either \( f_e(x) \downarrow \) or we establish that \( S_e \) is not dense and we can remove \( e \) from our list of functions.

As \( A \in \mathcal{GL}_2 \) and \( A \geq_T G \), we have that \( A \) computes a set whose principal function escapes domination by \( f \) and hence \( A \) computes an indifferent set for \( G \).

\[ \square \]
We will now look at another significant difference between indifferent sets for 1-generic sets and weakly 1-generic sets.

**Theorem 7.6.** Any $\Delta^0_2$ weakly 1-generic set computes a set it is indifferent to.

**Proof.** Let $G \leq_T \emptyset'$ be weakly 1-generic set with approximation $G_s$. To prove this theorem we will construct a Turing functional $\Gamma$ such that $\Gamma(G)$ is total and strictly increasing and that $\text{rng} \Gamma(G)$ is an indifferent set for $G$.

Take any $X \subseteq \text{rng} \Gamma(G)$. We will use the fact that $G$ is weakly 1-generic to ensure that $G[X]$ is also weakly 1-generic. For each c.e. set of strings $S$, we will enumerate our own c.e. set of strings $V$. If $S$ is dense, then we will threaten to make $V$ dense as well. Further we will construct $V$ in such a way that if $G$ meets it, then for all $X \subseteq \text{rng} \Gamma(G)$, $G[X]$ meets $S$. Our requirements are as follows:

- $I_e$: $\Gamma(G; e + 1) ≻ \Gamma(G; e)$,
- $S_e$ dense $\rightarrow \exists n([n, n + |g(n, e)|] \cap \text{rng} \Gamma(G) = \emptyset$, and $G \uparrow \downarrow n)g(n, e) - G]$. $\Gamma(G; e + 1)$ for all $e$.$S_e$

A requirement $I_e$ needs attention at stage $s$, if $\Gamma(G; e + 1)[s] \uparrow$. We act on such a requirement by finding the least $x > \Gamma(G; e)[s]$ that is not restrained by a higher priority requirement. We set $\Gamma(G_s; e + 1)[s + 1] = x$ with use $x$. We injure all lower priority requirements of the form $R_e$.

We will assign each requirement $R_e$, a c.e. set of strings $V_e$. Each time $R_e$ is injured we assign $R_e$ a new set $V_e$. We will define the following computable function $m : \omega^3 \rightarrow \omega$ by

$$m(e, i, s) = \max\{x + 1 : (\exists \rho \in 2^{<\omega})(x \in \text{rng} \Gamma(\rho)[s] \land \rho \not\geq G_s \uparrow i)\}.$$  

A requirement $R_e$ needs attention if at stage $s + 1$ if

1. $G_s$ does not meet $V_e$.
2. For some unused pair $(e, i)$, there exists an $s$ such that $g(m(e, i, s), e)[s] \downarrow$.

We act on a requirement $R_e$ by adding $\sigma g(m(e, i, s), e)$ to $V_e$ for all strings $\sigma$ such that:

1. $|\sigma| = m(e, i, s)$.
2. $\sigma \not\geq G_s \uparrow i$.

Following this, we injure all lower priority requirements and we restrain any $x \leq m(e, i, s) + g(m(e, i, s), e)$ with priority $e$. We declare the pair $(e, i)$ used.

The construction is simply this, at stage $s$ we find the highest priority requirement that needs attention and we act on that requirement.

**Verification.** Take any requirement, and assume that all higher priority requirements require attention finitely often. If the requirement is of the form $I_e$ then there is some maximum restraint $r$ placed on $I_e$ by higher priority requirements. Let $x$ be the least value that exceeds $r$ and, if $e \neq 0$, also exceeds $\Gamma(G; e - 1)$. The construction ensures that $\Gamma(G; e) = x$ with use $x$ and hence $I_e$ is met and further, $I_e$ requires attention finitely often.

If the requirement is of the form $R_e$, then first assume that $S_e$ is not a dense c.e. set of strings. In this case $R_e$ is met trivially. Further, $R_e$ cannot
need attention infinitely often because for some $x$ for all $y \geq x$, $g(y,e) \uparrow$. Now assume that $S_e$ is a dense c.e. set of strings. First we claim that for all $i$, there exists some $s$ such that $g(m(e,i,s),e)[s] \downarrow$. This claim holds because during the construction we only define $\Gamma$ on initial segments of $G_s$. Once our approximation settles on the first $i$ bits, then we will no longer define $\Gamma$ on strings that do not extend $G|_i$. This means that $\lim_s m(e,i,s)$ exists and so for some $s$, $g(m(e,i,s),e)[s] \downarrow$.

If $G$ did not meet $V_e$ then $R_e$ would act infinitely often and so $V_e$ would be dense. However as $G$ is weakly 1-generic then this would imply that $G$ does meet $V_e$. Hence we must conclude that at some point $G$ meets $V_e$ and so $R_e$ requires attention finitely often.

It remains to show that requirement $R_e$ is satisfied. Let $\sigma g(m(e,i,s),e)$ be the string in $V_e$ that $G$ meets, where $|\sigma| = m(e,i,s)$. At stage $s$, we had that $\sigma \neq G_s \upharpoonright i$ so $G \neq G_s \upharpoonright i$. Further for all $x$ such that

$$m(e,i,s) < x \leq m(e,i,s) + g(m(e,i,s),e)$$

we have that $x \notin \text{rng}(\rho)[s]$ if $\rho \not\geq G_s \upharpoonright i$. Thus for all

$$x \in [m(e,i,s), m(e,i,s) + g(m(e,i,s),e)],$$

$x \notin \text{rng}(G)$ because after stage $s$ all $x \leq g(m(e,i,s),e)$ are restrained with priority $e$. 

\[\square\]

8. Open questions

We conclude with some remaining questions.

**Question 8.1.** Does there exists an array computable set $A$, such that $A$ computes a 1-generic set $G$ and an indifferent set for $G$? If so, can $A$ be 1-generic? If this question has a positive answer it may well provide a new example of a degree with no strong minimal cover.

**Question 8.2.** In Corollary 7.2 we characterised indifferent sets for weakly 1-generic sets in terms of sparseness. Can indifferent sets for 1-generic sets be characterised by a sparseness requirement?

**Question 8.3.** Can the c.e. degrees that bound 1-generic sets that compute an indifferent set for themselves be characterised using the totally $\alpha$-c.e. degree hierarchy?

**Question 8.4.** In Section 7 we showed that any weakly 1-generic set that is either in $\mathbb{GL}_2$ or is $\Delta^0_2$ can compute an indifferent set for itself. However, we have not be able to answer the following question. Does there exists a weakly 1-generic set that cannot compute a set to which it is indifferent?

**Question 8.5.** What can be said about indifferent sets for (weakly) $n$-generic sets for $n > 1$?

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References


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