Instantaneous Singular Sets Associated to Spatial Motions

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1 Introduction

One of the oldest intuitions of kinematics is that when a plane lamina moves with *one* degree of freedom (dof) there is at any instant a point—the instantaneous centre of rotation—whose trajectory is singular at that instant: indeed the instantaneous centre is unique, provided the motion has a rotational component. Likewise, when a spatial body moves with one dof there is at any instant a line (the instantaneous axis of rotation) with the property that the trajectory of any point on that line is singular at that instant: again, the line is unique, provided the motion has a rotational component. In each case this key intuition gives rise to a fundamental connexion between two quite distinct geometric objects, namely the motion itself (a curve in the Euclidean group) and the associated family of trajectory curves. Despite the long history attached to this extremely fruitful connexion, little is known about its natural generalisation to Euclidean motions with *several* dof, which are fundamental in modern engineering robotics. The instantaneous centres and axes associated to motions with one dof are examples of the "instantaneous singular sets" of the title, which we associate to general Euclidean motions, and which provide a fundamental connexion (the envelope result of Section 4) between the geometry of the motion and the singularities of the family of trajectories. We feel it is important to make the point that such a connexion can *only* be made once a substantial body of singularity theory has been developed. Thus singularity theory does not only give rise to a possible viewpoint of kinematics, it represents an essential part of its development.

The generalizations of these ideas to motions with several dof is hardly touched upon in the classical literature on kinematics, though interestingly it is implicit in Hunt's book [16, Section 12.6]. We will set up a formal framework in which these intuitions can be made precise. Let $SE(p)$ denote the special Euclidean group, i.e.

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the Lie group of proper rigid motions of $\mathbb{R}^p$ with its standard Euclidean structure. By a Euclidean *motion* of $\mathbb{R}^p$ we mean a smooth mapping $\lambda : N \to SE(p)$, where N is a smooth manifold: we say that the motion has r–degrees of freedom (dof) when the maximal rank is r. Motions (or their germs) can be classified under $\mathcal{A}$–equivalence. For any choice of *tracing point* $w \in \mathbb{R}^p$, there is an *evaluation map* $ev_w : SE(p) \to \mathbb{R}^p$ given by $ev_w(g) = g.w$. Composition with a Euclidean motion λ yields a smooth mapping $M_{\lambda,w} : N \to \mathbb{R}^p$ defined by $t \mapsto \lambda(t)(w)$, the *trajectory* of w under λ. The formula $(t, w) \mapsto \lambda(t)(w)$ defines another smooth mapping $M_\lambda : N \times \mathbb{R}^p \to \mathbb{R}^p$, which we think of as a p–parameter family of trajectories. There are two quite distinct strands of thought here which, to our knowledge, have never been married.

1.1 Instantaneous Behaviour

¿From a historical perspective, the first strand in the development is the instantaneous behaviour of a spatial motion $\lambda : N \to SE(3)$ itself, at a point $t \in N$. Here it is natural to allow smooth changes of coordinates at the source t, and conjugation in the group at the target. It is no restriction to assume that the target is the identity 1 in the group. At the 1–jet level we have the differential $T_t\lambda : T_tN \to se(3)$, where $se(3)$ is the Lie algebra of $SE(3)$. Changes of coordinates at the source leave the image of the differential unchanged, whilst the derivative at the target of conjugation gives rise to the adjoint action of the group on its Lie algebra. The action is linear, so induces an action on subspaces of $se(3)$ of given dimension, and the problem is to list orbits under this action. Thinking projectively, these are the 'screw systems' of classical kinematics. The most significant early reference by far is [17]. This important work, virtually never quoted in the kinematics literature, contains the germ of an idea central to understanding screw systems, namely the way in which they sit relative to the 'Klein complex'. Despite the publication of Ball's treatise [1] in 1900, over a century was to elapse before the implications were understood. Screw systems were revived and explained to the engineering community in Hunt's seminal book [16]. Hunt makes a telling point, namely that the screw systems which arise in robotics are necessarily degenerate in some sense, thus raising the question of classification: further, he presents a list of 'special' screw systems, largely on the basis of physical intuition. A formal listing was obtained in [12], and described intrinsically in [5]. In [6] a complete description of the specialisations was given, and it was shown that the listing gave rise to natural Whitney stratifications. Rather little appears to be known about higher–order invariants of Euclidean motions: for instance, so far as we aware second–order invariants have yet to be described.

1.2 The Singular Viewpoint

For over a decade now the authors have applied the philosophy of René Thom to the study of Euclidean motions, namely that it is crucial to study singular behaviour in order to gain a true understanding of the general situation. That represents the second strand of thought in the development. The idea is to consider only "generic" motions, in the hope that the methods of singularity theory will produce exhaustive finite lists of local models for the singular behaviour of trajectories. This approach is based on a transversality lemma for Euclidean motions, established in [9], roughly speaking that a generic n–dof motion λ of p–space gives rise to general singularities on the resulting family M_λ of trajectories. More precisely, given an n–dof motion $\lambda : N \to SE(p)$, and positive integers r and k, the trajectory mapping $M_{\lambda,w} : N \to \mathbb{R}^p$ induces a multijet extension

$$ {}_rj^k M_{\lambda,w} : N^{(r)} \to {}_rJ^k(N, \mathbb{R}^p). $$

Since $M_{\lambda,w}$ depends smoothly on w, that yields the following smooth mapping, where the subscript '1' in ${}_rj_1^k$ indicates that we are taking jets with respect to the *first* term in the product

$$ {}_rj_1^k M_\lambda : N^{(r)} \times \mathbb{R}^p \to {}_rJ^k(N, \mathbb{R}^p). $$

Lemma 1 *Let $\mathcal{S}$ be a finite stratification of ${}_rJ^k(N, \mathbb{R}^p)$. The set of n–dof motions $\lambda : N \to SE(p)$ with ${}_rj^k M_\lambda$ transverse to $\mathcal{S}$ is residual in $C^\infty(N, SE(p))$, endowed with the Whitney C^∞ topology.* (Basic Transversality Lemma.)

One reason for the historical emphasis on motions with 1–dof is the fact that singularities of curves were well understood. However, real progress on motions with many dof could not take place until the systematic methods of listing singularities of smooth mappings (under natural equivalence relations) were available. For this reason the listing of local models for general motions is a relatively recent development. The programme of listing local models for general planar ($p = 2$) motions was initiated (the case $n = 1$) in [9], continued (the case $n = 2$) in [10], and completed (the case $n \geq 3$) in [7]. For general spatial ($p = 3$) motions the programme is not complete, and likely to remain so for some considerable time. The case $n = 1$ was covered in [9], whilst the next case $n = 2$ is covered by [11], modulo some of the multilocal models. Beyond this only the case $n = 3$ has received attention. There is a complete list of purely local models in [15]: however the list itself is little understood, and we are not aware of any study of naturally occuring examples of spatial motions with 3–dof based on this list. Nor is it clear that it is profitable to continue this listing process further. A natural consequence of adopting the singular viewpoint is that one associates to a motion $\lambda : N \to SE(p)$ its *bifurcation set* i.e. the set

$$ B(\lambda) = \{ w \in \mathbb{R}^p \,|\, \text{the trajectory } M_{\lambda,w} \text{ exhibits a non–stable multigerm} \}. $$

For instance, for general planar motions with 1–dof only three non–stable multigerms of codimension 1 can appear on the trajectories. These are the ordinary cusp, tacnode and ordinary triple point, giving rise to the moving centrode, the transition curve and the triple point curve of classical kinematics. A key problem is that of constructing robust algorithms rendering bifurcation curves and surfaces on a computer screen. For the simplest examples of engineering robotics this appears to be difficult [13, 20]. Beyond this, virtually nothing is known.

1.3 Instantaneous Singular Sets

The genesis of the material in this paper lies in a result proved in [14], roughly that two of the bifurcation curves associated to planar motions with 2–dof can be described as the envelope of a naturally appearing 2–parameter family of lines. These lines provide a special case of a key concept, relating the two strands of thought described above. Given a motion $\lambda : N \to SE(p)$ the *instantaneous singular set* (ISS) of λ at t is defined to be

$$I(\lambda, t) = \{w \in \mathbb{R}^p \,|\, \text{the trajectory } M_{\lambda,w} \text{ is singular at } t\}.$$

ISSes associated to planar motions are extremely simple objects. In the case of planar motions with 1–dof (the core concern of classical kinematics) the ISS at an instant t is just a point, the classical instantaneous centre of rotation. This appears to be the only case well known to the kinematics community. For an *immersive* planar motion germ with $\geq$ 2–dof, the ISS turns out to be a line [14] (the relatively little known “Polachsen” of [2, page 148]) whilst for a singular germ it is again a point. The result established in [14] is that the envelope of this 2–parameter family of lines is the union of the closures of the bifurcation curves associated to the lips and beaks singularities. The importance of such a result is that it opens up the possibility of rendering those curves on a computer screen, using computer algebra programs to determine an equation for the envelope. Some examples are presented in [14]. The picture changes somewhat in the spatial case. For spatial motions with 1–dof the ISS is generally the empty set, but may be a line, in which case it coincides with the classical instantaneous axis of rotation. However, for spatial motions with $\geq$ 2–dof the literature has little to say, despite the fact that these are the cases of real interest in applications to engineering robotics. That is hardly surprising: the image of the trajectory is no longer a curve, and the necessary singularity theory is relatively recent. Determining ISSes depends crucially on the classification of screw systems, a relatively recent formal result [5]: for that reason we use Section 2 to review this little–known area. In Section 3 we present a complete list of ISSes for spatial motions, up to proper rigid motions of 3–space: the most interesting case is represented by spatial motions with 3–dof where the general ISS is a 1–sheeted hyperboloid. Finally, in Section 4 we extend the envelope result of [14] to spatial motions with 3–dof. In view of this result it would undoubtedly be of interest to initiate projects in computer graphics to render the bifurcation sets associated to trajectory singularities of type A_1 for a range of examples of engineering interest.

2 A Review of Screw Systems

In order to describe the instantaneous singular sets of a spatial motion $\lambda : N \to SE(3)$ we require an understanding of its first–order behaviour. This is determined by the screw system associated to each point t of the configuration space N. The task is much simplified by employing the classification of screw systems under the natural relation of isometry equivalence. A full description of this classification can be found in [12, 5, 6] but we present here a brief review.

We start by considering n-dof motions of $\mathbb{R}^p$. Two n–dof motions $\lambda_1 : N_1, x_1 \to SE(p), \rho_1$ and $\lambda_2 : N_2, x_2 \to SE(p), \rho_2$ of $\mathbb{R}^p$ are said to be *$\mathcal{I}$–equivalent* (isometry equivalent) when there exist an invertible germ $g : N_1, x_1 \to N_2, x_2$ and isometries σ, τ such that $\lambda_2 = h^{-1}_{\sigma,\tau} \circ \lambda_1 \circ g$ where $h_{\sigma,\tau} : SE(p) \to SE(p)$ is the map defined by $\phi \mapsto \sigma\phi\tau$. It is a trivial remark that any local motion $\lambda : N, x \to SE(p), \rho$ is $\mathcal{I}$–equivalent to one of the form $\lambda : \mathbb{R}^n, 0 \to SE(p), 1$ where $n = \dim N$, and 1 is the identity element in $SE(p)$. And two such local motions λ_1, λ_2 are $\mathcal{I}$–equivalent when there exist an invertible germ g, and an isometry σ for which $\lambda_2 = h^{-1}_\sigma \circ \lambda_1 \circ g$ where h_σ denotes conjugation in $SE(p)$ by the element σ. Taking differentials we see that $T\lambda_2 = Th^{-1}_\sigma \circ T\lambda_1 \circ Tg$. Allowing the change of coordinates g means we are effectively concerned only with the image of the differential—a linear subspace of the Lie algebra $se(p)$. The mapping Th_σ defines the adjoint action of $SE(p)$ on its Lie algebra $se(p)$: this action is linear so induces an action on the Grassmannian of subspaces of $se(p)$ of any given dimension n.

Thus we see that the question of classifying 1–jets (first–order behaviour) of germs of motions λ under $\mathcal{I}$–equivalence reduces to the question of listing images of the differentials $T\lambda$ under the induced action on the Lie algebra. So one seeks natural finite stratifications of the Grassmannian, invariant under the action. From this point onwards we restrict our attention to the case $p = 3$. For $n = 1$ the objects of interest are lines in $se(3)$ so it is natural to consider the projectivised Lie algebra, whose elements are *screws*; the projective space itself is the *screw space*, and projective subspaces (corresponding to linear subspaces for $n \geq 2$) are *screw systems*. The convention in the engineering literature is that the screw system arising from a linear subspace of dimension n is called an n–system. We say that two n–systems are *equivalent* if they are equivalent under the induced action described above.

For a given choice of orthogonal coordinate system in $\mathbb{R}^3$, $SE(3)$ is isomorphic to the semidirect product $SO(3) \rtimes T(3)$ where $SO(3)$ is the special orthogonal group and $T(3)$ is the group of translations of $\mathbb{R}^3$: thus elements of $SE(3)$ can be identified with pairs (A, a) where A is an orthogonal 3×3 matrix, and a is a vector in $\mathbb{R}^3$. The Lie algebra $se(3)$ is then isomorphic to the semidirect product $so(3) \rtimes t(3)$ of the Lie algebras $so(3)$ and $t(3)$ of the factors. We identify elements of $so(3)$ with 3×3 skew symmetric matrices U or equivalently with the unique vector $u \in \mathbb{R}^3$ for which $Ux = u \wedge x$ for all $x \in \mathbb{R}^3$ (where $\wedge$ is the standard vector product on $\mathbb{R}^3$), and we identify elements of $t(3)$ with vectors v in $\mathbb{R}^3$. With these identifications, elements of $se(3)$ can be written in the form (u, v) where u, v are vectors in $\mathbb{R}^3$. In terms of these *motor coordinates* (u, v) the adjoint action of $SE(3)$ on $se(3)$ is given by

$$(A, a) \bullet (u, v) = (Au, Av - Au \wedge a).$$

In [5] it was shown that the ring of invariant polynomials for this action is generated by the Klein form $\langle u, v\rangle$ and the Killing form $\langle u, u\rangle$, where $\langle\ ,\ \rangle$ is the standard scalar product on $\mathbb{R}^3$. The ratio h defined by $h = \langle u, v\rangle/\langle u, u\rangle$ is the *pitch* of the motor (u, v). Since the invariant forms both have degree 2, pitch is well–defined for screws and is an invariant of the induced action, where now (u, v) refers to homogeneous coordinates in the screw space, a real projective 5–space. Screws with $u = 0$ are said to be of *infinite* pitch. The invariant h is a modulus for the space of orbits of screws under equivalence. To obtain a finite classification we must amalgamate orbits—physical experience suggests the classification of Table 1. This can be justified mathematically [6] by observing that the resulting classes are actually orbits of the action obtained by extending the group of isometries to that of similarities.

type	normal form	pitch
right-handed screw	$(1, 0, 0; h, 0, 0)$	$h > 0$
left-handed screw	$(1, 0, 0; h, 0, 0)$	$h < 0$
infinitesimal rotation	$(1, 0, 0; 0, 0, 0)$	$h = 0$
infinitesimal translation	$(0, 0, 0; 1, 0, 0)$	$h = \infty$

Table 1: Classification of 1–systems.

Before considering higher–dimensional screw systems we make some observations about the geometry of screw space. A screw (u, v) of pitch zero corresponds to a line in $\mathrm{P}\mathbb{R}^3$ with Plücker coordinates (u, v). The *axis* of a general screw (u, v) is defined to be the line in $\mathrm{P}\mathbb{R}^3$ with Plücker co–ordinates $(u, v - hu)$. In the screw space, the generators for the ring of invariant polynomials define the Klein quadric $\langle u, v\rangle = 0$, and the Killing quadric $\langle u, u\rangle = 0$, which (in the real case) is simply a generating α–plane for the Klein quadric. These two quadrics form a base for a real quadratic complex with general member

$$Q_h(u, v) = \langle u, v\rangle - h\langle u, u\rangle$$

whose zero set comprises all screws of pitch h (together with those of infinite pitch). For that reason Q_h is referred to as a *pitch quadric*. Two screws are *reciprocal* when they are orthogonal with respect to the (bilinear) Klein form defined by

$$B((u,v),(u',v')) = \frac{1}{2}(\langle u,v'\rangle + \langle u',v\rangle).$$

To any screw system S we can associate its *reciprocal* screw system $S^\perp$. Note that $(S^\perp)^\perp = S$. Thus an n–system ($n = 1,2,3$) is orthogonal to a $(6-n)$–system. It is easily verified that two screw systems S_1, S_2 are equivalent if and only if their reciprocal systems $S_1^\perp$, $S_2^\perp$ are equivalent. Thus the classification of 4– and 5– systems reduces to that of 2– and 1–systems, resulting in a considerable reduction of labour.

A principle for distinguishing 2– and 3–systems S is the way in which they lie in the screw space relative to the quadratic complex of pitch quadrics Q_h. The following table describes a crude subdivision of screw systems (of given dimension) by invariant geometric properties:

I:	S not contained in any Q_h	II:	S contained in a Q_h.
A:	S does not meet Q_∞	B:	S meets Q_∞ in a point
C:	S meets Q_∞ in a line	D:	S meets Q_∞ in a plane.

The notation combines in an obvious way—for example, IA refers to the case when S is neither contained in any Q_h nor meets Q_∞—leading to the broad classification of Table 2. In each case a "normal form" projective basis is listed.

type	basis	type	basis
IA	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$	IIA	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$
IB	$(1,0,0;0,0,0)$ $(0,0,0;1,p,0)$	IIB	$(1,0,0;h,0,0)$ $(0,0,0;0,1,0)$
		IIC	$(0,0,0;1,0,0)$ $(0,0,0;0,1,0)$

Table 2: Normal forms for 2–systems.

The normal forms involve invariants (or moduli) of two kinds: principal pitches h_α, h_β with $h_\alpha < h_\beta$ representing the the extreme pitches occurring in a 2–system of type A, and the [5] square invariant p in type IB, measuring the spread of axes, which may always be chosen to be non-negative. A natural refinement, similar to that employed for 1–systems above, is obtained in terms of the signs of these moduli. This leads to the classification of Table 3 which emphasises the special role of Q_0.

type	classes
IA	IA^{++}, IA^{+0}, IA^{+-}, IA^{0-},IA^{--}
IB	IB, IB^0
IIA	IIA^+, IIA^0, IIA^-
IIB	IIB^+, IIB^0, IIB^-
IIC	no finer subtypes

Table 3: Classification of 2–systems.

Finally we turn to 3–systems. The broad classification of type I systems in terms of A, B, C, D is amenable to a natural geometric refinement: the intersection of the 3–system (a projective plane) with the pitch quadrics gives rise to a pencil of real conics, whose projective classification is well-known and defines 5 distinct classes. One value of this is that it gives a readily calculable 'recognition' principle for types of 3–systems. Combining these 5 classes with those of type II results in the broad classification of Table 4. The principal pitches h_α, h_β, h_γ are such that $h_\alpha < h_\beta < h_\gamma$.

type	basis	type	basis
IA_1	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,1;0,0,h_\gamma)$	IIA	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,1;0,0,h)$
IA_2	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,1;0,0,h_\beta)$	IIB	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,0;0,0,1)$
IB_0	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,0;1,0,p)$	IIC	$(1,0,0;h,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;0,0,1)$
IB_3	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,0;0,0,1)$	IID	$(0,0,0;1,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;0,0,1)$
IC	$(1,0,0;0,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;1,0,p)$		

Table 4: Normal forms for 3–systems.

We may refine this classification by taking account of the signs of the moduli. Note that the square invariant p is non-negative so, in the list of refinements in Table 5, the sign of h distinguishes types $IB_0^{+,+}$, $IB_0^{+,0}$ and $IB_0^{+,-}$ in the case $p > 0$

while the other three types correspond to $p = 0$.

type	classes
IA_1	IA_1^{+++}, IA_1^{++0}, IA_1^{++-}, IA_1^{+0-}, IA_1^{+--}, IA_1^{0--}, IA_1^{---}
IA_2	$\mathrm{IA}_2^{(++)+}$, $\mathrm{IA}_2^{+(++)}$, $\mathrm{IA}_2^{(++)0}$, $\mathrm{IA}_2^{+(00)}$, $\mathrm{IA}_2^{(++)-}$ $\mathrm{IA}_2^{+(--)}$, $\mathrm{IA}_2^{(00)-}$, $\mathrm{IA}_2^{0(--)}$, $\mathrm{IA}_2^{(--)-}$, $\mathrm{IA}_2^{-(--)}$
IB_0	$\mathrm{IB}_0^{+,+}$, $\mathrm{IB}_0^{+,0}$, $\mathrm{IB}_0^{+,-}$, $\mathrm{IB}_0^{0,+}$, $\mathrm{IB}_0^{0,0}$, $\mathrm{IB}_0^{0,-}$
IB_3	IB_3^{++}, IB_3^{+0}, IB_3^{+-}, IB_3^{0-}, IB_3^{--}
IC	IC^{+}, IC^{0}
IIA	IIA^{+}, IIA^{0}, IIA^{-}
IIB	IIB^{+}, IIB^{0}, IIB^{-}
IIC	IIC^{+}, IIC^{0}, IIC^{-}
IID	no finer subtypes

Table 5: Classification of 3–systems.

The listing of screw systems has excellent properties. It is shown in [6] that the classes yield a finite Whitney regular stratification, giving rise to a genericity theorem. So for a typical 3–dof motion one would expect to observe only 3–systems of codimension ≤ 3, which are easily listed.

3 Instantaneous Singular Sets for Spatial Motions

A general observation is that if two motion germs are $\mathcal{I}$–equivalent, then the associated ISSes are related by an isometry of 3–space. Thus in principle the ISS associated to a given motion germ can be determined (up to isometries) from the normal forms under $\mathcal{I}$–equivalence given in Tables 1, 2 and 4. The explicit determination of ISSes is based on the following lemma.

Lemma 2 *Let $\lambda : \mathbb{R}^n, 0 \to SE(3), 1$ be a motion germ with d–dof, let S be the associated screw system, and for any tracing point $w \in \mathbb{R}^3$ let A_w be the generating α–plane in the Klein quadric Q_0 representing the bundle of lines through w. Then w lies in $I(\lambda, t)$ if and only if $S \cap A_w$ has projective dimension $\geq \max\,(0, d-3)$.*

Proof The condition for a point $w \in \mathbb{R}^3$ to lie in $I(\lambda, t)$ is that the rank of the derivative $T_0(ev_w \circ \lambda)$ is $< \min\,(n, 3)$, and hence that its kernel $\mathrm{im}\, T_0\lambda \cap \ker T_1 ev_w$ has dimension $> \max\,(0, d-3)$, given that ev_w is surjective and rank $T_0\lambda = d$. Projectively, the image im $T_0\lambda$ is the associated screw system S. The kernel of $T_1 ev_w$ is the tangent space at the identity to the (rotation) subgroup of $SE(3)$ fixing

w, and projectively this is the α–plane A_w in Q_0 comprising the bundle of lines through w. The result follows. □

Here are some elementary corollaries of Lemma 2. First, $I(\lambda, t)$ depends only on the associated screw system S, not on the particular motion germ λ used to define it — it is a first–order invariant of the motion. We write $I(S)$ for the ISS associated in this way to the screw system S. Second, the projective dimension of $S \cap A_w$ is the corank of the trajectory germ through w at t. Third, $I(S)$ is 'ruled', in the sense that for any point $w \in I(S)$, there is a line through w contained in $I(S)$. Fourth, for motion germs with ≤ 3–dof a point w is in $I(S)$ if and only if S contains a screw of pitch zero, and w lies on its axis. The next lemma tells us that these are the only screw systems we will need to consider.

Lemma 3 *Let S be a d–system, and let $S^\perp$ be the reciprocal $(6-d)$–system. Then $I(S) = I(S^\perp)$.*

Proof It suffices to show that $I(S) \subseteq I(S^\perp)$. While the form of the argument is similar for all d, its detail is simpler in the case $d \leq 3$, so we present that first. Let w be a point in $I(S)$. Then by Lemma 2 the intersection $S \cap A_w$ has projective dimension ≥ 0, so S contains a screw \$ of pitch zero. Write S as the projective join of $\{\$\}$ and a $(d-1)$–system S_0. Thus $S^\perp$ comprises all screws reciprocal both to \$ and to S_0. The reciprocal system $S_0^\perp$ has projective dimension $(6-d)$, and meets A_w in a projective subspace of dimension necessarily $\geq (6-d)+2-5 = 3-d$. However $S_0^\perp \cap A_w \subseteq S^\perp \cap A_w$, since *any* screw in A_w is reciprocal to \$. Thus $S^\perp \cap A_w$ likewise has projective dimension $\geq 3-d = (6-d)-3$, so by Lemma 2 the point w lies in $I(S^\perp)$, as required. In the case $d > 3$ we must replace the 1–system \$ by a $(d-3)$–system $S' \subseteq S \cap A_w$ and then continue as above. □

Lemma 2 translates into a procedure for determining equations for ISSes, starting from their normal forms under $\mathcal{I}$–equivalence. In view of Lemma 3 we can assume λ has d–dof, with $d \leq 3$. Write $w = (x, y, z)$. The evaluation map germ $ev_w : SE(3), 1 \to \mathbb{R}^3, w$ is submersive, with differential $T_1 ev_w : se(3) \to \mathbb{R}^3$ having a 3-dimensional kernel spanned by $(1,0,0,0,-z,y)$, $(0,1,0,z,0,-x)$, $(0,0,1,-y,x,0)$. Let $t = (t_1, \ldots, t_d)$ be coordinates at the source of λ. The condition for w to lie in the ISS associated to the parameter $t = 0$ is that the kernel should have a non–trivial intersection with the image of the differential of λ at $t = 0$. Write $\lambda_1, \ldots, \lambda_6$ for the components of λ, and λ_{ij} for the derivative of λ_i with respect to t_j evaluated at $t = 0$. Then the required conditon is that the following matrix should have rank

$< 3 + d$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -z & y \\ 0 & 1 & 0 & z & 0 & -x \\ 0 & 0 & 1 & -y & x & 0 \\ \lambda_{11} & \lambda_{21} & \lambda_{31} & \lambda_{41} & \lambda_{51} & \lambda_{61} \\ . & . & . & . & . & . \\ \lambda_{1d} & \lambda_{2d} & \lambda_{3d} & \lambda_{4d} & \lambda_{5d} & \lambda_{6d} \end{pmatrix}.$$

We now use these remarks to determine ISSes for the classes of screw system listed in Tables 1, 3, 5. The simplest case is that of 1–dof motion germs. The condition for the trajectory curve to be singular at a given instant is that the associated screw have pitch zero: the ISS is then a line, the classical instantaneous axis of rotation. However, for motions with 2–dof we obtain something new. In principle we expect a 2–system to contain two screws of pitch zero, and the resulting ISS to comprise two lines in 3–space. Table 6 gives a complete list of ISSes *as sets, up to isometries of 3–space* for the possible 2–systems in Table 2.

type of 2–system	ISS	max
IA^{++}, IA^{--}	empty	0
IA^{+0}, IA^{0-}	line	1
IA^{+-}	skew lines	1
IB	line	1
IIA^{+}, IIA^{-}	empty	0
IIA^{0}	plane	2
IIB^{+}, IIB^{-}	empty	0
IIB^{0}	plane	1
IIC	empty	0

Table 6: Instantaneous Singular Sets for 2–systems

The number in the third column in Table 6 represents the *maximal* corank of a trajectory germ at a point on the ISS, given by the excess of the dimension of the ISS over $\max(0, d-3)$. Indeed, for each type one expects a natural partition of the ISS by the corank of the germ at that point. There is only one type for which the corank fails to be constant on the ISS, namely the type IIA^0. In that case there is a line of screws of pitch zero, representing a planar pencil of lines in 3–space, whose union is a plane with a distinguished point at which the corank jumps from its generic value of 1 to the exceptional value 2. By contrast, type IIB^0 again has a line of screws of pitch zero, representing a planar pencil, but this time the vertex is at infinity, so there is no distinguished point in the resulting affine plane.

The most interesting case is provided by motions with 3–dof. The screws of pitch zero in a 3–system are obtained by intersecting the system with the Klein quadric

Q_0, so in principle lie on a conic, determining a (doubly) ruled quadric in 3–space: of course the conic, and hence the ISS, may be empty or degenerate. Table 7 gives a complete list of ISSes for 3–systems. The second column gives the intersection of the screw system with Q_0, the third column describes the associated ISS, and the fourth column gives the maximal corank of a trajectory germ at a point on the ISS.

type of 3–system	intersection	ISS	max
IA_1^{+++}, IA_1^{---}	empty	empty	0
IA_1^{++-}, IA_1^{+--}	conic	elliptic 1–sheet hyperboloid	1
IA_1^{++0}, IA_1^{0--}	point	line	1
IA_1^{+0-}	line pair	plane pair	2
$\mathrm{IA}_2^{(++)+}$, $\mathrm{IA}_2^{+(++)}$	empty	empty	0
$\mathrm{IA}_2^{(--)-}$, $\mathrm{IA}_2^{-(--)}$	empty	empty	0
$\mathrm{IA}_2^{(++)-}$, $\mathrm{IA}_2^{+(--)}$	conic	circular 1–sheet hyperboloid	1
$\mathrm{IA}_2^{(++)0}$, $\mathrm{IA}_2^{0(--)}$	point	line	1
$\mathrm{IA}_2^{+(00)}$, $\mathrm{IA}_2^{(00)-}$	repeated line	plane	2
IB_0^{+}, IB_0^{-}, $\mathrm{IB}_0^{0,+}$, $\mathrm{IB}_0^{0,-}$	conic	hyperbolic paraboloid	1
IB_0^{0}, $\mathrm{IB}_0^{0,0}$	line pair	plane pair	2
IB_3^{++}, IB_3^{--}	point in Q_∞	empty	0
IB_3^{+-}	line pair	parallel planes	1
IB_3^{+0}, IB_3^{0-}	line	plane	1
IC, IC^0	line	plane	1
IIA^+, IIA^-	empty	empty	0
IIA^0	α–plane	whole space	3
IIB^+, IIB^-	point in Q_∞	empty	0
IIB^0	β–plane	plane	2
IIC^+, IIC^-	line in Q_∞	empty	0
IIC^0	α–plane	whole space	1
IID	Q_∞	empty	0

Table 7: Instantaneous Singular Sets for 3–systems

For each ISS type in Table 7 one expects a natural partition by the corank of the trajectory germ at that instant. There are only four cases where the corank can increase above its generic value. (Figure 1.)

- For type IA_1^{+0-} the screw system intersects Q_0 in a line pair. Each line represents a planar pencil of lines in 3–space. Thus the ISS is the union of a pair of distinct planes, each with a distinguished point (the vertex of the pencil) on the line of intersection: at both points the corank increases from its generic value of 1 to 2. In particular, the partition of the ISS by corank is not a stratification.

- For types $\mathrm{IA}_2^{+(00)}$, $\mathrm{IA}_2^{(00)-}$ the screw system intersects Q_0 in a repeated line. The line represents a planar pencil of lines in 3–space. Thus the ISS is a plane, with a single distinguished point, the vertex of the pencil: at that point the corank increases from its generic value of 1 to 2.

- For type IB_0^0 the screw system intersects Q_0 in a line pair. Each line represents a planar pencil of lines in 3–space, one with vertex a finite point, and the other with vertex a point at infinity. Thus the ISS is a plane pair, the line of intersection being the unique line in the first pencil parallel to all the lines in the second. At the distinguished point on this line the corank increases from its generic value of 1 to 2.

- For type IIA^0 the screw system is a generating α–plane of Q_0 representing a bundle of lines through a point in 3–space. Thus the ISS is the whole of 3–space, with a distinguished point, at which the corank increases from its generic value of 1 to 3.

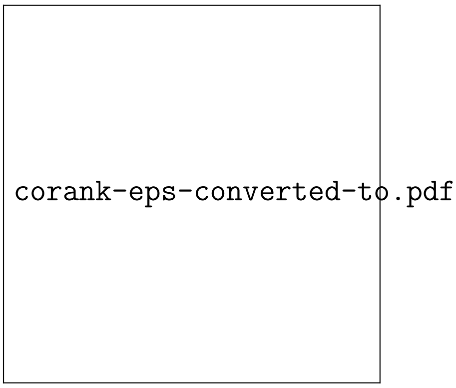

Figure 1: Corank increases on ISSes

4 An Envelope Result

The object of this section is to establish a key connexion between the geometry of a spatial motion, and that of the resulting family of trajectories. At root, the justification for adopting the singular viewpoint of kinematics is that it will lead to a genuine understanding of the local, and possibly global, structure of the bifurcation curves and surfaces associated to the family of trajectories. Ultimately, one seeks algorithms for rendering these sets on a computer screen. It is here that ISSes provide an interesting way forward, in that *some* of the bifurcation sets can be described in terms of their envelopes. It is interesting because the connexion is not present in the classical kinematics of 1–dof motions, where the ISSes are not hypersurfaces. The prototype result was obtained for generic planar motions with 2–dof in [14]: in that case the ISSes are lines in the plane, and it was shown that their envelope gave essentially the lips and beaks bifurcation curves. The attraction of such a result is that one can exploit computer algebra programs to render the bifurcation curves.

Our concern is with *generic* spatial motions having 3–dof. For such motions the trajectory singularities must occur in the list of germs $\mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ of codimension $\leq$ 3 under $\mathcal{A}$–equivalence. The only source we are aware of for this extensive list is the computer–aided classification by Hawes in [15]. However it needs to be said that several known lists have a considerable intersection with this list, such as Morin's listing of the stable germs [19], Bruce's listing of the codimension 1 germs [3], and Marar and Tari's listing [18] of the $\mathcal{A}$–simple germs of corank 1. For reasons which will be explained below, we require only the sublist of Hawes' list comprising germs of corank 1 *with non–smooth critical sets*. The sublist appears in Table 8. All the germs in the sublist turn out to be $\mathcal{A}$–simple. We know of no reason *a priori* why that should be the case: in particular they all appear in the listing [18] by Marar and Tari, which is why we have used their labels (with small obvious modifications) in the first column. It is not difficult to distinguish the germs in Table 8 by their discriminants.

label	cod	normal form
A_1^{++}	1	$(x, y, z^3 + y^2z + x^2z)$
A_1^{+-}	1	$(x, y, z^3 + y^2z - x^2z)$
A_1^{--}	1	$(x, y, z^3 - y^2z + x^2z)$
4_2^2	2	$(x, y, yz^2 + x^2z + z^4 + z^5)$
4_2^2	2	$(x, y, yz^2 + x^2z + z^4 - z^5)$
A_2^+	2	$(x, y, z^3 + x^3z + y^2z)$
A_2^-	2	$(x, y, z^3 + x^3z - y^2z)$
A_3^{++}	3	$(x, y, z^3 + y^2z + x^4z)$
A_3^{+-}	3	$(x, y, z^3 + y^2z - x^4z)$
A_3^{-+}	3	$(x, y, z^3 - y^2z + x^4z)$
A_3^{--}	3	$(x, y, z^3 - y^2z - x^4z)$
4_2^3	3	$(x, y, yz^2 + x^2z + z^4 + z^7)$
4_2^3	3	$(x, y, yz^2 + x^2z + z^4 - z^7)$

Table 8: Corank 1 germs of cod ≤ 3 with non–smooth critical sets

Lemma 4 *Let* $\lambda : \mathbb{R}^3, 0 \to SE(3), 1$ *be a generic motion germ with 3–dof whose associated ISS is a 1–sheeted hyperboloid. The envelope surface of the ISSes for parameters t close to 0, is the closure of the union of the bifurcation surfaces associated to the germs of type* A_1 *in Table 8.*

Proof We claim first that, for parameters t close to 0, the ISS remains a 1–sheeted hyperboloid whose equation depends smoothly on $t = (t_1, t_2, t_3)$. First, the 3–system types whose associated ISS is a 1–sheeted hyperboloid are stable under sufficiently small perturbations of the motion germ. Further, we can find bases depending smoothly on t for the image of the differential $T_t\lambda : \mathbb{R}^3, t \to SE(3), \lambda(t)$ and the kernel of the differential $T_{\lambda(t)}ev_w : SE(3), \lambda(t) \to \mathbb{R}^3, \lambda(t)(w)$. The determinant of the six resulting vectors in the tangent space to $SE(3)$ at $\lambda(t)$ then defines a smooth function $F(t, w)$ with the property that, firstly, for fixed t the relation $F(t, w) = 0$ defines the ISS associated to the germ of λ at t, and secondly, for fixed w the relation $F(t, w) = 0$ defines the critical set of the trajectory through w. By definition, the envelope surface of this 3–parameter family of surfaces $F(t, w) = 0$ (with t the parameter) is the set of points w in 3–space for which there exists a parameter t for which

$$F(t, w) = 0 : \quad \frac{\partial F}{\partial t_1}(t, w) = 0 : \quad \frac{\partial F}{\partial t_2}(t, w) = 0 : \quad \frac{\partial F}{\partial t_3}(t, w) = 0.$$

Let w be a point on the envelope surface. Then $F(t, w) = 0$ says that t lies on the critical set of the trajectory through w, whilst the vanishing of the partial derivatives says that t is a singular point of the critical set. Now the only stable singularities are the immersion, fold, cusp and swallowtail types, each having *smooth* critical sets: it follows that the germ at t of the trajectory through w is non–stable, and hence that w lies in the bifurcation set associated to that germ type. Since λ is assumed generic, the germ at t of the trajectory is of codimension ≤ 3: moreover it has corank 1 (since that is the only corank possible when the ISS is a 1–sheeted hyperboloid) and has a non–smooth critical set, so must be one of the types in Table 8. Thus w lies in one of the bifurcation sets associated to those germs. Conversely, let w be a point in the union of the bifurcation surfaces associated to the germ type in Table 8. Then there exists a parameter t for which the trajectory through w is critical at t, and has a non–stable germ at t. Moreover, the critical set of the germ at that point is non–smooth, so (t, w) satisfies the displayed equations above, and hence w lies on the envelope surface.

To establish the stated result, note first that since the motion germ λ is assumed to be generic, the family of trajectories versally unfolds all of the types in Table 8. According to the specialization results in [18, 4] a versal unfolding of any type in Table 8 contains at least one of the A_1–types. Thus the union of the bifurcations sets associated to the types in Table 8 is contained in the union of the closures of those associated to the A_1–types. For the converse implication we argue as follows. Each of the A_1–types is a corank 1 germ with zero intrinsic second derivative: thus any corank 1 germ in Hawes' list to which it specializes likewise has zero intrinsic second derivative, so has a normal form $(x, y, f(x, y, z))$ where $f(x, y, z)$ has zero 2–jet. It is an empirical fact that all such normal forms in Hawes' list have non–smooth critical sets, so appear in Table 8. □

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