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Singularities of robot manipulators: Lie groups and exponential products

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Abstract The kinematics of a robot manipulator are described by a relation between its joint space and work space, a subset of the Lie group of Euclidean motions $SE(3)$. Depending on the architecture of the manipulator, either the forward or the inverse kinematics may be a function which, for topological reasons, must have singularities. For the robot engineer, it is essential to know about these as they lead to a variety of phenomena such as loss of control, excessive joint acceleration or torque and trajectory branching. There is an extensive engineering literature on robot singularities but a much smaller literature applying singularity-theoretic techniques to understanding them.

For manipulators whose links are joined in series, the forward kinematics is a function, most succinctly described by a product of exponentials in the Lie group. The special form of these kinematic mappings leads to conditions for singularities and transversality in terms of the group's Lie algebra. A simple genericity result is derived for a broad class of manipulators and a computational approach to the presentation of global singular loci is described, based on a closed form expression for the exponential.

Keywords :

Euclidean group, robot kinematics, product of exponentials, genericity, Thom–Boardman singularity, regional manipulator

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1 Introduction

The analysis of singularities presents one of the most challenging problems in robot kinematics. Attempts to apply singularity-theoretic methods have been, at best, only partially successful. The difficulty arises

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from the very restrictive nature of kinematic mappings which, assuming that the components of a manipulator are rigid, are tied very strongly to the geometry of the Euclidean motion group. In this paper, the beginnings of an analysis of manipulator singularities based on the formulation of the kinematics as a product of exponentials in the Euclidean group are developed. In particular, transversality and genericity results are derived and a computational approach to the presentation of global singular loci is described based on a closed form expression for the exponential.

At a singularity there is a change in the expected freedom of movement of the manipulator that, at the simplest level, entails either some restriction on the way in which the robot can move or alternatively additional uncontrolled motion either infinitesimally or perhaps over some neighbourhood. In practice, control algorithms based on inverse methods may result in unrestricted accelerations or torques in the manipulator joints, leading to mechanical failure.

The recognition of singularities and the restrictions they impose on the effectiveness of robot manipulators originates in the work of Whitney [46] on the control of prosthetics. A growing number of works during the 80s describe and explore the phenomenon in greater detail, especially in relation to the simplest form of manipulator architecture, the serial arm. Over the succeeding 20 years singularities of parallel manipulators, whose architecture involves closed loops and passive joints, have become of great interest, particularly following the conceptual ideas introduced by Gosselin and Angeles [19] which uses implicit function methods and classify different singular phenomena, and Merlet [25] which introduces geometric algebra as a tool. The literature relating to singularities in robot kinematics is now extensive. For example, the database [8] features over 1,700 items. Predominantly, these concern the identification of the singularity loci of specific manipulator designs, strategies for avoidance and adaptation of control methods.

The idea of applying methods of singularity theory appears first in the works of Tchoń [39] and Pai and Leu [30]. Each considers questions of genericity, while the former, as well as seeking normal forms for manipulator singularities, notes that standard theorems on genericity may not apply because of the restricted form of the maps involved. As a result, the term 'generic' is often used simply to describe a manipulator whose singularities are 1-generic in the Thom–Boardman sense (see Section 4) but without explicit justification. Gibson and co-workers [13,14,15,16] specifically consider the point trajectories generated by multi-parameter rigid body motions, of which manipulator kinematics constitute a special set of examples. Here, it is possible to invoke some serious singularity theory and several new classifications of singularities of low-dimensional mappings (domain and range up to dimension 3) result. Nonetheless, only limited consequences of the restrictions imposed by the special form of the kinematics are taken into account. A review of this literature can be found in [9].

The significance of the Euclidean group in describing rigid body motion has been recognised, at least implicitly, for centuries. Its role in robot kinematics is of course more recent. The design of a serial manipulator is determined by its joint types and the relative placement of the joints within successive links, usually described by means of Denavit–Hartenberg (DH) parameters [7]. These are a minimal set of parameters that determine the transformation between coordinate systems in successive links and hence determine the specific design of the manipulator. Brockett [4] shows that the kinematics can equally well be written, without recourse to DH parameters, as a product of exponentials—a map $f : \mathbb{R}^k \to G$, G a Lie group, of the form:

$$f(q_1, \dots, q_k) = \prod_{i=1}^{k} \exp(q_i X_i). \tag{1}$$

where $X_1, \dots, X_k \in \mathfrak{g}$, the associated Lie algebra. Here, G is the Euclidean group, X_i represent joints and q_i the joint variables, for $i = 1, \dots, k$. The role of the design parameters is replaced by the infinitesimal motions X_i admitted by the manipulator joints. This formulation is employed in a number of papers on singularities, such as those of Karger [23] and Lerbet and Hao [24], whose approach is emulated here.

In this paper, Section 2 gives a brief introduction and overview of robot manipulators. Section 3 outlines the relevant background of the Euclidean group and its application to robot kinematics. References for more detailed discussion are Murray *et al* [28] and Selig [34]. The general machinery of Lie groups enables the product of exponentials to be expanded, providing an approach to the analysis of manipulator singularities and these are the subject of Section 4. In Section 5, an alternative approach via exponentials of adjoints yields a genericity theorem. An application of a method of Putzer leads to some specific consequences for a relatively simple but important class, the 3 degree-of-freedom regional manipulators in Section 6.

2 Robot manipulators

For our purposes, a robot manipulator consists of a set of rigid components termed *links*, connected by means of *joints* that admit continuous motion between the links. The *architecture* of a manipulator can be represented by a graph whose vertices are the links and whose edges represent joints between a pair of links [26]. A *serial manipulator* is an open chain consisting of a sequence of links $L_0, L_1, \dots, L_k$ pairwise connected by joints $J_1, \dots, J_k$. The initial link L_0 is referred to as the *base* and final link L_k as the *end-effector*, these only being connected to one other link. If, in addition, L_k and L_0 are directly connected to each other, the result is a *simple closed chain* whose architecture is a simple cycle. This is the most basic example of a *parallel manipulator*, that is a manipulator whose architecture includes at least one cycle. Examples of actual serial and parallel manipulators are shown in Figure 1. While restricted classes of manipulator, such as planar and spherical (in which the components are constrained to move relative to a fixed point in space), are of interest, we shall concentrate on the general spatial case.

Fig. 1 Canadarm serial manipulator and hexapod (Gough–Stewart) parallel manipulators.

Serial manipulator joints typically have one degree of freedom (dof) and each is *actuated*, that is to say their positions are controlled by motors. In 3 dimensions, there are three types of 1-dof joints: revolute (R), prismatic (P) and helical (H), illustrated in Figure 2. In practical manipulators, R and P joints are mostly used as these are simpler to manufacture and control. However, in parallel manipulators there are passive joints that frequently have more than 1-dof, for example 2-dof universal (U) joints (consisting of a pair of revolute joints with coincident axes) and 3-dof spherical (S) joints (shown in Figure 2).

Fig. 2 Revolute, prismatic and helical 1-dof and spherical 3-dof joints.

Associated with the architecture is a combinatorial invariant, the *(full-cycle) mobility* μ of the manipulator, which is its total internal number of degrees of freedom. This is given by the formula of

Chebyshev–Grübler–Kutzbach (CGK) [22]:

$$\mu = t(l-1) - \sum_{i=1}^{k}(t-\delta_i) = \sum_{i=1}^{k}\delta_i - t(k-l+1) \tag{2}$$

where t is the number of degrees of freedom of an unconnected link ($t = 6$ for spatial manipulators, as detailed in Section 3), k is the number of joints, l the number of links and δ_i the number of degrees of freedom admitted by the ith joint. The first expression in (2) represents the difference between the total freedom of the links and the constraints imposed by the joints. The second version emphasises that the mobility is the difference between the total joint dofs and the number of constraints as expressed by the dimension of the cycle space of the associated graph. The number of actuated joints is usually chosen to match the mobility.

A specific manipulator requires more information, determining the variable design parameters inherent in the architecture. The CGK formula (2) is generic in the following sense [27]: there may be specific realisations of an architecture for which the formula does not give the true mobility but for most choices of design parameters (e.g. Denavit–Hartenberg parameters) it is correct. However these issues are of interest mainly for parallel manipulator kinematics and we leave them to one side. For a serial manipulator, where $l = k+1$, and $\delta_i = 1$ for $i = 1, \dots, k$, the formula simply gives $\mu = k$. This does not preclude $k > 6$: such a manipulator is described as *redundant* since it has more internal degrees of freedom than are required for the motion of the end-effector. Nevertheless, redundancy, while introducing control problems, may have advantages in singularity avoidance.

3 The Euclidean group and kinematic mappings

The primary problem of kinematics is to describe the relation between the motion of the end-effector and the relative motion at each of the joints. We first clarify what the relevant domains should be.

The motion of an individual (rigid) link, relative to a pre-determined 'home' configuration, is given by a Euclidean isometry. The home configuration is fixed relative to another component of the manipulator, e.g. the base, end-effector or an adjacent link. Given a choice of origin and orthonormal frame in the (relatively) fixed and moving components, an isometry is uniquely given by a rigid rotation A about the origin and a subsequent translation $\mathbf{a}$. For a point with position vector $\mathbf{x}$ in moving coordinates, its fixed coordinates are then given by $A\mathbf{x} + \mathbf{a}$. The composition in this form is given by

$$(A_2, \mathbf{a}_2) \cdot (A_1, \mathbf{a}_1) = (A_2A_1, A_2\mathbf{a}_1 + \mathbf{a}_2) \tag{3}$$

Likewise, the n-dimensional Euclidean isometry group $SE(n)$ is isomorphic to the semi-direct product $SO(n) \ltimes \mathbb{R}^n$. and is a real Lie group of dimension $\frac{1}{2}n(n+1)$. For $n = 3$, this gives dimension 6. The pure translations form a normal subgroup. All other motions are rotations about *some* point in space.

The following homogeneous representation in 4 dimensions is widely used in the robotics literature:

$$\left(\begin{array}{c|c} A & \mathbf{a} \\ \hline \mathbf{0} & 1 \end{array}\right). \tag{4}$$

Three-dimensional motions are recovered by restricting to the affine subspace with 4th coordinate equal to 1.

The Euclidean Lie algebra $\mathfrak{se}(3)$, the tangent space at the identity to the group, can likewise be identified as a vector space with the direct sum $\mathfrak{so}(3) \oplus \mathbb{R}^3$. Elements of the Lie algebra are called *twists*. It is helpful to identify a skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) \tag{5}$$

with the 3-vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$. In other words, $\boldsymbol{\omega} \in \mathbb{R}^3$ is the unique vector that satisfies $\Omega\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$ (where $\times$ denotes the vector product in $\mathbb{R}^3$). Hence, relative to a choice of orthonormal frame, twists can be written as $X = (\boldsymbol{\omega}, \mathbf{v})$ where $\boldsymbol{\omega}, \mathbf{v}$ are 3-vectors. The coordinates of X are referred to as *Plücker coordinates*—they generalise the more familiar Plücker line coordinates. The homogeneous representation of a twist is:

$$\tilde{X} := \left(\begin{array}{c|c} \Omega & \mathbf{v} \\ \hline \mathbf{0} & 0 \end{array}\right). \tag{6}$$

The adjoint representation of $X \in \mathfrak{se}(3)$, in terms of Plücker coordinates, is given by:

$$\operatorname{ad} X := \left(\begin{array}{c|c} \Omega & O \\ \hline V & \Omega \end{array}\right), \tag{7}$$

where V is the skew-symmetric matrix related to $\mathbf{v} \in \mathbb{R}^3$. Alternatively, given $X_i = (\boldsymbol{\omega}_i, \mathbf{v}_i) \in \mathfrak{se}(3)$, $i = 1, 2$, their Lie bracket is the twisted form:

$$[X_1, X_2] := (\operatorname{ad} X_1).X_2 = \tilde{X}_1\tilde{X}_2 - \tilde{X}_2\tilde{X}_1 = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2). \tag{8}$$

The subspace of infinitesimal translations $(\mathbf{0}, \mathbf{v}) \in \mathfrak{se}(3)$ is abelian. The infinitesimal rotations about three orthogonal axes cyclically permute under the bracket. However, as can be seen from (8), the Lie algebra does not decompose into its vector space summands.

Since the representations of the Euclidean group and its Lie algebra are dependent on a choice of coordinate frame, one should seek invariants with respect to the adjoint action, corresponding to change of coordinates. There are two invariant quadratic forms on the Lie algebra [10], the Killing form $\boldsymbol{\omega}.\boldsymbol{\omega}$ and the Klein form $\boldsymbol{\omega}.\mathbf{v}$. These give rise to a pencil of quadratics with matrix form:

$$Q_h(X) := X^t \left(\begin{array}{c|c} -hI_3 & \frac{1}{2}I_3 \\ \hline \frac{1}{2}I_3 & O \end{array}\right) X = \boldsymbol{\omega}.\mathbf{v} - h\,\boldsymbol{\omega}.\boldsymbol{\omega}, \tag{9}$$

the Killing form being denoted Q_∞. Note that the Killing form is degenerate, corresponding to the fact that the infinitesimal translations $\{\mathbf{0}\} \oplus \mathbb{R}^3$ form a solvable subalgebra, so the group is not semi-simple. The pitch forms Q_h, h finite, are non-degenerate but are indefinite of signature $(3, 3)$.

For any $X \in \mathfrak{se}(3)$, the ratio of the invariants $\boldsymbol{\omega}.\mathbf{v}/\boldsymbol{\omega}.\boldsymbol{\omega}$ is the parameter h for which $X^tQ_hX = 0$; h is termed the *pitch* of the twist (and indeed, for $h \neq \infty$, it corresponds to the displacement along the unique invariant axis arising from any point travelling 2π along the corresponding Killing vector field on $\mathbb{R}^3$).

The significance for manipulator kinematics is that, relative to coordinate frames in adjacent links L_{i-1}, L_i that coincide in the home configuration, the motion admitted by the 1-dof joint J_i is given by the exponential map:

$$q_i \mapsto \exp(q_iX_i) \in SE(3) \tag{10}$$

For given coordinates, the defining twist X_i of the joint J_i is defined only up to a non-zero constant. In other words, J_i corresponds to an element of the projective Lie algebra, termed a *screw*. The pitch is an invariant of screws under the action induced from the adjoint action. The three joint types are characterised by:

$$\text{R: } h = 0 \qquad \text{P: } h = \infty \qquad \text{H: } h \neq 0, \infty$$

For H joints the sign of h distinguishes right and left-handed screws. For an R joint, the joint motion (10) is periodic, with period 2π so its joint variable can be taken in the unit circle S^1, whereas for H and P joints, the joint variable lies in $\mathbb{R}$. In practice, there will be bounds on the joint variables that, for the development of the theory, we overlook.

By way of amplification, the screws of infinite pitch form a 3-dimensional subspace in $\mathfrak{se}(3)$, $\boldsymbol{\omega} = 0$, that is a subspace of the quadric $Q_h(X) = 0$ for every finite h. Projectively, the latter quadrics have two sets of rulings by (projective) planes, one of which contains the plane $Q_\infty(X) = 0$. The invariants do not separate orbits of infinite pitch twists. On the other hand, projectively, the infinite pitch screws form a single orbit [10].

For a k-joint serial manipulator, the *joint space* is the product of the (actuated) individual joint spaces, hence has the form

$$\mathcal{J} \cong (S^1)^r \times \mathbb{R}^{k-r}$$

where r is the number of revolute joints. The *work space* $\mathcal{W}$ is the configuration space of the end-effector, generally taken to be $SE(3)$, or some subset of it.

The primary kinematic problem splits into two parts:

- the *forward* kinematics $\mathcal{J} \to \mathcal{W}$: to determine workspace configuration in terms of the joint variables;
- the *inverse* kinematics $\mathcal{W} \to \mathcal{J}$: to determine the (active) joint variables in terms of workspace configuration.

For a serial manipulator, the forward kinematics is given by a function $f : \mathcal{J} \to \mathcal{W}$. It follows from the discussion above that, as Brockett [4] demonstrates, f can be expressed as a product of exponentials as in (1).

Associated with this kinematic mapping is a family of trajectories that are the evaluation of the map on fixed points $\mathbf{c}$ in the end-effector:

$$f_{\mathbf{c}} : \mathcal{J} \to \mathbb{R}^3, \quad (q_1, \ldots, q_k) \mapsto \exp(q_1 X_1) \cdots \exp(q_k X_k) \cdot \mathbf{c} \tag{11}$$

In the case that the number of degrees of freedom $k = 3$, associated manipulators are often referred to as *regional* (see, for example, [5,21]). They are used as positioning devices for a 3-dof hand that is spherical, the point of attachment being referred to as the *wrist centre*. Such 'wrist-partitioned' 6-dof arms are widely used because it is possible to solve explicitly their inverse kinematics [31]. The trajectory map (11) determines the motion of the wrist centre.

4 Manipulator singularities

Given the forward kinematic mapping $f : \mathcal{J}^{(k)} \to SE(3)$ of a k-dof serial manipulator, a singularity occurs at $\mathbf{q} \in \mathcal{J}$ when $\operatorname{rank} Df(\mathbf{q}) < \rho = \min\{k, 6\}$. In the case that $f(\mathbf{q}) = e \in SE(3)$, the identity, the derivative $Df(\mathbf{q})$ is a linear map into the Lie algebra $\mathfrak{se}(3)$. Its Jacobian matrix with respect to joint-variable parametrisation of $\mathcal{J}$ and Plücker coordinates for $\mathfrak{se}(3)$ is the matrix whose columns are the joint twists:

$$Jf(\mathbf{q}) = \left(X_1 | X_2 | \cdots | X_k \right) \tag{12}$$

The geometric interpretation of this matrix is that it determines the generalised (linear and angular) velocities of the end-effector in terms of the joint velocities. In the robotics literature this matrix is often referred to as the *geometric Jacobian* [33]. In particular, a singularity occurs when the joint twists span a subspace of the Lie algebra of dimension $< \rho$. In the theory of rigid body motion, such subspaces are called *screw systems*. They are the subject of Ball's great treatise [3] and have been much studied since Hunt's revival of the subject in [22]. Their classification is given firm mathematical foundation by Gibson and Hunt [17] and regularity properties of the stratification of screw systems are established in [11].

The simplest invariant of a manipulator singularity is its corank and the singularity locus decomposes into the singularity sets $\Sigma^r f$ of singular points of corank $r < 6$. In the case that f is *1-generic*, that is the 1-jet extension is transverse to the Thom–Boardman manifolds Σ^r, the codimension of $\Sigma^r f$ is $r(|k-\rho|+r)$ if it is non-empty. One immediate consequence for the design of manipulators relates to redundancy. For $k = 6$, typically the set $\Sigma^1 f$ would constitute a codimension 1 submanifold of $\mathcal{J}$ but, for a redundant manipulator having $k = 7$, $\operatorname{codim} \Sigma^1 f = 2$. That permits the existence of singularity-free paths in joint space connecting any pair of non-singular end-effector configurations (assuming $\mathcal{J}$ connected). From the practical point of view, there would however be an additional computational cost arising from the redundancy in determining such a path. This prompts the question of when a serial manipulator kinematic mapping is indeed 1-generic and whether this property is truly generic [27].

One approach is to make use of the Baker–Campbell–Hausdorff (BCH) formula (see, for example, [12]) that, on a neighbourhood of $\mathbf{0} \in \mathfrak{se}(3)$, determines Z such that $\exp Z = \exp X . \exp Y$ in terms of X, Y and the Lie bracket. It must take into account the possible non-commutativity of X, Y. Repeated application of the BCH formula to a general product of exponentials gives:

$$f(q_1, \ldots, q_k) = \exp\left(\sum_{i=1}^{k} q_i X_i + \tfrac{1}{2} \sum_{1 \le i < j \le k} q_i q_j [X_i, X_j] + O(3) \right), \tag{13}$$

where the coefficients of order 3 and higher terms in the joint variables q_i involve nested brackets of the twists X_i. Since the exponential function is a local diffeomorphism at the origin, it provides a coordinate parametrisation for $SE(3)$ on a neighbourhood of the identity. That is to say, the expression in the exponential in (13) provides a local representation, $\tilde{f}$, of the kinematic mapping f. A local representation of the 1-jet extension of f can be found by expanding $\tilde{f}(\mathbf{q}+\mathbf{a})$, differentiating with respect to the coordinates $a_1, \ldots, a_k$ of $\mathbf{a}$ and setting $\mathbf{a} = \mathbf{0}$. This gives:

$$\begin{aligned} j^1 f(\mathbf{q}) = \Bigg(\mathbf{q}, f(\mathbf{q}), \Big(X_1 + \tfrac{1}{2} \sum_{j=2}^{k} q_j [X_1, X_j] \Big| \ldots \Big| X_l - \tfrac{1}{2} \sum_{j=1}^{l-1} q_j [X_l, X_j] + \tfrac{1}{2} \sum_{j=l+1}^{k} q_j [X_l, X_j] \Big| \\ \ldots \Big| X_k - \tfrac{1}{2} \sum_{j=1}^{k-1} q_j [X_k, X_j] \Big) + O(2) \Bigg) \in \mathbb{R}^k \times \mathfrak{se}(3) \times L(\mathbb{R}^k, \mathfrak{se}(3)) \end{aligned} \tag{14}$$

where vertical bars separate column vectors in the matrix form of the linear map, the joint variable-dependent Jacobian.

Necessary and sufficient conditions for $j^1 f \pitchfork_{\mathbf{q}} \Sigma^s$ can be found in, for example, Golubitsky and Guillemin [18], Chapter VI. Denote by $\Upsilon_l \in L(\mathbb{R}^k, \mathfrak{se}(3))$ the derivative with respect to q_l, $l = 1, \ldots, k$, of the component of the 1-jet extension (14) in the fibre over $\mathbb{R}^k \times \mathfrak{se}(3)$. Then

$$\Upsilon_l = \tfrac{1}{2} \Big([X_1, X_l] \Big| \cdots \Big| [X_{l-1}, X_l] \Big| \mathbf{0} \Big| -[X_{l+1}, X_l] \Big| \cdots \Big| -[X_k, X_l] \Big) \tag{15}$$

The following transversality conditions can be determined as a result.

Theorem 1 *Let $f : \mathcal{J} \to SE(3)$ be a serial manipulator kinematic mapping given by the product of exponentials (1) and suppose $\mathbf{q} \in \mathcal{J}$ satisfies $f(\mathbf{q}) = e$ and $j^1 f(\mathbf{q}) \in \Sigma^s$. Then $j^1 f \pitchfork_{\mathbf{q}} \Sigma^s$ if and only if:*

(a) $s = 1$, $k \geq 5$ and if $\mathbf{c}_r \in T_{\mathbf{q}}\mathcal{J} \cong \mathbb{R}^k$, $r = 1, \ldots, R = 1 + \min\{|k-6|, 0\}$ span the kernel of $Df(\mathbf{q})$, then the vectors

$$X_1, \ldots, X_k, \Upsilon_1.\mathbf{c}_1, \ldots, \Upsilon_k.\mathbf{c}_R \tag{16}$$

span $\mathfrak{se}(3)$;

(b) $s = 2$, $k = 6$ and if $\mathbf{c}_r \in T_{\mathbf{q}}\mathcal{J} \cong \mathbb{R}^k$, $r = 1, \ldots, R = 2 + \min\{|k-6|, 0\}$ span the kernel of $DF(\mathbf{q})$, then the 6×2 matrices (where $|$ separates columns)

$$\begin{aligned} &(X_i \,|\, X_j), \quad (1 \leq i < \cdots < j \leq k), \\ &(\Upsilon_l.\mathbf{c}_r \,|\, \Upsilon_l.\mathbf{c}_t), \quad (l = 1, \ldots, k,\ 1 \leq r < t \leq R) \end{aligned} \tag{17}$$

span $L(\mathbb{R}^2, \mathfrak{se}(3))$.

For $s = 1$, $k \leq 4$, for $s = 2$, $k \neq 6$ and for $s \geq 3$, any k, transversality is not possible.

The condition $\mathbf{c} \in \ker Df(0)$ means that $Df(0).\mathbf{c} = \sum_{i=1}^k c_i X_i = \mathbf{0}$. Hence, for $l = 1, \ldots, k$:

$$\begin{aligned} \Upsilon_l.\mathbf{c} &= \tfrac{1}{2} \left[c_1 X_1 + \cdots + c_{l-1} X_{l-1} - c_{l+1} X_{l+1} - \cdots - c_k X_k, X_l\right] \\ &= \left[c_1 X_1 + \cdots + c_{l-1} X_{l-1}, X_l\right] \\ &= -\left[c_{l+1} X_{l+1} + \cdots + c_k X_k, X_l\right] \end{aligned} \tag{18}$$

and the terms (16), (17) in Theorem 1 can be rewritten. In particular, note that $\Upsilon_1 \mathbf{c} = \Upsilon_k \mathbf{c} = 0$. It is this that rules out transverse intersection with Σ^1 for $k = 4$ and Σ^2 for redundant manipulators with $k = 7, 8$, even though on dimensional grounds transversality appears possible.

The dependency of the transversality condition on the Lie bracket in $\mathfrak{se}(3)$ gives, as a corollary, the necessary condition that the Lie subalgebra Δ_∞ spanned by the joint twists must be the whole of $\mathfrak{se}(3)$. In general, there is a nested sequence of subspaces of the Lie algebra:

$$\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_r = \Delta_\infty$$

where Δ_0 is the subspace spanned by $X_1, \ldots, X_m$ and $\Delta_{i+1} = \Delta_i + [\Delta_0, \Delta_i]$ for $i \geq 0$, terminating in the subalgebra spanned by the twists [20,34]. The sequence is independent of the spanning set of twists for Δ_0, so it depends only on the screw system spanned by them. Moreover the sequence is invariant under the adjoint action. For example, let X_1, X_2, X_3 span a 3–system of Gibson–Hunt type IB_2 [17] with pitch modulus $h_\beta = 0$, so that a spanning set of twists, in Plücker coordinates, is

$$(1,0,0,0,0,0), \quad (0,1,0,0,0,0), \quad (0,0,0,1,0,0),$$

then Δ_1 is a 4-dimensional subspace and $\Delta_2 = \mathfrak{se}(3)$. This would rule out transversality for a manipulator with $k \geq 4$ twists spanning such a screw system at a singularity. However, we already know from the dimensional argument that transversality is impossible. Of course, if Δ_∞ is a proper subalgebra then transversality is precluded but in this case the workspace should be restricted to the corresponding Lie subgroup and a modified version of Theorem 1 will hold.

The subalgebras of $\mathfrak{se}(3)$ are well known and there is none of dimension 5 so that for $k = 6$ and $\mathbf{q} \in \Sigma^1 f$ so that the twists span a 5–dimensional subspace, $\Delta_\infty = \mathfrak{se}(3)$. However this is not sufficient for transversality. For suppose X_1, X_2, X_3 span a 3–dimensional subalgebra $L \subset \mathfrak{se}(3)$, $X_1 + X_4 = 0$ and X_5, X_6 span an independent 2–dimensional subspace. Then the kernel of the derivative is spanned by $\mathbf{c} = (1, 0, 0, 1, 0, 0)$ and from (18):

$$\begin{aligned}
\Upsilon_1.\mathbf{c} &= \Upsilon_6.\mathbf{c} = \mathbf{0} \\
\Upsilon_2.\mathbf{c} &= [X_1, X_2] \in L \\
\Upsilon_3.\mathbf{c} &= [X_1, X_3] \in L \\
\Upsilon_4.\mathbf{c} &= [X_1, X_4] = \mathbf{0} \\
\Upsilon_5.\mathbf{c} &= [X_1 + X_4, X_5] = \mathbf{0},
\end{aligned}$$

so that transversality fails.

5 The geometric Jacobian

The representation of the kinematic mapping as a product of exponentials requires a choice of home position at which the joint variables have value zero and the end-effector is in the identity configuration. As the robot manipulator undergoes motion the twists defining the joints vary, except in respect of a coordinate system fixed in one of the links connected by that joint. Since the geometric Jacobian (12) has the advantage that its columns span a screw system in $\mathfrak{se}(3)$, it is valuable to determine how the geometric Jacobian itself varies as a function of the joint variables.

Given a matrix representation of the Euclidean group $SE(3)$, its tangent space at an element g is itself a vector space of matrices. The analytic Jacobian of the product of exponentials kinematic mapping has as its columns the vectors representing partial derivatives with respect to the joint variables. Recalling that:

$$\frac{d}{ds}\exp(sX) = X \cdot \exp(sX) = \exp(sX) \cdot X, \tag{19}$$

the derivative of a product of exponentials can be written as:

$$\frac{\partial}{\partial q_i} e^{q_i X_i} e^{q_{i+1} X_{i+1}} = e^{q_i X_i} X_i e^{q_{i+1} X_{i+1}} = e^{q_i X_i} e^{q_{i+1} X_{i+1}} \mathrm{Ad}\left(e^{-q_{i+1} X_{i+1}}\right)(X_i) \tag{20}$$

for any $i = 1, \ldots, k-1$. For any Lie group, the adjoint of an exponential can be written in terms of the operator exponential of the induced adjoint action of the Lie algebra:

$$\mathrm{Ad}\left(\exp(qX)\right) = \mathrm{Exp}\left(q\,\mathrm{ad}\,X\right) = \sum_{n=0}^{\infty} \frac{q^n}{n!} (\mathrm{ad}\,X)^n, \tag{21}$$

Here, Exp is used to distinguish the exponential map on $GL(\mathfrak{se}(3))$ from the exponential map on $\mathfrak{se}(3)$ itself. Using (21), the right-hand side of (20) can be written as:

$$e^{q_i X_i} e^{q_{i+1} X_{i+1}} \mathrm{Exp}\,(-q_{i+1}\,\mathrm{ad}\,X_{i+1})\,X_i. \tag{22}$$

By extension, it follows that the ith column of the analytic Jacobian, for $i = 1, \ldots, k$, has the form:

$$f(q_1, \ldots, q_k) \cdot \mathrm{Exp}\,(-q_k\,\mathrm{ad}\,X_k) \cdots \mathrm{Exp}\,(-q_{i+1}\,\mathrm{ad}\,X_{i+1}) X_i \tag{23}$$

To obtain the *(extended) geometric Jacobian*, pull back the tangent vectors at $f(\mathbf{q})$ to the identity via left multiplication by $f(\mathbf{q})^{-1}$. Then the ith column becomes the home joint X_i, operated on by a product of exponentials:

$$\mathrm{Exp}\,(-q_k\,\mathrm{ad}\,X_k) \cdots \mathrm{Exp}\,(-q_{i+1}\,\mathrm{ad}\,X_{i+1}) X_i. \tag{24}$$

This is also known as the *body Jacobian* [28], as (24) is the expression for the ith joint twist expressed in end-effector coordinates. One could similarly use right multiplication to obtain the geometric Jacobian in base coordinates. There is an advantage in working in end-effector coordinates for regional manipulators (the subject of Section 6) since then the coordinates of the wrist centre are fixed.

While the transversality condition in Theorem 1 is purely local, it is now possible to obtain a genericity theorem via a simple transversality argument. This makes use of the elementary transversality theorem (see, for example, Golubitsky and Guillemin [18], Section 2.4) asserting that if a parametrised family of functions is collectively transverse to a closed submanifold, or indeed a Whitney regular family of submanifolds [41], then for almost all parameter values (an open and dense set) the individual functions in the family are transverse.

Let $B_k = (\mathfrak{se}(3) - \{\mathbf{0}\})^k$ denote the set of k-tuples of non-zero twists or *k–multi-twists.*

Theorem 2 *For any $k \geq 1$, there is an open and dense set of k–multi-twists for which the 1–jet extension of the associated product of exponentials is transverse to Σ^s for all $0 \leq s \leq \min\{k, 6\}$.*

Proof Denote by $F : \mathbb{R}^k \times B_k \to SE(3)$ the product of exponentials (1) with joint variables $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$, parametrised by multi-twists $\mathbf{X} = (X_1, \ldots, X_k) \in B_k$. The submanifolds $\Sigma^s \subset J^1(\mathbb{R}^k, SE(3))$ are invariant under left and right translation in $SE(3)$. The parametrised family of 1-jet extensions $\Phi : \mathbb{R}^k \times B_k \to J^1(\mathbb{R}^k, SE(3))$ is therefore transverse to the first-order Thom–Boardman strata if and only if the same is true for the associated map $\Psi : \mathbb{R}^k \times B_k \to \mathbb{R}^k \times B_k \times L(\mathbb{R}^k, \mathfrak{se}(3))$ whose third component is the geometric Jacobian as in (24):

$$(\,\mathrm{Exp}\,(-q_k\,\mathrm{ad}\,X_k) \cdots \mathrm{Exp}\,(-q_2\,\mathrm{ad}\,X_2) X_1 \mid \cdots \mid \mathrm{Exp}\,(-q_k\,\mathrm{ad}\,X_k) X_{k-1} \mid X_k\,)\,. \tag{25}$$

Since the tangent space to each of the submanifolds Σ^s projects onto the base $\mathbb{R}^k \times SE(3)$ of the jet bundle, it is only necessary to ensure the transversality condition holds after projection to the fibre $L(\mathbb{R}^k, \mathfrak{se}(3))$. Since each exponential $\mathrm{Exp}\,(-q_j\,\mathrm{ad}\,X_j)$, $j = 2, \ldots, k$ is an isomorphism, it is clear, by considering only the parameter space B_k, that Ψ projects submersively on the fibre and so is automatically transverse to any submanifold. The theorem follows.

It remains an open question as to whether the manipulators having k revolute joints that are transverse to the first-order Thom–Boardman strata form an open and dense set, though it is conjectured to hold at least for $k \leq 9$.

5.1 Putzer's method

Given the matrix form of the adjoint (7), the following theorem of Putzer [32] enables the variable twists in the geometric Jacobian to be written in a closed form. This is particularly helpful for computational purposes and enables images of singular sets to be obtained readily, at least in the case of the kinematic mapping of a regional manipulator, described at the end of Section 3.

Theorem 3 (Putzer's Method) *If A is a square matrix with minimal polynomial $x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$, then*

$$e^{tA} = \sum_{i=0}^{m-1} b_i(t) A^i \tag{26}$$

where the functions $b_i(t)$, $i = 0, \ldots, m-1$, are the solutions of the differential equation:

$$\sum_{r=0}^{m} a_r b_i^{(r)} = 0, \tag{27}$$

satisfying the initial conditions $b_i^{(j)} = \delta_{ij}$, $j = 0, \ldots, m-1$.

Selig [34] shows that if $X = (\boldsymbol{\omega}, \mathbf{v}) \in \mathfrak{se}(3)$ has finite pitch then the characteristic polynomial for ad X is $x^5 + 2r^2x^3 + r^4x$, where $r = \|\boldsymbol{\omega}\|$. Since we may take any non-zero multiple of a twist X, it is possible to choose $r = 1$. It follows, in this case, that the functions arising in Putzer's method are:

$$\begin{aligned} b_0(t) &= 1 \\ b_1(t) &= \tfrac{1}{2}(3\sin t - t\cos t) \\ b_2(t) &= \tfrac{1}{2}(4 - 4\cos t - t\sin t) \\ b_3(t) &= \tfrac{1}{2}(\sin t - t\cos t) \\ b_4(t) &= \tfrac{1}{2}(2 - 2\cos t - t\sin t). \end{aligned} \tag{28}$$

If the pitch of X is infinite, so that $\boldsymbol{\omega} = 0$, then the minimal polynomial of ad X is simply x^2 and

$$\operatorname{Exp}(t\,\operatorname{ad} X) = I + t\,\operatorname{ad} X, \tag{29}$$

that is, $b_0(t) = 1$, $b_1(t) = t$. Repeated application of Putzer's theorem leads to the following form for the ith column of the geometric Jacobian (24) in the case that there are no prismatic joints:

$$\sum_{r_{i+1},\ldots,r_k=0}^{4} b_{r_{i+1}}(-q_{i+1}) \cdots b_{r_k}(-q_k) [X_k^{r_k} \cdots X_{i+1}^{r_{i+1}} X_i] \tag{30}$$

where $[X_k^{r_k} \cdots X_{i+1}^{r_{i+1}} X_i]$ denotes the nested Lie bracket:

$$\underbrace{[X_k, [\cdots, [X_k}_{r_k}, \cdots \underbrace{[X_{i+2}, [\cdots, [X_{i+2}}_{r_{i+2}}, \underbrace{[X_{i+1}, [\cdots, [X_{i+1}}_{r_{i+1}}, X_i] \cdots].$$

With the geometric Jacobian in this form it is possible, at least in principle, to determine the complete singularity locus of the kinematic mapping. Note that none of the columns of the body Jacobian depend on the first joint variable. It follows that the singularity locus is also independent of q_1 when viewed in end-effector coordinates (or q_k in base coordinates). In the next section this form of the geometric Jacobian is exploited for the analysis of regional manipulators.

6 Regional manipulators and trajectory singularities

The singularity loci of regional manipulators having three revolute (3R) joints have been much studied since the late 1980s [5,21,29,35,36,43,44]. The approach followed in most of the cited work involves solving the inverse kinematics problem. The connected components of the complement of the singular image are regions in which the position $(x, y, z) \in \mathbb{R}^3$ of the wrist centre has a fixed number of pre-images in the joint space, usually referred to as *poses* of the manipulator. In this view, the singular image is the bifurcation set for the inverse kinematics problem.

In the 3R case, exploiting the rotational symmetry of the singular image with respect to the axis of the first joint, it is possible to eliminate joint variables so that the inverse kinematics are determined by a polynomial equation (see, for example, [1] Section 4.4):

$$F(t, \rho, z; d_1, \ldots, d_k) = 0, \tag{31}$$

where $t = \tan(q_3/2)$, $\rho = x^2 + y^2$ and $d_1, \ldots, d_k$ are Denavit–Hartenberg parameters. The equation is quartic in t, so that there are up to four poses for a given location of the wrist centre. Thus, the singular image can be represented by the apparent contour of the projection onto the ρz-plane of the surface in $t\rho z$-space defined by (31), that is, the set of (ρ, z) for which $F = \partial F/\partial t = 0$ for some t. The question arises as to whether it is possible to construct a path in the joint space, which does not intersect the singular locus, connecting two distinct poses. Wenger and El Omri [45] show that this is possible precisely when the inverse kinematics has a location with three poses, corresponding to a cusp point of the projection (where $\partial^2 F/\partial t^2 = 0$ as well). Recent work [40] identifies regional manipulators exhibiting beaks, lips and swallowtail bifurcations in the family F, for specific design parameters.

Here, however, a more general approach is followed that encompasses manipulators with non-revolute joints, using the product of exponentials approach. The kinematic mapping (11) for a regional manipulator's wrist centre can be thought of as the composition $f_{\mathbf{c}} = \epsilon_{\mathbf{c}} \circ f$, where $\epsilon_{\mathbf{c}} : SE(3) \to \mathbb{R}^3$ is the evaluation map $g \mapsto g \cdot \mathbf{c}$. Applying the chain rule, $Jf_{\mathbf{c}}(\mathbf{q}) = J\epsilon_{\mathbf{c}}(f(\mathbf{q})) \circ Jf(\mathbf{q})$. It follows that the rank of the Jacobian $Jf_{\mathbf{c}}(\mathbf{q})$ is less than 3, so that $f_{\mathbf{c}}$ has a singularity at $\mathbf{q}$, if and only if one of the following occurs:

i. rank $Jf(\mathbf{q}) < 3$, i.e. f itself has a singularity at $\mathbf{q}$, or
ii. the kernel $J\epsilon_{\mathbf{c}}(f(\mathbf{q}))$ has non-trivial intersection with the image of $Jf(\mathbf{q})$.

In case (i), the joint twists themselves are linearly dependent in the Lie algebra. It follows from Theorems 1 and 2 that this does not occur generically. In case (ii), the kernel of the derivative of $\epsilon_{\mathbf{c}}$ at the identity is precisely the 3-dimensional subspace of pitch-zero twists (lines or revolute joints) whose axes pass through $\mathbf{c}$, as these are the twists whose Killing field fixes $\mathbf{c}$. This leads to strong geometric conditions on the form of the singularity locus of wrist centres in a given configuration, in terms of the screw system spanned by the joint twists [6]. Here, we pursue the other point of view: for a fixed wrist centre determine the singularity locus over the joint space, and how it varies in terms of design parameters.

The Jacobian of the evaluation map $J\epsilon_{\mathbf{c}}$, with respect to Plücker coordinates on $\mathfrak{se}(3)$, has the form $(\, \Gamma \mid I \,)$, where Γ is the skew-symmetric matrix corresponding to $-\mathbf{c}$, as in (5). The columns of $Jf(\mathbf{q})$ are the joint twists $X_i = (\boldsymbol{\omega}_i, \mathbf{v}_i)$, $i = 1, 2, 3$. Thus the composition is the matrix:

$$\left(X_1.\mathbf{c} \middle| X_2.\mathbf{c} \middle| X_3.\mathbf{c} \right), \tag{32}$$

where the action of $X = (\boldsymbol{\omega}, \mathbf{v}) \in \mathfrak{se}(3)$ on $\mathbf{c} \in \mathbb{R}^3$ is $X.\mathbf{c} = \boldsymbol{\omega} \times \mathbf{c} + \mathbf{v}$. This form extends in the obvious way to regional manipulators with k joints. Moreover, taking into account the effect of the manipulator's internal motion on the joint twists, the geometric Jacobian for the regional manipulator becomes:

$$\left(\textstyle\sum_{i,j=0}^{4} b_i(-q_2) b_j(-q_3) [X_3^j X_2^i X_1].\mathbf{c} \middle| \textstyle\sum_{k=0}^{4} b_k(-q_3)[X_3^k X_2].\mathbf{c} \middle| X_3.\mathbf{c} \right). \tag{33}$$

The singular locus is simply given by the vanishing of the determinant of this Jacobian and can be written in the form:

$$g(q_2, q_3) = \sum_{i,j,k=0}^{4} \gamma_{\mathbf{X},\mathbf{c}}^{ijk} \, b_i(-q_2) b_j(-q_3) b_k(-q_3) = 0 \tag{34}$$

where the coefficients $\gamma_{\mathbf{X},\mathbf{c}}^{ijk}$ are determinants of matrices independent of the joint variables. Note however that the functions $b_i(-q_j)$ are only independent of the twist X_i when the rotational part $\boldsymbol{\omega}_i$ of the twist is chosen as a unit vector. Hence, the expression (34) only varies continuously with the choice of joints so long as they have finite pitch, even though the twists of infinite pitch lie in the closure of the pitch quadrics $Q_h(X) = 0$ for all finite h.

Example 1. The 3R regional manipulator with

$$X_1 = (0,0,1,0,0,0), \quad X_2 = \left(1,0,0,0,-1,\tfrac{1}{2}\right), \quad X_3 = \left(0,0,1,-\tfrac{3}{2},1,0\right)$$

and wrist centre at $\mathbf{c} = (0,0,t)$ has its joint axes successively orthogonal, this angle being invariant under the motion of the manipulator. The singularity loci of such *orthogonal manipulators* has been studied in depth by Baili *et al* [2]. The formula (34) yields this equation for the singularity locus:

$$\begin{aligned} &-4\sin q_2(\cos q_3 + 5\sin q_3)t + 14\cos q_2 \cos^2 q_3 - 34\cos q_2 \sin q_3 \cos q_3 \\ &\quad + 20\cos q_2 \sin q_3 + 4\cos q_2 \cos q_3 + 4\sin q_2 \cos q_3 + 20\sin q_2 \sin q_3 \\ &\quad + 12\cos^2 q_3 - 5\sin q_3 \cos q_3 - 20\cos q_2 - 4\cos q_3 + 6\sin q_3 = 6 \end{aligned} \tag{35}$$

The curve has period 2π in both q_2 and q_3. Figure 3(a) shows an animation of the singularity locus as the wrist centre varies from 0.8 to 1.2. In this interval there are 3 values of t at which the 1-jet of the kinematic mapping fails to be transverse to the singularity stratum Σ^1 so that the singular locus is not a smooth manifold.

Fig. 3 Singular loci for (a) an orthogonal 3R and (b) a non-orthogonal 3R manipulator, with varying wrist centres.

Example 2. The 3R regional manipulator with

$$X_1 = (0,0,1,0,0,0,0), \quad X_2 = \left(\tfrac{4}{13}, \tfrac{12}{13}, \tfrac{3}{13}, 1, 0, -2\right), \quad X_3 = \left(0,0,1,0,0,\tfrac{1}{2},0\right)$$

and wrist centre at $\mathbf{c} = (1,1,t)$. The singular locus is illustrated in Figure 3(b) for t from 1.5 to 2.5. In most examples, it seems that bifurcation through non-generic manipulators is via a crunode or a degeneration such as a tacnode. However this example shows that the locus may also encounter an acnode bifurcation.

Fig. 4 Singular loci for a RHR manipulator, with varying pitch for joint 2.

Example 3. While singularity loci of 3R manipulators have been well studied, the formula (34) is effective for manipulators with non-revolute joints. In this example, the effect of altering the pitch of the middle joint in a RHR manipulator is illustrated.

$$X_1 = (0,0,1,0,0,0)\,, \quad X_2 = \left(1,0,0,h,-1,\tfrac{1}{2}\right)\,, \quad X_3 = (0,0,1,-2,1,0)$$

and wrist centre at $\mathbf{c} = (0,0,1)$. The singular locus is illustrated in Figure 4 for h from -1 to 1.

To locate $\Sigma^{1,1}$ singularities (cusps) in the case that $f_{\mathbf{c}}$ is transverse to Σ^1, the singularities of the restriction of the kinematic mapping to the singular locus can be found by the method used for constrained optimisation (Lagrange multipliers). One requires the rank of the 4×3 matrix, given by the Jacobian and extended by the row of partial derivatives of its determinant, to have corank 1 (given that the Jacobian itself already has corank 1). This codimension 2 condition is typically satisfied at isolated points. Figure 5 shows a typical configuration of 8 cusp points ($\diamond$) on the singular locus of the 3R manipulator in Example 1 with wrist centre $\mathbf{c} = (1,0,1)$. In fact, each of these corresponds to a circular cuspidal edge in the image of the singular set, parametrised by the joint variable q_1.

Fig. 5 Cusp points ($\diamond$) on the singular locus of a 3R manipulator.

7 Conclusion

The extensive literature on robot manipulator singularities demonstrates that they constitute a field of genuine interest for engineers. Much of the research concentrates on the special geometry of specific manipulators: those with revolute and prismatic joints, wrist-partitioned 6–joint serial manipulator, or orthogonal 3R regional manipulators. In many cases, these enable effective computation of the kinematics where it would, in general, be overwhelming or impossible. Here, an approach has been presented that places the kinematics in the context of the Euclidean Lie group, employing the product-of-exponentials approach that enables the use of classical results of Lie theory such as the Baker–Campbell–Hausdorff formula. As a result, the relation of singularities to the structure of the group and its Lie algebra becomes clear.

Tension remains between the natural questions that arise when one takes a singularity-theoretic approach and those that concern engineers. Whereas genericity theorems are important mathematically, it is frequently the special or non-generic geometry that provides both useful manipulators and computable control. Nevertheless, it is to be hoped that, by developing a sufficiently general approach, common ground can be forged which will provide both mathematically interesting results and practical insight into singularities and their role in manipulator design.

References

1. Angeles, J.: Fundamentals of Robotic Mechanical Systems. Springer, New York (1997)
2. Baili, M., Wenger, P., Chablat, D.: A classification of 3R orthogonal manipulators by the topology of their workspace. In Proc. IEEE Int. Conf. Robot. Autom., pp. 1933–1938. IEEE, Piscataway NY (2004)
3. Ball, R. S.: A Treatise on the Theory of Screws. Cambridge University Press, Cambridge (1998)
4. Brockett, R.: Robotic manipulators and the product of exponentials formula. In Fuhrman, P. (ed.) Proc. Math. Theory Netw. Syst., Beer-Sheva, Israel pp. 120–129. Springer, Berlin/Heidelberg (1984)
5. Burdick, J. W.: A Classification of 3R regional manipulator singularities and geometries. Mech. Mach. Theory **30**, 71–89 (1995)
6. Cocke, M. W., Donelan, P. S., Gibson, C. G.: Instantaneous singular sets associated to spatial motions. In Real and Complex Singularities, São Carlos, 1998, pp. 147–163. Chapman and Hall/CRC Press, Boca Raton FL (2000)
7. Denavit, J., Hartenberg, R. S.: A kinematic notation for lower pair mechanisms based on matrices. J. Appl. Mech. **22**, 215–221 (1955)
8. Donelan, P., Azzato, J.: Singularities in Robot Kinematics - A Publications Database, http://homepages.ecs.vuw.ac.nz/ donelan/cgi-bin/rs/main (2013). Accessed 9 June 2013
9. Donelan, P.: Singularity-theoretic methods in robot kinematics. Robotica **25**, 641–659 (2007)
10. Donelan, P. S., Gibson, C. G.: First–order invariants of Euclidean motions. Acta Appl. Math. **4**, 233–251 (1991)
11. Donelan, P. S., Gibson, C. G.: On the hierarchy of screw systems. Acta Appl. Math. **32**, 267–296 (1993)
12. Faraut, J.: Analysis on Lie Groups. Cambridge University Press, Cambridge (2008)
13. Gibson, C. G.: Kinematic singularities—a new mathematical tool. In Proc. 3rd Int. Workshop on Adv. Robot Kinemat., Ferrara, Italy, pp. 209–215 (1992)
14. Gibson, C. G., Hobbs, C. A.: Local models for general one–parameter motions of the plane and space. Proc. Royal Soc. Edinb., Section A: Math. **125**, 639–656 (1995)
15. Gibson, C. G., Hobbs, C. A.: Singularity and bifurcation for general two–dimensional planar motions. N. Z. J. Math. **25**, 141–163 (1996)
16. Gibson, C. G., Hobbs, C. A., Marar, W. L.: On versal unfoldings of singularities for general two–dimensional spatial motions. Acta Appl. Math. **47**, 221–242 (1996)
17. Gibson, C. G., Hunt, K. H.: Geometry of screw systems I & II. Mech. Mach. Theory **25** 1–27 (1990)
18. Golubitsky, M., Guillemin, V.: Stable Mappings and Their Singularities. Springer Verlag, New York (1973)
19. Gosselin, C., Angeles, J.: Singularity analysis of closed–loop kinematic chains. IEEE Trans. Robot. Autom. **6**, 281–290 (1990)
20. Hao, K.: Dual number method, rank of a screw system and generation of Lie subalgebras. Mech. Mach. Theory **33**, 1063–1084 (1998)
21. Hsu, M.-S., Kohli, D.: Boundary surfaces and accessibility regions for regional structures of manipulators. Mech. Mach. Theory **22**, 277–289 (1987)
22. Hunt, K. H.: Kinematic Geometry of Mechanisms. Clarendon Press, Oxford (1978)
23. Karger, A.: Singularity analysis of serial robot–manipulators. J. Mech. Des. **118**, 520–525 (1996)
24. Lerbet , J., Hao, K. : Kinematics of mechanisms to the second order—application to the closed mechanisms. Acta Appl. Math. **59**, 1–19 (1999)
25. Merlet, J. P.: Singular configurations of parallel manipulators and Grassmann geometry. Int. J. Robot. Res. **8**, 45–56 (1989)
26. Müller, A.: A conservative elimination procedure for permanently redundant closure constraints in MBS-models with relative coordinates. Multibody Syst. Dyn. **16**, 309–330 (2006)
27. Müller, A.: Generic mobility of rigid body mechanisms. Mech. Mach. Theory **44**, 1240–1255 (2009)
28. Murray, R. M., Li, Z., Shastry, S. S.: A Mathematical Introduction to Robotic Manipulation. CRC Press, Boca Raton (1994)
29. Ottaviano, E., Ceccarelli, M., Husty, M.: Workspace topologies of industrial 3R manipulators. Int. J. Adv. Robot. Syst. **4**, 355–364 (2008)
30. Pai, D. K., Leu, M. C.: Genericity and singularities of robot manipulators. IEEE Trans. Robot. Autom. **8**, 545–559 (1992)
31. Pieper, D., Roth, B.: The kinematics of manipulators under computer control. In Proc. 2nd World Cong. Theory Mach. Mech., Zakopane, Poland, **2**, pp. 159–169 (1969)
32. Putzer, E. J.: Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients. Amer. Math. Mon. **73**, 2–7 (1966)
33. Sciavicco, L., Siciliano, B.: Modelling and Control of Robot Manipulators. Springer, London (2000)
34. Selig, J.: Geometric Fundamentals of Robotics. Springer, New York (2005)

35. Smith, D. R., Lipkin, H.: Analysis of fourth order manipulator kinematics using conic sections. In Proc. Int. Conf. Robot. Autom., Cincinnati, OH, 1990, pp. 274–278. IEEE, Piscataway NY (1990)
36. Stanišić, M. M., Engelberth, J. W.: A geometric description of manipulator singularities in terms of singular surfaces. In Proc. 1st Int. Workshop Adv. Robot Kinemat., Ljubljana, Slovenia, pp. 132–141 (1988)
37. Sugimoto, K., Duffy, J., Hunt, K. H.: Special configurations of spatial mechanisms and robot arms. Mech. Mach. Theory **17**, 119–132 (1982)
38. Tchoń, K.: Differential topology of the inverse kinematic problem for redundant robot manipulators. Int. J. Robot. Res. **10**, 492–504 (1991)
39. Tchoń, K., Muszynski, R.: Singularities of nonredundant robot kinematics. Int. J. Robot. Res. **16**, 71–89 (1997)
40. Thomas, F., Wenger, P.: On the topological characterization of robot singularity loci. A catastrophe-theoretic approach. In Proc. Int. Conf. Roboti. Autom., Shanghai, China, pp. 3940–3945. IEEE, Piscataway NY (2011)
41. Wall, C. T. C.: Regular stratifications. In Manning, A. (ed.) Dynamical Systems, Warwick, 1974. Lecture Notes in Math. **468**, pp. 332–344. Springer, New York (1975)
42. Wang, S. L., Waldron, K. J.: A study of the singular configurations of serial manipulators. J. Mech., Transm. Autom. Des. **109**, 14–20 (1987)
43. Wenger, P.: Classification of 3R positioning manipulators. J. Mech.l Des. **120**, 327–332 (1998)
44. Wenger, P.: Cuspidal and noncuspidal robot manipulators. Robotica **25**, 677–689 (2007)
45. Wenger, P., El Omri, J.: Changing posture for cuspidal robot manipulators. In Proc. IEEE Int. Conf. Robot. Autom., Minneapolis MN, pp. 3173–3178. IEEE, Piscataway NY (1993)
46. Whitney, D. E.: Resolved motion rate control of manipulators and human prostheses. IEEE Trans. Man–Mach. Sys. **10**, 47–53 (1969)