Singularities of Regional Manipulators Revisited

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Abstract. The workspace singularities of 3R regional manipulators have been much analyzed. The presence of cusps in the singularity locus is known to admit singularity-avoiding posture change. Cusps arise in singularity theory as second-order phenomena—specifically they are $\Sigma^{1,1}$ Thom–Boardman singularities. The occurrence of such singularities requires that the kinematic mapping be generic (in the sense of Pai and Leu [1]). Genericity and the occurrence of higher-order singularities in families of regional manipulators are investigated using Lie-theoretic properties of the Euclidean group.

Key words: regional manipulator, cusp, Thom–Boardman singularity, genericity

1 Introduction

A spatial serial 3-link manipulator is frequently termed a *regional manipulator* in recognition of its use as the positioning component for the wrist-centre of a wrist-partitioned 6-dof industrial manipulator. An example is illustrated in Figure 1. The importance of such a design is that the inverse kinematics reduces to solving a degree 4 polynomial [2]. Hsu and Kohli [3] used this to show that there are, for a typical regional manipulator, surfaces in the joint space that divide it into regions corresponding to different numbers of poses. These regions were also studied, in a more general setting, by Burdick [4]. Taking a different perspective, Stanišić and Engelberth [5] demonstrated, using screw systems, that the wrist-positioning subassembly gives rise to a singularity of the whole manipulator when the wrist centre lies on a certain surface, dependent on the subassembly configuration. This was referred to as an *instantaneous singular set* in [6] and was subsequently used as the basis for a singularity metric [7].

A number of researchers have used ideas from the mathematical theory of singularities in the study of manipulators [1, 8, 9]. Pai and Leu examined the stratification of the singularity locus by the *corank* of the singularity, that is, the instantaneous loss of degrees of freedom (dofs). In particular, they introduced the concept of *generic* manipulator, to describe one whose kinematic mapping has a nice singularity locus.

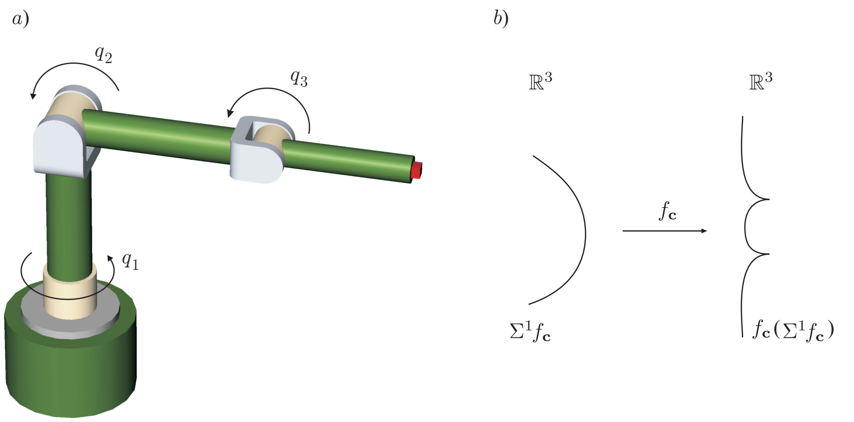

Fig. 1 a) Ortho-parallel regional manipulator [10], b) Visualisation of cusp singularities.

In singularity theory, the term 'generic' is used to describe a property that pertains for a topologically large set (for example, open and dense, residual, or having complement of measure zero) in a given parametrised family of mappings. In this setting, the family could be the set of 3R manipulators, or it could be the set of wrist-centres for a given manipulator, or the entire family with both the serial structure and wrist-centre as parameters. Generic properties are typically realised via *transversality* to a given family of manifolds—an intersection condition satisfied when certain vectors span a given space. In this case, the given family of manifolds can be interpreted as the sets Σ^r of Jacobian matrices of fixed corank r.

In much of the literature on this subject, the term 'generic' is used to describe a manipulator for which the transversality condition holds rather than the property itself. A more accurate term is *transverse-regular* [11]. There is, as pointed out by Tchoń [8], no certainty that the transversality condition will indeed hold for most manipulators in a given class. Regardless of whether the condition holds for most manipulators, whenever it does hold it guarantees that the singular loci $\Sigma^r f$, corresponding to singularities of fixed rank, are themselves manifolds in the joint space of the kinematic mapping f. On the other hand, the singular image in the workspace can exhibit singularities, such as cusps. For further details of genericity and concepts of singularity theory, see [11, 12, 13, 14, 15].

In the context of regional manipulators, algebraic conditions for genericity (i.e. transversality) were obtained in [1, 9]. In particular, they showed that it is not possible to encounter Σ^2 transversely, so that rank 1 singularities are ruled out. Burdick [16] gave an alternative geometric criterion to the algebraic equations of [3]: when a 3R regional manipulator is in a singular configuration there exists a screw of pitch zero whose axis passes through the wrist centre and intersects the axes of each joint screw. He observed that for an open set of 3R regional manipulators, there exist trajectories in joint space that do not intersect the singular locus, yet effect a change of posture. Such manipulators have been termed *cuspidal* and have been explored in detail by Wenger et al. [17, 18, 19]. Smith and Lipkin [20, 21] showed that the inverse kinematics of a given wrist-centre for a 3R regional manipulator can be encoded by a pencil of conics. Exceptional pencils in which the conics possess some tangency correspond to singular configurations, while high-order tangency (3rd or 4th order or paired) correspond to higher-order singularities, including cusps. Recent classifications focus on specific classes of manipulator, for example orthogonal [22], where a closed form expression for the Jacobian can be found, and on workspace topologies [10].

Selig [23] analysed the kinematics and singularities of 3R manipulators using product-of-exponentials formulation for the kinematics and results of Lie theory. It is this approach that we pursue. A cusp arises as a singular point of the restriction of the kinematic mapping to the singular locus: in the notation of Thom–Boardman singularities [24] it is $\Sigma^{1,1}$. Our aim is to develop the singularity analysis of regional manipulators in a reasonably broad context, provisionally allowing for 1-dof joints of any sort, and deriving local descriptions of singular loci using methods of Lie groups and Lie algebras. In this setting the two different aspects of the singularity problem for regional manipulators—choice of the underlying serial manipulator structure and of the wrist centre—can be developed together.

2 The kinematic mapping

The motion associated with each 1-dof joint of a manipulator can be represented by a non-zero twist X—an element of the Lie algebra $\mathfrak{se}(3)$ of the group $SE(3)$ of Euclidean isometries. The motion itself is given by the exponential $\exp(qX)$, a path in the group of transformations, where q is the joint variable. The twist is relative to a given choice of coordinates in the link and the ambient space; under a change of coordinates represented by an isometry $g \in SE(3)$, the twist transforms by conjugacy and this is the *adjoint* action of the Lie group on its Lie algebra:

$$X \mapsto \mathrm{Ad}\,(g)(X) = gXg^{-1}, \tag{1}$$

where the elements of the group and twists can be written in matrix form. In a given coordinate frame, the twist may be replaced by any non-zero multiple, a twist of the same pitch, the joint variable being scaled by the inverse of the multiple. In other words the joint is really represented by a *screw*. The kinematic mapping of a serial manipulator with k 1-dof joints can then be written as a product of exponentials

$$f(q_1,\ldots,q_k) = \exp(q_1X_1)\cdots\exp(q_kX_k), \tag{2}$$

where X_i, $i = 1,\ldots,k$, is the twist representing the ith joint, in a chosen home configuration with respect to given space (or base) coordinates, and $q_i \in \mathbb{R}$ is the joint variable. Again, exponentials can be evaluated as matrices via the standard series formula. The image of each exponential map is the one-parameter subgroup of Euclidean transformations in $SE(3)$, parametrised by the joint variable. The twists X_i

can be equivalently represented by either: a 6-vector $(\boldsymbol{\omega}_i, \mathbf{v}_i)$ comprised of two 3-vectors corresponding to infinitesimal rotation and translation, or a 4×4 matrix partitioned as

$$\left(\begin{array}{c|c} \widetilde{\omega} & \mathbf{v} \\ \hline \mathbf{0}^t & 0 \end{array}\right), \tag{3}$$

where the identification of $\boldsymbol{\omega} \in \mathbb{R}^3$ and the 3 skew-symmetric matrix $\widetilde{\omega}$ proceeds with

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \leftrightarrow \widetilde{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{4}$$

If the joint X_i is revolute then $\boldsymbol{\omega}_i \cdot \mathbf{v}_i = 0$, while if it is prismatic then $\boldsymbol{\omega}_i = \mathbf{0}$. A priori, there is no need to assume that the joints are either of these types, that is, they may have pitch $\boldsymbol{\omega}_i \cdot \mathbf{v}_i / \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_i \neq 0, \infty$. While the exponential map is defined independent of the representation used for the Lie algebra, in the matrix form it can be computed by the usual exponential series.

For a regional manipulator $k = 3$, and there is a choice of wrist-centre $\mathbf{c} \in \mathbb{R}^3$ (in body coordinates for the third link). The kinematic mapping for the wrist centre is the function

$$f_{\mathbf{c}} : \mathbb{R}^3 \to \mathbb{R}^3; \quad f_{\mathbf{c}}(q_1, q_2, q_3) = \exp(q_1 X_1) \exp(q_2 X_2) \exp(q_3 X_3) \cdot \mathbf{c}. \tag{5}$$

The 'evaluation' map $\varepsilon_{\mathbf{c}} : SE(3) \to \mathbb{R}^3$ is given by the action of the group on the wrist-centre $\mathbf{c}$, that is for $g \in SE(3)$, $\varepsilon_{\mathbf{c}}(g) = g \cdot \mathbf{c}$. Then $f_{\mathbf{c}}$ is the composition of $\varepsilon_{\mathbf{c}}$ with the manipulator kinematic mapping f in (2).

3 Jacobian Matrices

A kinematic mapping f has a singularity at $\mathbf{q}$ when the rank of its derivative $Df(\mathbf{q})$ drops below its maximum possible value, which is the minimum of the dimensions of the joint-space and the configuration space. The derivative is represented by the (analytic) Jacobian matrix of partial derivatives. This represents a linear mapping into the tangent space at the image $f(\mathbf{q}) \in SE(3)$ rather than into the Lie algebra, so that the columns are not themselves, in general, twists. However the group structure can be used to 'pull back' the tangent space to the identity by either left multiplication (corresponding to body coordinates) or right translation (space coordinates) to give a more familiar geometric Jacobian. For the manipulator mapping f in (2) with $k = 3$ the matrix is therefore 6×3. To find an explicit form requires the derivative of an exponential:

$$\frac{d}{dq} \exp(qX) = X \cdot \exp(qX) = \exp(qX) \cdot X, \tag{6}$$

where the operations between the transformation $\exp(qX)$ and the twist X can be realised by matrix multiplication. Following [23], if $g \in SE(3)$ can be written as $g = (R, \mathbf{t}) \in SO(3) \times_s \mathbb{R}^3$ (where $\times_s$ denotes semi-direct product), then g is represented by the 6×6 partitioned matrix

$$\left(\begin{array}{c|c} R & O \\ \hline \widetilde{t}R & R \end{array}\right) \tag{7}$$

with the skew-symmetric matrix $\widetilde{t}$ defined in (4). Note that the exponential mapping commutes with its defining twist. However, it does not commute with a general twist and we require:

$$Y \exp(qX) = \exp(qX) \exp(-qX) Y \exp(qX) = \exp(qX) \mathrm{Ad}\,(\exp(-qX))(Y). \tag{8}$$

Differentiating Ad as in (1) with respect to $g \in SE(3)$ gives the adjoint representation of the Lie algebra $\mathfrak{se}(3)$ on itself. This also determines the *Lie bracket* operation in the Lie algebra:

$$\mathrm{ad}\,(Y)(X) = [Y, X] \tag{9}$$

In matrix terms $[Y, X] = YX - XY$, while in screw coordinates

$$[(\boldsymbol{\omega}_1, \mathbf{v}_1), (\boldsymbol{\omega}_2, \mathbf{v}_2)] = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2). \tag{10}$$

It is a theorem of matrix Lie groups that

$$\mathrm{Ad}\,(\exp(qX)) = \mathrm{Exp}(q\,\mathrm{ad}\,(X)) = \sum_{n=0}^{\infty} \frac{q^n}{n!} (\,\mathrm{ad}\,X)^n, \tag{11}$$

where the exponential Exp is an operator on the Lie algebra. We obtain the analytic Jacobian of f as follows (where vertical dots separate column vectors):

$$Jf(\mathbf{q}) = \Big(\exp(q_1X_1)X_1 \exp(q_2X_2)\exp(q_3X_3) \vdots \exp(q_1X_1)\exp(q_2X_2)X_2\exp(q_3X_3) \vdots \\ \exp(q_1X_1)\exp(q_2X_2)\exp(q_3X_3)X_3\Big) \tag{12a}$$

$$= \Big(f(q_1,q_2,q_3) \cdot \mathrm{Exp}(-q_3\,\mathrm{ad}\,X_3)\mathrm{Exp}(-q_2\,\mathrm{ad}\,X_2)X_1 \vdots \\ f(q_1,q_2,q_3) \cdot \mathrm{Exp}(-q_3\,\mathrm{ad}\,X_3)X_2 \vdots f(q_1,q_2,q_3) \cdot X_3\Big). \tag{12b}$$

The second expression is obtained by applying (8) and the effect of $f(q_1,q_2,q_3)$ on each term is to translate the twists in the tangent space at the identity (the Lie algebra $se(3)$) to the tangent space at the given configuration. The corresponding geometric Jacobian, consisting of the instantaneous joint screws in end-effector coordinates, is therefore:

$$J_{\mathrm{geom}} = \left(X_1' \vdots X_2' \vdots X_3'\right) \tag{13}$$

where $X_1' := \mathrm{Exp}(-q_3\,\mathrm{ad}\,X_3)\mathrm{Exp}(-q_2\,\mathrm{ad}\,X_2)X_1$, $X_2' := \mathrm{Exp}(-q_3\,\mathrm{ad}\,X_3)X_2$, $X_3' := X_3$.

An important object for establishing transversality of the kinematic mapping at a given configuration is the Lie subalgebra generated by the joint screws X_i since it contains the the subspace spanned by X_i at any configuration. Assuming that we are interested in the configuration $\mathbf{q} = \mathbf{0}$ then (12b) reduces to (13). Expanding (13) as a series in q_1, q_2, q_3 by means of (11) gives:

$$\begin{aligned} X_1' &= X_1 + q_2[X_1, X_2] + q_3[X_1, X_3] + O(2) \\ &= (\boldsymbol{\omega}_1, \mathbf{v}_1) + q_2(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \mathbf{r}_{12}) + q_3(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_3, \mathbf{r}_{13}) + O(2) \end{aligned} \tag{14a}$$

$$\begin{aligned} X_2' &= X_2 + q_3[X_2, X_3] + O(2) \\ &= (\boldsymbol{\omega}_2, \mathbf{v}_2) + q_3(\boldsymbol{\omega}_2 \times \boldsymbol{\omega}_3, \mathbf{r}_{23}) \end{aligned} \tag{14b}$$

$$X_3' = X_3 = (\boldsymbol{\omega}_3, \mathbf{v}_3), \tag{14c}$$

where $\mathbf{r}_{ij} = \boldsymbol{\omega}_i \times \mathbf{v}_j + \mathbf{v}_i \times \boldsymbol{\omega}_j$.

4 The Singular Locus

For the wrist-centre kinematic mapping $f_\mathbf{c}$ in (5), the Jacobian is 3×3. Its columns are the result of applying the columns of (12b), considered as elements of $SE(3)$, to $\mathbf{c}$. However, recalling that $f_\mathbf{c} = \varepsilon_\mathbf{c} \circ f = f \cdot \mathbf{c}$, and applying the chain rule to get $Df_\mathbf{c} = D\varepsilon_\mathbf{c} \circ Df$, it is clear that the rank of the derivative of $f_\mathbf{c}$ is less than 3, i.e. $f_\mathbf{c}$ has a singularity, if and only if one of the following occurs [6]:

i. f itself has a singularity;
ii. the kernel of the derivative of the evaluation map $\varepsilon_\mathbf{c}$ has non-trivial intersection with the image of the derivative of f.

Case (i) corresponds to Burdick's 'extra branch singularities' [16]. Here, the defining screws X_1, X_2, X_3 are linearly dependent in the Lie algebra. We assume none are zero and no two adjacent joint screws are permanently linearly dependent (since then the manipulator effectively only has 2 dof). In particular the screws must span a 2-dimensional subspace so correspond to a Σ^1 singularity. It follows from Theorem 3.1 in [13] that the singularity occurs transversely so long as the the subspace is not a subalgebra (i.e. closed under the Lie bracket), in which case that manipulator would have 2 dof only. The 2-dimensional subalgebras are the algebra of cylindrical motion and pure 2 dof translations.

In case (ii), the kernel of the derivative of $\varepsilon_\mathbf{c}$ at the identity is precisely the set of pitch-zero twists whose axes pass through $\mathbf{c}$: in the terminology of the Klein quadric, this is an α-plane. This provides the principle for determining the instantaneous singular sets in terms of the screw system in [5, 6]: a singularity can be detected when the determinant of the 6×6 matrix, whose columns are 3 twists spanning the α-plane and 3 from the Jacobian of f, vanishes.

This can be exploited by choosing coordinates in the end-effector so that $\mathbf{c}$ is the origin. Then the twists spanning the α-plane can be taken as infinitesimal rotations about the coordinate axes and the Jacobian has the partitioned form

$$\left(\begin{array}{c|c} I & J_{\boldsymbol{\omega}} f(\mathbf{q}) \\ \hline O & J_{\mathbf{v}} f(\mathbf{q}) \end{array}\right), \tag{15}$$

where $J_{\boldsymbol{\omega}} f(\mathbf{q}), J_{\mathbf{v}} f(\mathbf{q})$ denote the projections of the Jacobian $Jf(\mathbf{q})$ onto the subspaces of infinitesimal rotations about the origin and infinitesimal translations, respectively. The determinant is simply equal to that of the lower right 3×3 block coming from the 'translational' part of f. Expanding this as a triple scalar product of its columns and using (14) gives the following expression for $\det J_{\mathbf{v}} f(\mathbf{q})$:

$$h(\mathbf{q}) := \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) + q_2 \mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{r}_{12}) + q_3 \mathbf{v}_3 \cdot (\mathbf{v}_1 \times \mathbf{r}_{23} + \mathbf{v}_2 \times \mathbf{r}_{13}) + O(2) \tag{16}$$

The condition $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) = \mathbf{0}$ (equivalently, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly dependent) affirms that $\mathbf{q} = \mathbf{0}$ itself is a singular point of $f_{\mathbf{c}}$. Indeed, if $\mathbf{v}$ lie in the orthogonal complement to the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ then the twist $(\mathbf{v}, \mathbf{0})$ passes through the wrist centre and is clearly reciprocal to each joint. In particular, if the joints are revolute then the line intersects their axes [25], giving Burdick's geometric criterion mentioned in Section 1.

The form (16) gives an equation $h(\mathbf{q}) = 0$ for the singularity locus in a neighbourhood of $\mathbf{q} = \mathbf{0}$. By the Implicit Function Theorem, if either of the coefficients of the q_i, $i = 2, 3$ in (16) is non-zero, then the singular locus is a 2-dimensional submanifold of the joint space in a neighbourhood of $\mathbf{0}$. Since each coefficient is a polynomial in the screw coordinates of the three joints, their zero sets are closed subspaces (affine varieties) and so there is an open set of 3 dof (not necessarily 3R) manipulators for which a given wrist centre has a smooth singular locus. In other words, this family is generic with respect to transverse regularity.

5 Cusps

The defining equation for the singular locus enables us to deduce a criterion for the wrist centre to be a cusp point in the case that the locus is a manifold. In the notation of the Thom–Boardman singularities [24, 26] a cusp point is a point of $\Sigma^{1,1} f_{\mathbf{c}}$; that is, a point in the joint space at which the restriction of the kinematic mapping $f_{\mathbf{c}}$ to its singular locus itself has a corank 1 singularity. Computationally, this is similar to a Lagrange multiplier problem—find the singular points of a function constrained to a given submanifold. In Section 4 we have not produced a closed form expression for the singular locus—it is only known as a series expansion up to first-order, though in principle more terms can be calculated and indeed there are closed form expressions for $\exp(q \operatorname{ad} X)$ [27]. However this is sufficient for the criterion we seek.

The required condition is that the derivative (or Jacobian matrix) for the augmented map $(f_{\mathbf{c}}, h) : \mathbb{R}^3 \to \mathbb{R}^4$ should have corank 1, in addition to the underlying requirement that $Df_{\mathbf{c}}(\mathbf{q})$ itself has corank 1. Since we have only expanded the singular locus about $\mathbf{q} = \mathbf{0}$, it is only possible to apply this criterion at that point, where we have enough information to determine the Jacobian of $(f_{\mathbf{c}}, h)$. The 4×3 matrix arising from differentiating (5) and (16) is:

$$\begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \\ 0 & \mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{r}_{12}) & \mathbf{v}_3 \cdot (\mathbf{v}_1 \times \mathbf{r}_{23} + \mathbf{v}_2 \times \mathbf{r}_{31}) \end{pmatrix} \tag{17}$$

where $\mathbf{v}_i = (v_{i1}, v_{i2}, v_{i3})^T$ for $i = 1,2,3$, and a sufficient condition for this to have rank 2 is that all 3×3 submatrices have determinant zero. Taking the first three rows automatically ensures that $\mathbf{q} = \mathbf{0} \in \Sigma^1 f_{\mathbf{c}}$. While there are three further submatrices, only one condition is algebraically independent: if the kinematic mapping is well behaved (satisfies an appropriate transversality condition) then $\Sigma^{1,1} f_{\mathbf{c}}$ will be a 1-dimensional submanifold (curve) in joint space.

6 Example: ortho-parallel manipulator

For an ortho-parallel manipulator, take as home configuration the one shown in Figure 1, where the parallel joints and the wrist centre all lie in a plane orthogonal to the axis of the first joint. Then with the wrist-centre as origin of coordinates and suitable choice of axes, the kinematics (5) can be defined using the following screw coordinates $(\boldsymbol{\omega}_i, \mathbf{v}_i)$, $i = 1,2,3$ for the joints:

$$\begin{aligned} X_1 &= (0,0,1,-(a_1+a_2+a_3), d_2+d_3, 0) \\ X_2 &= (1,0,0,0,0,a_2+a_3) \\ X_3 &= (1,0,0,0,0,a_3), \end{aligned} \tag{18}$$

where the a_i, d_i are DH parameters (see, for example, [10]). In the home configuration the joints are linearly independent. However $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) = 0$ so this is a singular configuration. Locally, the singular locus is defined by

$$h(\mathbf{q}) = -a_2 a_3 (a_1 + a_2 + a_3) q_3 + O(2) \tag{19}$$

Thus, the singular locus is locally a surface for non-trivial DH parameters unless $a_1 + a_2 + a_3 = 0$. The condition for a cusp is that the matrix

$$\begin{pmatrix} -(a_1+a_2+a_3) & 0 & 0 \\ d_2+d_3 & 0 & 0 \\ 0 & a_2+a_3 & a_3 \\ 0 & 0 & -a_2 a_3 (a_1+a_2+a_3) \end{pmatrix} \tag{20}$$

has rank 2, which is clearly not the case provided the conditions above hold. Notice that for the manipulator in Figure 1 we have $a_1 = d_2 = 0$. While it is of limited practical value, such a local singularity analysis remains straightforward for general screw joints of non-zero pitch.

It is also possible to analyse the effect of varying the wrist centre to a point $\mathbf{c} = (c_1, c_2, c_3)^T$. One way to do this is to transform coordinates by means of a translation so that the wrist centre remains the origin. The joint twist $X_i = (\boldsymbol{\omega}_i, \mathbf{v}_i)$ transforms to $X_i = (\boldsymbol{\omega}_i, -\boldsymbol{\omega}_i \times_i \mathbf{c} + \mathbf{v}_i)$. The linear part of the equation for the singularity locus of an ortho-parallel manipulator becomes:

$$(a_1 + a_2 + a_3 - c_2)c_3 a_2 + c_3^2 a_2 q_2 \\ - \left[(a_2 + a_3 - c_2)c_3^2 - (a_3 - c_2)((a_1 + a_2 + a_3 - c_2)a_2 + c_3^2)\right] q_3. \quad (21)$$

In particular, a shift of wrist centre parallel to the base axis (c_3-direction) moves the wrist centre off the singular locus.

7 Conclusions

Traditionally kinematic singularities of robotic manipulators have mostly been studied as a first order phenomenon in the sense of the Thom–Boardman singularity theory. It was realized that a 3R regional manipulator can change its posture without meeting a singularity if it exhibits cusp singularities (a second-order phenomena). Therefore, and because the 3 dof manipulator kinematics is accessible to symbolic manipulations, cusp singularities of 3R regional manipulators have been the subject of extensive studies. Today much is known in the most important cases employed in industrial manipulators, in particular for orthogonal and, to some extent, for ortho-parallel manipulators. The significance of higher order singularities of general manipulators remains an open question. Here we have revisited the general problem for regional manipulators in terms of the manipulator screw system using the adjoint action of the Euclidean group and the Thom–Boardman singularity theory. This provides the basis for exploring genericity conditions of regional manipulators without resorting to DH parameters. With the approach taken here it shall be possible to derive geometrically interpretable second order genericity conditions for regional manipulators with general screw joints (ensuring that $\Sigma^{1,1}$ is a manifold). Eventually we shall be able to determine second order genericity of general manipulators.

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