Trajectory Singularities of General Planar Motions

P. S. Donelan
Department of Mathematics, Victoria University of Wellington,
PO Box 600, Wellington, New Zealand.
C. G. Gibson
Department of Mathematical Sciences, University of Liverpool,
PO Box 147, Liverpool L69 3BX, England.
W. Hawes [1]
School of Mathematics and Statistics, Middlesex University,
Bounds Green Road, London N11 2NQ, England.

March 29, 2019

Abstract

Local models are given for the singularities which can appear on the trajectories of general motions of the plane with more than two degrees of freedom. Versal unfoldings of these model singularities give rise to computer generated pictures describing the family of trajectories arising from small deformations of the tracing point, and determine the local structure of the bifurcation curves. [2]

[1]Supported by a grant from the Science and Engineering Research Council.
[2]AMS 1991 Classification: 70B10, 57R45

1 Introduction

This paper is the fourth [?, ?, ?] in a series devoted to a basic question in theoretical kinematics, namely that of describing the surprisingly complex geometry associated to singularities of trajectories for *general* motions of Euclidean spaces. More precisely, one seeks an exhaustive finite list of local models, and complete descriptions of their bifurcations. Planar and spatial motions are of particular interest because of the enormous range of potential application to engineering robotics, with the planar case providing a natural starting point. One–dimensional planar motions (where the motion is generically an immersion) were discussed in [?], and 2–dimensional planar motions (where the motion is generically an immersion with a discrete set of cross–caps) were dealt with in [?]. In this paper we present a complete solution of the above problem for planar motions of arbitrary dimension ≥ 3.

The initial problem is that of listing smooth germs $\mathbb{R}^n, 0 \to \mathbb{R}^2, 0$ of "codimension" ≤ 2 in the case $n \geq 3$. That produces (up to a non–degenerate quadratic form in further variables) the identical list of corank 1 germs associated to 2–dimensional motions in [?], plus two new $\mathcal{A}$–unimodular families, which (for reasons that will become apparent later) we call *ephemera*. Having listed strata and specializations we are then faced with the question of whether the normal forms we obtain can actually appear as germs of trajectories for general planar motions. For a motion germ of fold type we obtain all the generic trajectory types except ephemera: and for motion germs of cusp or swallowtail types we obtain a very limited sublist. As a result it is possible [?] to extend a result for 2–dimensional motions in [?], describing two of the bifurcation curves as envelopes of instantaneous singular lines. We would like to acknowledge our debt to Professor J. W. Bruce for numerous helpful conversations, and to Dr. F. Tari who clarified the mechanics of computing the discriminant matrix in Section ??.

2 The General Framework

The general framework is as follows. Let $SE(p)$ be the Lie group of proper rigid motions of $\mathbb{R}^p$. By an *n–dimensional motion* of $\mathbb{R}^p$ we mean a smooth map $\lambda : N \to SE(p)$, where N is a smooth manifold of dimension n. The formula $(t, w) \mapsto \lambda(t)(w)$ defines another smooth mapping $M_\lambda : N \times \mathbb{R}^p \to \mathbb{R}^p$: and for any fixed choice of *tracing point* $w \in \mathbb{R}^p$ it yields a smooth mapping $M_{\lambda,w} : N \to \mathbb{R}^p$ defined by $t \mapsto \lambda(t)(w)$, which we refer to as the *trajectory* of $w \in \mathbb{R}^p$ under λ. We think of M_λ as a p–parameter family of trajectories. The first paper in the series [?] established a

Basic Transversality Lemma, roughly speaking that a generic n–dimensional motion of p–space gives rise to general singularities on the family of resulting trajectories. More precisely, given an n–dimensional motion $\lambda : N \to SE(p)$, and positive integers r and k, the trajectory mapping $M_{\lambda,w} : N \to \mathbb{R}^p$ induces a multijet extension

$$_rj^kM_{\lambda,w} : N^{(r)} \to {_rJ^k}(N, \mathbb{R}^p)$$

and since $M_{\lambda,w}$ depends smoothly on w, that yields a smooth mapping

$$_rj_1^kM_\lambda : N^{(r)} \times \mathbb{R}^p \to {_rJ^k}(N, \mathbb{R}^p).$$

Lemma 2.1. (Basic Transversality Lemma.) *Let $\mathcal{S}$ be a finite stratification of $_rJ^k(N, \mathbb{R}^p)$. The set of n–dimensional motions $\lambda : N \to SE(p)$ with $_rj^kM_\lambda$ transverse to $\mathcal{S}$ is residual in $C^\infty(N, SE(p))$, endowed with the Whitney C^∞ topology.*

In this situation there is a codimensional restriction on the relevant strata of $\mathcal{S}$, which we need to make precise. Let X be an $\mathcal{A}$–invariant smooth submanifold of $_rJ^k(n, p)$, giving rise in a natural way to an $\mathcal{A}$–invariant smooth submanifold Y of $_rJ^k(N, \mathbb{R}^p)$. Suppose that $_rj^kM_\lambda$ is transverse to $\mathcal{S}$, and that Y is one of the strata. Then a necessary condition for the pull–back of Y under $_rj^kM_\lambda$ to be non–void is that Y has codimension $\leq rn + p$, the dimension of the domain. However an easy argument using local triviality of the multijet bundle shows that the codimensions of X, Y in their respective multijet spaces differ by $(r - 1)p$. Thus the condition on X is that its codimension should be $\leq 2p + r(n - p)$. Given n, p the guiding principle for classification is to list strata in the multijet space up to this figure. In particular, for a generic motion, non–stable $\mathcal{A}$–simple multigerms of trajectories will be of $\mathcal{A}_e$–codimension $\leq p$. For potential applications to robotics the important cases are $p = 2$ (of planar motions) and $p = 3$ (of spatial motions) where it is realistic to expect complete solutions.

3 Listing Monogerms

In the case $r = 1$ of monogerms we seek to list $\mathcal{A}$–invariant strata in $J^k(n, p)$ of codimension $\leq n + p$: and in particular, in the planar case $p = 2$ we seek to list $\mathcal{A}$–invariant strata in $J^k(n, 2)$ of codimension $\leq n + 2$. Of course the $\mathcal{A}$–simple germs on the list appear in the complete classification of $\mathcal{A}$–simples in [**?**]: however our list cannot be derived from that paper. The result is:

Theorem 3.1. *For $k \geq 7$ there exists an $\mathcal{A}$–invariant stratification of the jet space $J^k(n,2)$ with the property that each stratum of positive codimension $\leq n+2$ is either the $\mathcal{A}^{(k)}$–orbit of a normal form in Table* **??**, *or comprises the $\mathcal{A}^{(k)}$–orbits of the $\mathcal{A}$–unimodular family given by the last entry. All the germs in the table have degree of $\mathcal{A}$–determinacy ≤ 7. In each entry $q(t)$ is a non–degenerate quadratic form in the other variables t_1, t_2, ...*

Name	Normal Form	$\mathcal{A}$–Cod	$\mathcal{A}_e$–Cod
fold	$(x, y^2 + q(t))$	$n-1$	0
cusp	$(x, xy + y^3 + q(t))$	n	0
lips	$(x, y^3 + x^2y + q(t))$	$n+1$	1
beaks	$(x, y^3 - x^2y + q(t))$	$n+1$	1
swallowtail	$(x, xy + y^4 + q(t))$	$n+1$	1
goose	$(x, y^3 + x^3y + q(t))$	$n+2$	2
gulls	$(x, xy^2 + y^4 + y^5 + q(t))$	$n+2$	2
butterfly	$(x, xy + y^5 \pm y^7 + q(t))$	$n+2$	2
ephemera	$(x, xy + y^3 + by^2z + z^3 \pm y^5 + q(t))$	$n+3$	3

Table 1: Normal forms for Monogerms with $n \geq 3$.

We start the listing process with a very simple result:

Lemma 3.1. *Let $n \geq 3$. Any $\mathcal{A}$–invariant submanifold X of the jet space $J^k(n,2)$ composed entirely of corank 2 jets has codimension $> n+2$.*

Proof. In $J^k(n,2)$ the jets of corank 2 form a linear subspace of codimension $2n$. X is contained in this subspace, so has codimension $\geq 2n > n+2$. □

Since we are only interested in $\mathcal{A}$–invariant submanifolds of the jet space of codimension $\leq n+2$ this means that we need only concern ourselves henceforth with germs of corank 1. The next generality is a Splitting Lemma, which probably first appeared in [**?**]. We phrase it (quite deliberately) as a lemma about $\mathcal{R}$–equivalence, for reasons which will appear later. Like its better known namesake for function germs it is easily established by an inductive argument on the k–jet (starting at $k = 2$) using the Complete Transversal Lemma of [**?**] for the induction step.

Lemma 3.2. (Splitting Lemma) *Let $F : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$ be an $\mathcal{A}$–finite germ of corank 1, transverse to the direction $u = 0$, where u,v are local coordinates at $0 \in \mathbb{R}^2$.*

There exist local coordinates x, y_1, ..., y_r, t_1, ..., t_s at $0 \in \mathbb{R}^n$ where $r, s \geq 0$ and $r + s + 1 = n$ with the following property. F has a "prenormal form" $F(x, y, t) = (x, \lambda x + f(x, y) + q(t))$ under $\mathcal{R}$–equivalence, where λ is a scalar, $y = (y_1, \ldots, y_r)$, $t = (t_1, \ldots, t_s)$, $q(t) = t_1^2 \pm \ldots \pm t_s^2$ and $f(x, y)$ is a polynomial with zero 1–jet for which $f(0, y)$ has zero 2–jet. Moreover, the $\mathcal{A}$–codimensions of $F(x, y, t)$ and $(x, \lambda x + f(x, y))$ differ by s. Likewise if F is transverse to the direction $v = 0$ we obtain a similar prenormal form for F, with the two components interchanged.

Under the hypotheses of the lemma the kernel of the differential of F has dimension $(n-1)$, and the cokernel has dimension 1, so the second intrinsic derivative [**?**, page 213] is represented by a single quadratic form, whose rank is the integer s: indeed $q(t)$ is a normal form for that quadratic form under linear changes of coordinates. Thus r and s are $\mathcal{A}$–invariants of F. Working with $\mathcal{A}$–equivalence, the prenormal form in the statement of the Splitting Lemma improves to $F(x, y, t) = (x, f(x, y) + q(t))$: indeed a linear change of coordinates at the target is sufficient to get the 1–jet into the required form, and then we proceed as before. We need to know that to classify germs $F(x, y, t)$ up to $\mathcal{A}$–equivalence it suffices to classify the *residual* germs $(x, f(x, y))$ up to $\mathcal{A}$–equivalence: that is provided by the following uniqueness result, stated in [**?**, page 172].

Lemma 3.3. *Let $F_1(x, y, t) = (x, f_1(x, y) + q(t))$, $F_2(x, y, t) = (x, f_2(x, y) + q(t))$ be prenormal forms, as above. Then $(x, f_1(x, y))$, $(x, f_2(x, y))$ are $\mathcal{A}$–equivalent if and only if $F_1(x, y, t)$, $F_2(x, y, t)$ are $\mathcal{A}$–equivalent.*

In the situation we are studying there are very strong restrictions on the invariant r, provided by the following result.

Lemma 3.4. *Let $n \geq 3$ and let X be an $\mathcal{A}$–invariant submanifold of the jet space $J^k(n, 2)$ of codimension $\leq n + 2$ comprising corank 1 jets with constant $\mathcal{A}$–invariant r. Then $r = 0$, $r = 1$ or $r = 2$.*

Proof. By the Splitting Lemma the 2–jet of every k–jet in X lies in the same orbit as a 2–jet of the form $(x, f(x, y) + q(t))$ described in the Splitting Lemma. Projecting X into $J^2(n, 2)$ we see that its codimension is bounded below by that of the orbit through the 2–jet. We compute the latter codimension explicitly.

Suppose first that $f(x, y)$ has zero 2–jet, so the 2–jet is in the same orbit as $(x, q(t))$. By inspection, a supplement for the tangent space to the $\mathcal{A}^{(2)}$–orbit is spanned by the linearly independent vectors $(0, y_i)$, $(0, y_i y_j)$, $(0, x y_i)$ with $1 \leq i, j \leq r$ and $(0, t_k)$ with $1 \leq k \leq s$. It follows that the $\mathcal{A}^{(2)}$–orbit has codimension $(n-1) +$

$\frac{1}{2}r(r+3)$. The hypothesis that X has codimension $\leq n+2$ then implies that $\frac{1}{2}r(r+3) \leq 3$ so $r=0$ or $r=1$.

Now suppose that $f(x,y)$ has non–zero 2–jet, but that still $f(0,y)$ has zero 2–jet. Then an obvious $\mathcal{A}$–equivalence allows us to assume that the 2–jet is in the same orbit as $(x, xy_1 + q(t))$. By inspection, a supplement for the tangent space to the $\mathcal{A}^{(2)}$–orbit is spanned by the linearly independent vectors $(0, y_i)$, $(0, y_i y_j)$ with $1 \leq i, j \leq r$ and $(0, t_k)$ with $1 \leq k \leq s$. It follows that the $\mathcal{A}^{(2)}$–orbit has codimension $(n-1) + \frac{1}{2}r(r+1)$. The hypothesis that X has codimension $\leq n+2$ then implies that $\frac{1}{2}r(r+1) \leq 3$ so $r=0$, $r=1$ or $r=2$. □

The next step is to pursue the trichotomy provided by the possibilities $r=0$, $r=1$ or $r=2$ arising from this result.

Lemma 3.5. *Let $n \geq 3$, $k \geq 2$ and let X be an $\mathcal{A}$–invariant submanifold of the jet space $J^k(n,2)$ comprising corank 1 jets with $r=0$. Then any k–jet in X is sufficient, and in the same orbit as the k–jet of one of the stable germs $(x, t_1^2 \pm \ldots \pm t_{n-1}^2)$.*

Proof. Standard determinacy results show that the 2–jet $(x, t_1^2 \pm \ldots \pm t_{n-1}^2)$ is sufficient. The germ has $\mathcal{A}$–codimension $(n-1)$, and $\mathcal{A}_e$–codimension zero. □

Lemma 3.6. *Let $n \geq 3$, $k \geq 7$ and let X be an $\mathcal{A}$–invariant submanifold of the jet space $J^k(n,2)$ of codimension $\leq n+2$, comprising corank 1 jets with $r=1$. Then any k–jet in X is sufficient, and in the same orbit as the k–jet of one of the germs in Table* **??**. *In each entry $q(t) = t_1^2 \pm \ldots \pm t_{n-2}^2$.*

Name	Normal Form	$\mathcal{A}$–Cod	$\mathcal{A}_e$–Cod
cusp	$(x, xy + y^3 + q(t))$	n	0
lips	$(x, y^3 + x^2y + q(t))$	$n+1$	1
beaks	$(x, y^3 - x^2y + q(t))$	$n+1$	1
swallowtail	$(x, xy + y^4 + q(t))$	$n+1$	1
goose	$(x, y^3 + x^3y + q(t))$	$n+2$	2
gulls	$(x, xy^2 + y^4 + y^5 + q(t))$	$n+2$	2
butterfly	$(x, xy + y^5 \pm y^7 + q(t))$	$n+2$	2

Table 2: Monolocal models with $r=1$.

Proof. In view of the Uniqueness result for the Splitting Lemma we are reduced to the well–known $\mathcal{A}$–classification [**?**, **?**, **?**, **?**, **?**] of corank 1 germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ up to strata of codimension 4 in the jet space. □

Lemma 3.7. *Let $n \geq 3$, $k \geq 5$ and let X be an $\mathcal{A}$–invariant submanifold of the jet space $J^k(n,2)$ of codimension $\leq n+2$, comprising corank 1 jets with $r = 2$. Then any k–jet in X is sufficient, and in the same orbit as the k–jet of a germ*

$$f_b^{\pm}(x,y,z,t) = (x, xy + y^3 + by^2z + z^3 \pm y^5 + q(t))$$

where the modulus b is distinct from two "exceptional" values $b = 0$ and $b = b_0$ where $4b_0^3 + 27 = 0$. These germs are all 5–$\mathcal{A}$–determined, and form two $\mathcal{A}$–unimodular families.

First, some preliminary remarks on the techniques used in the proof. We work with the subgroup $\mathcal{G}$ of the Mather group $\mathcal{A}$ with Lie algebra

$$L\mathcal{G} = L\mathcal{A}_1 \oplus \{x\partial/\partial y, x\partial/\partial z, y\partial/\partial z\} \oplus \{v\partial/\partial u\}$$

where x, y, z are the coordinates at the source, and u, v are the coordinates in the target. We use the fact [**?**] that a Complete Transversal Lemma can be associated to $\mathcal{G}$. Also, we use the following consequence of the determinacy results in [**?**], namely that a germ $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ is k–$\mathcal{G}$–determined if and only if

$$\mathcal{M}_3^{k+1}\mathcal{E}(3,2) \subseteq L\mathcal{G}.f + \mathcal{M}_3^{k+1} f^\star(\mathcal{M}_2)\mathcal{E}(3,2) + \mathcal{M}_3^{2k+2}\mathcal{E}(3,2).$$

Proof. In view of the Uniqueness result for the Splitting Lemma we are reduced to the $\mathcal{A}$–classification of corank 1 germs $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ with $r = 2$ up to strata of codimension 5 in the jet space. We write x, y, z for the coordinates at the source. The proof of Lemma **??** shows that we can assume that the 2–jet is (x, xy). We will suppress the computations of complete transversals and degrees of determinacy.

A complete transversal for the 2–jet (x, xy) is $(x, xy + ay^3 + by^2z + cyz^2 + dz^3)$. Suppose $d = 0$. Then, by inspection, a supplement for the tangent space in $J^3(3,2)$ to the $\mathcal{A}^{(3)}$–orbit through j^3F is spanned by $(0,y)$, $(0,z)$, $(0,y^2)$, $(0,yz)$, $(0,z^2)$, $(0,z^3)$: thus the orbit has codimension 6, which is too high. We can therefore assume $d \neq 0$. Replacing z by $z + \lambda y$ and choosing λ appropriately we can force $c = 0$: and scaling z in the source we can force $d = 1$, reducing the 3–jet to the form $(x, xy + ay^3 + by^2z + z^3)$. Suppose $a = 0$. Then, by inspection, a supplement for the tangent space in $J^3(3,2)$ to the $\mathcal{A}^{(3)}$–orbit through j^3F contains the subspace spanned by $(0,y)$, $(0,z)$, $(0,y^2)$, $(0,yz)$, $(0,z^2)$, $(0,y^3)$: so the orbit has codimension ≥ 6, which is too high. Thus we can assume $a \neq 0$, and by scaling (in source and target) that the 3–jet has the form $(x, xy + y^3 + by^2z + z^3)$. Note that b is a modulus as the vector $(0, y^2z)$ is not in the $\mathcal{A}^{(3)}$ tangent space. Computation verifies that the 3–jet has a trivial complete transversal in the space of 4–jets unless $b(4b^3 + 27) = 0$: that gives two exceptional values of the modulus b which we exclude from further consideration, as the $\mathcal{A}^{(4)}$–orbits have codimension 6. ($4b^3 + 27 = 0$ is the condition for the binary cubic $C(y,z) = y^3 + by^2z + z^3$ to have a repeated root.) With that proviso, further computation shows that in the space of 5–jets a complete transversal is given by $(x, xy + y^3 + by^2z + z^3 + ey^5)$. For a fixed value b, not all these 5–jets lie in the same orbit: indeed the vector $(0, y^5)$ lies in the $\mathcal{A}^{(5)}$ tangent space to the orbit for $e \neq 0$, but fails to do so for $e = 0$. We can discard the case $e = 0$ since the $\mathcal{A}^{(5)}$–orbit has codimension 6, and assume henceforth that $e \neq 0$. Scaling the coordinates x, y, z in the source by λ^2, λ, λ respectively, and the coordinates in the target by λ^{-2}, λ^{-3} respectively, we obtain the $\mathcal{A}$–equivalent germ $(x, xy + y^3 + by^2z + z^3 + e\lambda^2y^5)$, and then an obvious choice of λ yields the 5–jet $(x, xy + y^3 + by^2z + z^3 \pm y^5)$. The criterion for k–$\mathcal{G}$–determinacy mentioned above now shows that all these 5–jets are $\mathcal{G}$–sufficient. (We were unable to complete this computation by hand: it was checked using the TRANSVERSAL program [?] developed in Liverpool by Dr. N. P. Kirk.) □

4 Distinguishing Ephemera

It is worth remarking that (for a fixed choice of modulus b) the $\mathcal{A}$–types of the germs f_b^+, f_b^- are not distinguished by the topology of the fibres over target points. The transitions in the topology (as a target point crosses the discriminant) are easily described. In principle the germ of f_b^+ or f_b^- at a point on the critical curve are of fold type $(x, y^2 \pm z^2)$: the + case corresponds to the birth of a loop, and the − case to a self–crossing. Computation verifies that for $b < b_0$ both branches of the discriminant locus correspond to self–crossings, whilst for $b_0 < b < 0$ one branch gives rise to self–crossings, and the other to births of loops. The changes in these cases are illustrated in Figs 1a, 1b. For $b > 0$ the discriminant is a single point, so the topology of the fibre is constant away from that point – indeed the fibre is diffeomorphic to a line.

A natural choice of $\mathcal{A}$–invariant for the germs $f_b^\pm$ is offered by the $\mathcal{K}$–type (of the germ at the origin) of the discriminant locus. More formally, to such a germ can be associated [?] a "discriminant matrix", whose determinant defines the discriminant locus of the complexification, and whose $\mathcal{K}$–type is an invariant of the original germ. According to the prescription in [?] we require a basis for the module $\mathcal{E}_3/J_f + (\pi \circ f)\mathcal{E}_3$, where $f = f_b^\pm$, and π is projection onto one of the target coordinates. We choose projection onto the first coordinate, giving

$$\mathcal{E}_3/ < 3y^2 + 2byz \pm 5y^4,\ by^2 + 3z^2 > .$$

Figure 1: a) Fibres for f_b with $b < b_0$.   b) Fibres for f_b with $b_0 < b < 0$.

Provided that b is non–exceptional a suitable basis is given by $\{1, y, z, y^2\}$. Writing $f(x, y, z) = (x, g(x, y, z))$, we need to express $g \cdot 1$, $g \cdot y$, $g \cdot z$ and $g \cdot y^2$ as linear combinations of the basis elements, modulo $T\mathcal{A}.f$. Note that

$$g \cdot 1 = \frac{1}{3} \left\{ (yg_y + zg_z) + 2xy \mp 2y^5 \right\}$$

so it suffices to write y^5 in terms of our basis elements. Computation yields

$$y^5 = \frac{3}{\Delta^2} \left\{ 9(27 - 4b^3)x^2y - 36bx^2z \right\}$$

where $\Delta = 27 + 4b^3$, so that

$$g \cdot 1 \sim \frac{2}{3\Delta^2} \left\{ \Delta^2 xy \mp 3(27 - 4b^3)x^2y \pm 108bx^2z \right\}.$$

Similar calculations may be made for the other germs. The coefficients of each basis term are displayed in the following table, where $\Delta' = 27 - 4b^3$.

	$g \cdot 1$	$g \cdot y$	$g \cdot z$	$g \cdot y^2$
coeff of 1	0	0	$-\dfrac{1}{3b}x^2$	0
coeff of y	$\dfrac{2x}{3} \mp 2\dfrac{\Delta'}{\Delta^2}x^2$	0	0	$-\dfrac{6}{\Delta}x^2$
coeff of z	$\pm\dfrac{72b}{\Delta^2}x^2$	0	0	$\dfrac{4b}{\Delta}x^2$
coeff of y^2	0	$\dfrac{2}{3}x$	$-\dfrac{1}{b}x$	0

The entries in this table form a 4×4 matrix P. The required discriminant matrix D is formed by taking the target coordinates as (X, Y) and forming $D = Y.I - P$: thus

$$D = \begin{bmatrix} Y & 0 & \frac{1}{3b}X^2 & 0 \\ -\frac{2}{3}X \pm 2\frac{\Delta'}{\Delta^2}X^2 & Y & 0 & \frac{6}{\Delta}X^2 \\ \mp\frac{72b}{\Delta^2}X^2 & 0 & Y & -\frac{4b}{\Delta}X^2 \\ 0 & -\frac{2}{3}X & \frac{1}{b}X & Y \end{bmatrix}$$

The required discriminant is the determinant of D. A MAPLE computation verifies that it has the form

$$(AX^6 + BX^3Y^2 + CY^4) \pm (DX^7 + EX^4Y^2) + \{\text{higher order terms}\}$$

where A, B, C depend only on the modulus b, and D, E are constants. Assigning the weights 1/6, 1/4 to the variables X, Y the first braces enclose terms of weight 1, the second enclose terms of weight 7/6, and the higher order terms encompass all terms of weight $> 7/6$. Explicit computation verifies that the principal part is non–degenerate, so the singularity is of Arnold type W_{15} in the listings of [?]. By [?, Theorem 5] the $\mathcal{K}$–type does not depend on the higher order terms. Moreover, by explicit $\mathcal{K}$–equivalences the polynomial can be reduced to a form $x^6+\lambda x^3y^2+y^4\pm x^4y^2$ representing two $\mathcal{K}$–inequivalent germs. That concludes the proof that for any non–exceptional value of the modulus b the germs f_b^+, f_b^- are $\mathcal{A}$–inequivalent.

5 Listing Multigerms

Recall that for generic n–dimensional motions of p–space the objective is to list strata in the multijet space ${}_rJ^k(n,p)$ of codimension $\leq 2p+r(n-p)$ so we are free to discard (semialgebraic) subsets of the multijet spaces of codimension $> 2p+r(n-p)$. First, this imposes (sharp) bounds on the number r of multigerm branches. In the space ${}_rJ^1(n,p)$ of 1–jets the singular multigerms (in the sense that all the branches are singular) form a semialgebraic set of codimension $r(n-p+1)$, which is $> 2p+r(n-p)$ if and only if $r > 2p$. In particular, for planar motions only multigerms with ≤ 4 branches are relevant. Very similar reasoning shows that in the planar case we need only consider singular r–germs which satisfy the conditions listed below: in each case it is easily checked that the 1–jets of r–germs which fail to satisfy the stated condition form a semialgebraic subset of too high codimension.

- $r=2$: both branches have corank 1.
- $r=3$: all branches have corank 1, and at most two critical images are tangent.
- $r=4$: all branches have corank 1, and no two critical images are tangent.

In each case we can be more specific about the geometric possibilities via the fact that the $\mathcal{A}$–codimension of a multigerm bounds the sum of the $\mathcal{A}$–codimensions of its branches. Thus when $r=2$ we can discard strata of codimension $> 2n$ and have two possibilities: either one branch is a fold, and the other is of fold, cusp, lips, beaks or swallowtail type, or both branches are of cusp type, necessarily transverse. When $r=3$ we can discard strata of codimension $> 3n-2$, and have again two possibilities: either all three branches are folds with at most one tangency, or two are folds and the third is of cusp type, with no tangencies. Finally, when $r=4$ we can discard strata of codimension $> 4n-4$ yielding a single possibility, namely that all four branches are folds with no tangencies. The next task is to obtain normal forms for these geometric types.

We use the ideas of [?] to reduce the problem of finding normal forms under $\mathcal{A}$–equivalence for multigerms $\mathbb{R}^n,0 \to \mathbb{R}^2,0$ to the case $n=2$ dealt with in that publication. The definitions are as follows. For a smooth germ $g:\mathbb{R}^n,0 \to \mathbb{R}^p,0$

let Σg be the (germ at the source of the) critical set, i.e. the set of points at which the differential (of some representative) of g has rank $< p$: the *discriminant* Δ_g of g is the image $g(\Sigma g)$. Further, let $\mathcal{L}_g$ denote the subgroup of the group $\mathcal{L}$ of coordinate changes at the target which preserve Δ_g setwise, and let $\mathcal{A}_g = \mathcal{R} \times \mathcal{L}_g$. These definitions extend in an obvious way to multigerms $g : \mathbb{R}^n, S \to \mathbb{R}^p, 0$ where S is a finite set of points. Such a multigerm is *nice* when for any $\mathcal{L}_g$–equivalence k there exists an $\mathcal{R}$–equivalence h for which $g = k \circ g \circ h^{-1}$. Thus a multigerm is nice if and only if all its branches are nice. Now suppose (inductively) that we have a normal form under $\mathcal{A}$–equivalence for the nice r–germ $g = (g_1, \dots, g_r)$. Then the $(r+1)$–germs (g, f), (g, f') are $\mathcal{A}$–equivalent if and only if the monogerms f, f' are $\mathcal{A}_g$–equivalent, yielding the *classification principle* of [**?**]. The usefulness of this principle is based on the following result [**?**, Theorem 2.1].

Proposition 5.1. *Let $g_1, g_2 : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be critical normalisations having the same discriminant. If either germ has corank $\neq 1$ they are $\mathcal{R}$–equivalent. Suppose both germs have corank 1, so the first derivatives $dg_1(0)$, $dg_2(0)$ have the same image, and the second intrinsic derivatives $d^2g_1(0)$, $d^2g_2(0)$ have the same rank: give the cokernels of $dg_1(0)$, $dg_2(0)$ the same orientation. If $d^2g_1(0)$, $d^2g_2(0)$ have the same index then g_1, g_2 are $\mathcal{R}$–equivalent.*

This has the immediate consequence that any critical normalisation $g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is a nice germ. The definition of the term 'critical normalization', and the basic properties of this class of germs can be found in [**?**]. All we need to know is that any stable germ is automatically a 'critical normalization'. By the above that means that for r–germs g with stable branches the $\mathcal{A}$–classification of $(r+1)$–germs (g, f) reduces to the $\mathcal{A}_g$–classification of monogerms f. In this paper we need only consider the case when the branches are the simplest stable germs, namely folds and cusps.

The Splitting Lemma extends from $\mathcal{R}$–equivalence to $\mathcal{A}_g$–equivalence, at least for those germs $g : \mathbb{R}^n, S \to \mathbb{R}^2, 0$ of immediate interest to us: the principle is to use $\mathcal{L}_g$–equivalences to bring the 1–jet of the germ to a normal form, and then to apply the Splitting Lemma.

To implement these generalities, consider first the case of bigerms. Assume first that one branch is a fold so can be put into the normal form $g(x, y, t) = (x, y^2 + q(t))$ of Lemma **??**. Here the discriminant Δ_g is the line given by $v = 0$, and $\mathcal{L}_g$ is the subgroup of invertible germs in $\mathcal{L}$ preserving this line. Write f for the second branch of the germ. Of course, if the two branches are tangent, the 1–jet of f is tangent to the line $v = 0$: and if the branches are transverse we can assume (via $\mathcal{L}_g$–equivalences) that f is tangent to the line $u = 0$. Either way, the Splitting Lemma reduces the problem to the case $n = 2$, and the arguments of [**?**] yield all the normal forms for bigerms in Table ??, except the last. In each case X, Y, T_1, ..., T_{n-2} are local coordinates at the source of the second branch, and $q(T) = \pm T_1^2 \pm \ldots \pm T_{n-2}^2$.

name	normal form for f	cod
node fold	$(X^2 + q(T), Y)$	0
++ tacnode fold	$(X, X^2 + Y^2 + q(T))$	1
+− tacnode fold	$(X, X^2 - Y^2 + q(T))$	1
−− tacnode fold	$(X, -X^2 - Y^2 + q(T))$	1
cusp plus fold	$(XY + X^3 + q(T), Y)$	1
cusp plus tangent	$(X, XY + Y^3 + q(T))$	2
flecnode fold	$(X, Y^2 + X^3 + q(T))$	2
lips plus fold	$(X^3 + XY^2 + q(T), Y)$	2
beaks plus fold	$(X^3 - XY^2 + q(T), Y)$	2
swallowtail plus fold	$(XY + X^4 + q(T), Y)$	2
cusp plus cusp	$(XY + X^3 + q(T), Y)$	2

Table 3: Normal forms for Bigerms with $n \geq 3$.

To complete the discussion of bigerms we need to consider the remaining case when both branches are of cusp type, with transverse cuspidal tangents. In that case the first branch can be put into the normal form $g(x, y, t) = (x, xy + y^3 + q(t))$ of Table ??. Here the discriminant Δ_g is the cuspidal cubic $4u^3 + 27v^2 = 0$, and $\mathcal{L}_g$ is the subgroup of invertible germs in $\mathcal{L}$ preserving this curve. The first branch is tangent to the line $v = 0$, so the second branch is transverse to that line. By a result of Arnold in [?] we can assume (via $\mathcal{L}_g$–equivalences) that the second branch f is tangent to the line $u = 0$. The Splitting Lemma then reduces the problem to the case $n = 2$, treated in [?] and yielding the last normal form in Table ??.

For trigerms we can assume (as above) that two of the branches are transverse folds yielding a bigerm of nodefold type with normal form g given by the first entry in Table ??. The discriminant Δ_g is the line–pair $uv = 0$, and $\mathcal{L}_g$ is the subgroup of invertible germs in $\mathcal{L}$ preserving it. If the third branch f is transverse to the other two we can assume (via $\mathcal{L}_g$–equivalences) that f is tangent to the line $u = v$: otherwise f is tangent either to $u = 0$ or to $v = 0$. The Splitting Lemma then reduces the problem to the case $n = 2$, treated in [?] and yielding the first five normal forms in Table ??. For quadrigerms we can assume (as above) that all four branches are mutually transverse folds: thus any three yield a trigerm of triplefold type with normal form g given by the first entry in Table ??. The discriminant Δ_g is the concurrent line–triple $uv(u-v)=0$, and $\mathcal{L}_g$ is the subgroup of invertible germs in $\mathcal{L}$ preserving it. Via $\mathcal{L}_g$–equivalences we can assume that the fourth branch f is tangent to a line $v=cu$ with $c\neq 0,1,\infty$ and proceed as before to obtain the last entry in Table ??.

name	normal form for f	cod
triple fold	$(X, X+Y^2+q(T))$	1
node fold plus cusp	$(X, X+XY+Y^3+q(T))$	2
++ tacnode fold plus fold	$(X, X^2+Y^2+q(T))$	2
+− tacnode fold plus fold	$(X, X^2-Y^2+q(T))$	2
−− tacnode fold plus fold	$(X, -X^2-Y^2+q(T))$	2
quadruple fold	$(X, cX\pm Y^2+q(T))$	3

Table 4: Trigerms and Quadrigerms.

6 Ephemera Bifurcations

The bifurcation set for the family $M_\lambda : N\times\mathbb{R}^p\to\mathbb{R}^p$ comprises the set of tracing points $w\in\mathbb{R}^p$ for which the trajectory $M_{\lambda,w}$ exhibits a non–stable singularity. Its importance in the kinematic situation is that it represents the boundary between different qualitative types of mechanical behaviour. The Basic Transversality Lemma of [?] ensures that the local structure of the bifurcation set is determined by versal unfoldings of the generic singularity types. In the case of planar motions the bifurcation set is the union of a number of curves, corresponding to the generic singularity types of codimension 1. For 1–dimensional motions these are classical, namely the cusp curve (or moving centrode), the tacnode (or transition) curve and the triple point curve. But for multi–dimensional motions there are several bifurcation curves, none of which (to the best of our knowledge) have ever been observed in the kinematics literature. The bifurcations for the monogerms in Table ?? with $r=0$ or $r=1$ are well–documented, and recalled in [?] together with the bifurcations for all the multigerms, so we need only consider the unimodular families $f_b^\pm$. Miniversal $\mathcal{A}_e$–unfoldings of $f_b^\pm$ are given by

$$F_b^\pm(x,y,z)=(x, xy+y^3+by^2z+z^3\pm y^5+cz+dy^2)=(x,\phi(x,y,z,b,c,d))$$

We conjecture that this is a topologically versal unfolding of the family, but are unaware of any existing technique which would establish such a result. In the circumstances we are forced to consider specialisations of the $\mathcal{A}$–simple strata on a case–by–case basis. We claim that:

Lemma 6.1. *The only singularities which appear in the ephemera unfoldings $F_b^\pm$ are the fold, cusp, lips, beaks, swallowtail and triple fold types.*

The starting point is the observation that all the germs in Table **??** can be viewed as 1–parameter unfoldings (with x as the unfolding parameter) of singular function germs $\phi(0, y, z, b, c, d)$ in one fewer variable, with the $\mathcal{A}$–type of the former determining [**?**, page 222] the $\mathcal{K}$–type of the latter. The $\mathcal{A}$–simple germs give rise to function germs in the A–series: the fold corresponds to A_1, the cusp, lips, beaks and goose to A_2, the swallowtail and gulls to A_3, and the butterfly to A_4. The ephemera correspond to the hyperbolic umbilic D_4^- for $b > b_0$, and the elliptic umbilic D_4^+ for $b < b_0$. When we specialize to germs in the unfolding of the ephemera the situation is rather simpler than might be expected.

Lemma 6.2. *The critical set $\Sigma F_b^\pm$ of the unfolding $F_b^\pm$ is smooth if and only if $c \neq 0$. When $c = 0$ the only non–smooth point is $(0, 0, 0)$, at which the germ is of lips type for $b > 0$, and beaks type for $b < 0$. Neither the goose, the gulls nor the butterfly types can appear in the unfolding.*

Proof. $\Sigma F_b^\pm$ is the zero–set of the map $(x, y, z) \mapsto (X, Y)$ where $X = \partial\phi/\partial y$, $Y = \partial\phi/\partial z$. This map drops rank if and only if $y = z = 0$; in that case the conditions $X = 0$, $Y = 0$ imply that $x = 0$ and $c = 0$. When $c = 0$ the origin is a non–smooth point of the critical set: for $b > 0$ it is isolated, and for $b < 0$ it is the (non–tangential) intersection of two smooth branches. Suppose now that $c = 0$. (And $d \neq 0$, else the unfolding is an ephemera.) We will determine the $\mathcal{A}$–type of the germ of the unfolding at $(0, 0, 0)$. An $\mathcal{R}$–equivalence of the form $(x, y, z) \mapsto (x, y+\lambda x, z)$, followed by an $\mathcal{L}$–equivalence of the form $(u, v) \mapsto (u, v+p(u))$, reduces the unfolding to $(x, dy^2 + y\alpha(x, y, z) + \beta(x, z))$ with $\alpha(x, y, z)$ having zero 1–jet, and $\beta(x, z) = z^3 + b\lambda^2x^2z$. $\mathcal{R}$–equivalences now allow us to kill the term $y\alpha(x, y, z)$ without altering the 3–jet of the other terms. Thus the 3–jet of the unfolding is $\mathcal{A}$–equivalent to $(x, dy^2 + z^3 + b\lambda^2x^2z)$, which is of lips type for $b > 0$, and of beaks type for $b < 0$.

The goose and gulls types are excluded since their critical sets fail to be smooth: the critical set for the goose type has a cusp; and the critical set for the gulls type has a point at which two smooth curves meet non–tangentially. And the butterfly type is excluded because an A_4 singularity cannot appear in the unfolding of a D_4. □

Figure 2: a) Bifurcation of D_4^+. b) Bifurcation of D_4^-.

Thus, restricting ourselves to monogerms which may appear in an unfolding of the ephemera, the fold corresponds to A_1, the cusp, lips, and beaks to A_2, and the swallowtail to A_3. Further progress is based on the observation that $\phi(x, y, z, b, c, d)$ is a $\mathcal{K}_e$–versal unfolding of the umbilic $y^3 + by^2z + z^3 \pm y^5$. The (full) bifurcation sets for the D_4^+ and D_4^- singularities are illustrated in Figs 2a, 2b, taken from [**?**]. The A_3–type appears in both umbilic unfoldings, so the swallowtail type appears in the ephemera unfoldings for all non–exceptional values of b. Of course Lemma **??** exhibits the lips and beaks types, so (trivially) the fold and cusp types appear as well. This line of thought throws light on the multigerms appearing in the ephemera unfoldings.

Lemma 6.3. *The only multigerm which can appear in the ephemera unfoldings $F_b^{\pm}$ is the triple fold type, when $b < b_0$.*

Proof. For a multigerm in the ephemera unfolding, the function germs ϕ must appear at the same level. A careful reading of [**?**, Table 1] shows that neither the A_2–nor the A_3–types can appear *at the same level* with any other type, precluding the appearance of any multigerm having a branch of cusp, lips, beaks or swallowtail type. Thus we need only consider multigerms in the unfoldings each of whose branches is of fold type.

We claim that such multigerms cannot have tangent branches. At any point (x, y, z) in $\Sigma F_b^{\pm}$ we have $\phi_y = 0$, $\phi_z = 0$ and the image of the differential of $F_b^{\pm}$

at (x, y, z) is the line spanned by the vector $(1, y)$, the first column in the Jacobian matrix: when the germ is of fold type this line is the tangent to the fold curve. Thus if $F_b^{\pm}$ is of fold type at distinct sources (x_1, y_1, z_1), (x_2, y_2, z_2) with the same target, giving rise to tangent fold curves, then $x_1 = x_2 = x$ and $y_1 = y_2 = y$. However the relations $\phi_y(x, y, z_1, b, c, d) = 0$, $\phi_y(x, y, z_2, b, c, d) = 0$ yield $by(z_1 - z_2) = 0$, hence $y = 0$ and $x = 0$: but then the relation $\phi_z(0, 0, z, b, c, d) = 0$ shows that z_1, z_2 satisfy $3z^2 + c = 0$, so $\phi(x, y, z_1, b, c, d)$, $\phi(x, y, z_2, b, c, d)$ take distinct values, a contradiction.

Thus only the triple and quadruple folds need to be considered, corresponding respectively to the multigerm types $3A_1$ and $4A_1$ for the function ϕ. The former only appear in the versal unfolding of the elliptic umbilic, but the latter appear in the unfoldings of neither umbilic. That completes the proof of Lemma **??**. □

For a fixed value of the modulus b the bifurcations of the ephemera can be neatly illustrated by the classic device of a "clock diagram" in the (c, d)–plane. Fig 3 illustrates the bifurcations for $b < b_0$, Fig 4 for $b > 0$ and Figs 5a, 5b for $b_0 < b < 0$. The nomenclature for these unimodular families arises from the distinctive bifurcation in Fig 5b corresponding to the negative c–axis: it shows a remarkable resemblance to illustrations in natural history books of *Ephemera danica*, the common Mayfly.

Figure 3: Bifurcations for $b < b_0$.

Figure 4: Bifurcations for $b > 0$.

Figure 5: a) Bifurcations of f_b^+ with $b_0 < b < 0$. b) Bifurcations of f_b^- with $b_0 < b < 0$.

7 General Planar Motions

We are now in a position to return to the kinematic problem which provided the genesis for the listing process pursued above. Having established for $n \geq 3$ a list of germs $\mathbb{R}^n, 0 \to \mathbb{R}^2, 0$ of "codimension" ≤ 2 we ask whether these germs can actually appear as germs of trajectories associated to n–dimensional planar motions. The first thing to point out is that *when $n \geq 3$, a singular point on a trajectory can only arise from a singular point of the motion* λ, since the trajectory $M_{\lambda,w}$ is the composite of a submersion (the evaluation map $ev_w : SE(2) \to \mathbb{R}^2$) with λ. Moreover the nature of the trajectory singularity will depend on the nature of the motion singularity. It is therefore of importance that for a *general* n–dimensional planar motion $\lambda : N \to SE(2)$ with $n \geq 3$ there exist normal forms for the $\mathcal{A}$–equivalence class of the germ at any point. Indeed for a general motion λ the germ at any singular point is stable, and by results of Morin [**?**], described for instance in [**?**, pages 202–203], normal forms up to $\mathcal{A}$–equivalence are given by the 2–parameter unfoldings of the A_1–, A_2– and A_3–singularities in Table **??**, where $q(t) = \pm t_1^2 \pm \ldots \pm t_{n-3}^2$ for $n \geq 4$.

name	normal form for f	cod
fold	$(x, y, z^2 + q(t))$	1
cusp	$(x, y, z^3 + yz + q(t))$	2
swallowtail	$(x, y, z^4 + yz + xz^2 + q(t))$	3

Table 5: Stable Germs of Motions.

Our next objective is to determine for each of these motion types the types of generic trajectory singularity which can arise. We start with the fold case.

Lemma 7.1. *For any $\mathcal{A}$–simple type $\mathcal{S}$ in Table* **??** *there exists a germ of an n–dimensional planar motion $\lambda : N, x \to SE(2), 1$ of fold type for which the given type can appears as a trajectory singularity associated to λ. The ephemera cannot appear in this way.*

Proof. Consider a standard fold $\zeta : \mathbb{R}^n, 0 \to \mathbb{R}^3, 0$ given (in local coordinates x, y, z, t_1, …, t_{n-3} at the source, and u, v, w at the target) by $u = x$, $v = y$, $w = z^2 + q(t)$. Since any two submersions (with identical source and target dimensions) are $\mathcal{R}$–equivalent, it suffices to produce any submersive germ $\pi : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ for which the composite $\pi \circ \zeta$ is of type $\mathcal{S}$. Observe that any $\mathcal{A}$–simple type in Table **??** has a normal form $(x, f(x, y) + q(t))$ for some polynomial $f(x, y)$ with zero 1–jet. Then, writing X, Y for the components of π, we see that the submersive germ defined by $X = u$, $Y = f(u, v) + w$ has the required property. It remains to show that the $\mathcal{A}$–unimodular types cannot appear in this way. Clearly, it suffices to show that the composite of a standard fold with an *arbitrary* submersive germ $\pi : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ cannot be one of these types. Consider therefore a submersive germ π given by a formula

$$X = au + bv + cw + \ldots, \qquad Y = a'u + b'v + c'w + \ldots. \tag{1}$$

Then the composite $\pi \circ \zeta$ is the germ given by

$$X = ax + by + c(z^2 + q(t)) + \ldots \qquad Y = a'x + b'y + c'(z^2 + q(t)) + \ldots$$

We can suppose this germ has rank 1, and by a change of coordinates at the source that

$$X = x \qquad Y = Ax + C(z^2 + q(t)) + \dots$$

for some scalars A, C with $C \neq 0$. The quadratic part of Y then has the shape

$$\alpha x^2 + \beta xy + \gamma y^2 + C(z^2 + q(t))$$

for some scalars α, β, γ. It follows that the $\mathcal{A}$–invariant r (in the Splitting Lemma) is ≤ 1, excluding the ephemera, for which $r = 2$. □

The following lemmas show that when the motion germ is of cusp or swallowtail type, only a very limited range of trajectory types with $r = 1$ can occur. On the other hand, in each case the ephemera types can arise.

Lemma 7.2. *For an n–dimensional motion germ $\lambda : N, x \to SE(2), 1$ ($n \geq 3$) of cusp type, only germs of the (non-definite) fold, cusp and ephemera types in Table* **??** *can occur as trajectory singularities.*

Proof. To begin with, we proceed as in Lemma **??** by assuming λ to be in standard cusp form as in Table **??** and composing with an arbitrary submersive germ (**??**). Then the composite has the form

$$X = ax + by + c(yz + z^3 + q(t)) + \dots, \qquad Y = a'x + b'y + c'(yz + z^3 + q(t)) + \dots$$

This must have corank 1 (if singular and in the list of Table **??**) so $a \neq 0$ or $a = 0, b \neq 0$. When $a \neq 0$ changes of coordinates at source and target reduce the germ to

$$X = x, \qquad Y = C(yz + q(t)) + Exy + Fy^2 + \dots$$

with $C \neq 0$, which is of non–definite fold type $X = x$, $Y = y^2 - z^2 + q(t)$. When $a = 0$, $b \neq 0$ a similar reduction gives

$$X = y, \qquad Y = C(yz + z^3 + q(t)) + Dx^2 + Exy + Fy^2 + \dots$$

When $D \neq 0$ explicit changes of coordinates reduce the germ to the form $X = y$, $Y = yz + z^3 \pm x^2 + q(t) + O(4)$, which (by determinacy) is of cusp type: but when $D = 0$, the $\mathcal{A}$–invariant $r = 2$, excluding all the other types in Table **??** with $r = 1$. The ephemera are exhibited as the composite of $\lambda(x, y, z) = (x, xy + y^3 + by^2z + z^3 \pm y^5, z)$ with projection onto the first two coordinates. It suffices therefore to show that this λ is $\mathcal{A}$–equivalent to the cusp type in Table **??**. Replacing y by $y + \alpha z$ (for appropriately chosen α) and using obvious changes of coordinates in the target, reduces the germ to the shape $(x, xy + y^3 + Byz^2 + O(4), z)$ and then replacing x by $x + \beta z^2$ (for appropriately chosen β) yields $(x, xy + y^3 + O(4), z)$ which (by determinacy) is of cusp type. (Note that if we choose local coordinates at the identity on $SE(2)$ so that the first two represent translation and the third rotation, then the submersion used in this construction is precisely the evaluation map for the origin in $\mathbb{R}^2$.) □

Lemma 7.3. *For an n–dimensional planar motion $\lambda : N, x \to SE(2), 1$ ($n \geq 3$) of swallowtail type, only germs of the (non–definite) fold, swallowtail and ephemera types in Table* **??** *can occur as trajectory singularities.*

Proof. This follows the same lines as the proof of Lemma **??**. Start with the motion in normal form $\lambda(x, y, z, t) = (x, y, yz + xz^2 + z^4 + q(t))$ and compose with a general submersive germ (**??**). Given the composite is singular we again distinguish the cases $a \neq 0$ and $a = 0$. When $a \neq 0$ the composite yields a non-definite fold, as in Lemma **??**. When $a = 0$ the composite has $\mathcal{A}$–invariant $r = 1$ or $r = 2$: in the former case it is easy to show that the composite is of swallowtail type. It remains to show that the ephemera can indeed occur when $r = 2$. To this end, consider the motion λ given by

$$(u, v, w) = (x, y - \alpha z - \beta z^2, xy + y^3 + by^2 z + z^3 \pm y^5).$$

Composition with the submersion given by projection onto (u, w) clearly gives an arbitrary ephemera type, so it suffices to show that λ is $\mathcal{A}$–equivalent to a swallowtail. Replacing y by $y - \alpha z - \beta z^2$, and removing terms in w independent of z, yields $u = x$, $v = y$, and an expression for w in which the coefficients of the monomials yz^2, z^3 are respectively $\alpha(3\alpha + 2b)$, $\alpha^3 + \alpha^2 b + 1$. Choose $\alpha \neq 0$ so that the coefficient of z^3 vanishes; the coefficient of yz^2 is then non–zero, since we exclude the exceptional value of the modulus b defined by $4b^3 + 27 = 0$. With that choice

$$w = \alpha xz + \gamma y^2 z + \beta xz^2 + \alpha\delta yz^2 + 3\beta y^2 z^2 + 2\beta\delta yz^3 + \alpha\beta\delta z^4 + O(5)$$

where $\gamma = 3\alpha + b$, $\delta = 3\alpha + 2b$. Consider now the change of coordinates at the origin in $\mathbb{R}^3$ defined by $\phi(x, y, z) = (x', y', z')$ where $x' = \alpha x + (3\alpha + b)y^2$, $y' = \alpha(3\alpha + 2b)y + \beta$, $z' = z$. Applying ϕ at the source and ϕ^{-1} at the target gives $u = x$, $v = y$, $w = xz + yz^2 + O(4)$: among the quartic terms those in x^2z^2, xyz^2, y^2z^2, xz^3 , yz^3 can all be absorbed into lower order terms, leaving only a term in z^4 whose coefficient can be set to unity by choosing β so that $\alpha\beta\delta = 1$. The desired result follows, as the swallowtail is 4–$\mathcal{A}$–determined. □

Some consideration of the way in which singular trajectories arise illuminates these results. The composition of a map $\mathbb{R}^3 \to \mathbb{R}^3$ with a submersion $\mathbb{R}^3 \to \mathbb{R}^2$ has a singularity if and only if the 1–dimensional tangent to the fibre of the submersion is contained in the 2–dimensional image of the derivative of the map at a singular point. In the above lemmas with the motion in normal form this corresponds to the condition $ab' - a'b = 0$ on the coefficients of the submersion. In the fold case, there is no distinguished direction in the tangent space to the critical image but for the cusp and swallowtail we can identify the directions tangent to the cuspidal edge and at the cusp point in the cuspidal edge, respectively. These both correspond to the extra condition $a = a' = 0$. The restrictions required for the ephemera can be seen as imposing extra conditions on the contact between these special features in the discriminant surface and the fibre of the submersion.

8 The Spatial Case

It would be of interest to extend our results to the spatial case. This seems to be a substantial challenge. One–dimensional spatial motions received an exhaustive treatment in [**?**]. Bifurcations of trajectories associated to 2–dimensional spatial motions are discussed in [**?**]: here there is a complete listing of the monolocal models [**?**], extended to multilocal models in [**?**]. Three–dimensional spatial motions represent a wide field of study, offering a range of pertinent engineering examples. The third author's thesis [**?**] is a first attempt at studying such motions, coming within a hairsbreadth of a complete listing of monolocal models via extensive use of computer algebra. At the time of writing it is probably not practical to develop lists of multilocal models.

References

[1] Arnold, V. I. (1974) Normal Forms of Functions in the Neighbourhoods of Degenerate Critical Points. *Russian Math. Surveys* **29** . 11-49.

[2] Arnold, V. I. (1976) Wavefront Evolution and Equivariant Morse Lemma. *Comm. Pure Appl. Math. Vol 29* 557–582.

[3] Arnold, V. I. (1979) Indices of Singular points of 1–Forms on a Manifold with Boundary, Convolution of Invariants of Reflection Groups, and Singular Projections of Smooth Surfaces. *Russian Math. Surveys* **34** *(2)*. 3-38.

[4] Bruce, J.W., Kirk, N.P. and du Plessis, A.A. (1996) Complete Transversals and the Classification of Singularities. *Nonlinearity*, **9**, 1–23.

[5] Bruce, J. W. (1981) On the Degree of Determinacy of Smooth Functions. *Bull. London Math. Soc., (13)*. 55–58.

[6] Bruce, J. W., du Plessis, A. A. and Wall, C. T. C. (1987) Determinacy and Unipotency. *Inventiones Mathematicae* **88** . 521–554.

[7] Cocke, M. W., Donelan, P. S. & Gibson, C. G. Instantaneous Singularity Sets for Planar and Spatial Motions. In preparation

[8] Gaffney, T. (1983) The structure of $T\mathcal{A}(f)$, classification and an application to differential geometry, *AMS – Proceedings of Symposia in Pure Mathematics.* **40** *(Singularities) Part 1*, 409-427.

[9] Gaffney, T., du Plessis, A. A. and Wilson, L. C. (1990) Map–Germs Determined by their Discriminant. To appear in the Proceedings of the Hawaii–Provence Singularities Conferences, Travaux en Cours, Hermann, Paris.

[10] Gaffney, T. and Wilson, L. (1983) Equivalence Theorems in Global Singularity Theory. *AMS Proceedings of Symposia in Pure Mathematics.* **40** *(Singularities) Part 1*, 439–447.

[11] Gibson, C. G. Singular Points of Smooth Mappings. (1979) *Research Notes in Mathematics.* Pitman, London.

[12] Gibson, C. G. & Hobbs, C. A. (1995) Local Models for General One–Parameter Motions of the Plane and Space. *Proceedings of the Royal Society of Edinburgh*, **125A**, 639–656.

[13] Gibson, C. G. and Hobbs, C. A. (1995) Singularity and Bifurcation for General Two–Dimensional Planar Motions. *New Zealand Journal of Mathematics.* **25** (1996) 141–163.

[14] Gibson, C. G., Hobbs, C. A. & Marar, W. L. (1995) On Versal Unfoldings of Singularities for General Two–Dimensional Spatial Motions. To appear in *Acta Applicandae Mathematicae.*

[15] Gibson, C. G., Marsh, D. and Xiang, Y. (1996) Singular Aspects of Generic Planar Motions with Two Degrees of Freedom. Preprint, Napier University.

[16] Hawes, W. (1995) Multi–Dimensional Motions of the Plane and Space. *Ph. D. Thesis, University of Liverpool.*

[17] Hobbs, C. A. and Kirk, N. P. (1996) On the Classification and Bifurcation of Multigerms of Maps from Surfaces to 3–Space. Preprint, Oxford–Brookes University.

[18] Kirk, N. P. (1993) Computational Aspects of Singularity Theory. *Ph. D. Thesis, University of Liverpool.*

[19] Martinet, J. (1982) Singularities of Smooth Functions and Maps. London Mathematical Society Lecture Note Series, 58, Cambridge University Press.

[20] Mather, J. N. (1970) Stability of C^∞ Mappings: VI The Nice Dimensions. *Springer Lecture Notes in Mathematics, 192, Proceedings of Liverpool Singularities – Symposium I* 207–253.

[21] Mond, D. M. Q. (1985) On the Classification of Germs of Maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. *Proc. London Math. Soc.*, **50** (3), 333–369.

[22] Morin, B. (1965) Formes canoniques des Singularités d'une Application Différentiable. *Compt. Rendus Acad. Sci. Paris* **260** 5662–5665, 6503–6506.

[23] du Plessis, A. A. (1980) On the Determinacy of Smooth Map–Germs. *Invent. Math.* **58** 107–160.

[24] du Plessis, A. A. and Wilson, L. (1985) On Right–Equivalence. *Math. Z.* **190** 163–205.

[25] Rieger, J. H. and Ruas, M. A. S. (1987) Classification of $\mathcal{A}$–simple germs from k^n to k^2. *Comp. Math.* **79** 99–108.

[26] Wall, C. T. C. (1980) Affine Cubic Functions III. The Real Plane. *Math. Proc. Camb. Phil. Soc.* **87** (1) 1–14.

[27] Whitney, H. (1955) On Singularities of Mappings of Euclidean Space I: Mappings of the Plane to the Plane. *Ann. Math.* **62** 374–410.