Genericity Conditions for Serial Manipulators

Peter Donelan

School of Mathematics, Statistics and Computer Science
Victoria University of Wellington, New Zealand, e-mail: peter.donelan@vuw.ac.nz

Abstract. A generic, or more properly 1-generic, serial manipulator is one whose forward kinematic mapping exhibits singularities of given rank in a regular way. In this paper, the product-of-exponentials formulation of a kinematic mapping together with the Baker–Campbell–Hausdorff formula for Lie groups is used to derive an algebraic condition for the regularity.

Key words: Generic manipulator, transversality, singularity, Lie group, product of exponentials

1 Introduction

The idea of a generic manipulator was introduced by Pai and Leu [11], who observed that mathematical singularity theory (e.g. [4]) may provide powerful tools for exploring robot singularities. For a given manipulator one wants to understand the singularity locus, how it partitions the joint space and how its image appears in the workspace. A simple invariant of a singularity is its rank—the property of *1-genericity* requires certain regularity or, more precisely, transversality conditions for the forward kinematic mapping of a serial manipulator which ensure that the locus of singularities of fixed rank must be a submanifold (smooth subset of the joint space) whose dimension is determined by the number k of joints and the rank. In particular, there will be no singularities if the rank is greater than a certain number depending on k (either 1 or 2). In particular, only for $k = 6$ did Pai and Leu explicitly determine a general condition for 1-genericity of a kinematic mapping and went on to show that the property held for certain partitioned manipulators where one could consider separately rotational and translational kinematics. Subsequent work on generic manipulators has been undertaken by Tsai et al. [13] who extended these results to full 6-dof serial manipulators, Burdick [1] who obtained specific equations for the bifurcation sets where the regularity fails in the case of regional manipulators, and Lerbet and Hao [6] who, in studying over-constrained closed loop mechanisms, showed how the transversality condition relates explicitly to the algebra of the joint screws. The practical relevance of genericity was realised by Narasimhan and Kumar [10] who introduced a manipulator control algorithm adapted to handle generic singularities.

In this paper, a different approach is presented to obtain results like those of Hao and Lerbet, putting them in an algebraic context which shows how they can be generalised and potentially used to tackle substantial questions about the singularities of classes of manipulator. For example, in singularity theory, the term *generic* is used to describe any property of a family of mappings that is common to almost all members of the family. 'Almost all' has a topological meaning—the set should be open and dense. Openness ensures that any mapping with the property is surrounded by a neighbourhood of mappings that share the property, so that the property is stable. Density means that if a mapping fails to have the property, then arbitrarily close to it is one that does—there are no open regions where every mapping fails to have the property.

There is a specific set of tools that are used to guarantee genericity: transversality theorems. A survey of the technical details can be found in [2], where some of these results first appeared. There is one set of transversality theorems that guarantee genericity when one is interested in the infinite-dimensional space of *all* smooth mappings between given manifolds. In this setting, 1-genericity is automatically generic. However, families of serial manipulators are only ever finite-dimensional and there is no certainty that the special associated families of kinematic mappings sit nicely within the very large space of all mappings. In this setting, an Elementary Transversality Theorem is required and a transversality hypothesis must be satisfied before a conclusion of genericity can be drawn. The transversality condition established here would seem to be quite applicable in establishing theorems of this kind for 1-generic manipulators.

2 Jet Extensions of Kinematic Mappings

Suppose the forward kinematics of a serial manipulator with m joints is given by the smooth function $f : M^{(m)} \to SE(3)$, where M is the m-dimensional joint manifold (the superscript denoting the dimension) and $SE(3)$ is the 6-dimensional Lie group of Euclidean isometries describing the motion of the end-effector. A given set of joint variables $x \in M$ is a singularity of the robot kinematics if the rank of the derivative $Df(x)$ falls below the maximum possible value, i.e. the minimum of m and 6. The number $\min\{m,6\} - \operatorname{rank} Df(x)$ is called the *corank* of the singularity. We briefly present the key definitions; more details can be found in [2, 4, 7, 11].

One way to recognise a singularity is to consider the first-order Taylor expansion, with respect to some choice of coordinates for M and $SE(3)$, at points $x \in M$. This includes information on the derivative $Df(x)$ so it tells us whether there is a singularity. Regarding x as a variable, there is a smooth map $j^1 f$, called the *1-jet extension* of f, which assigns to each $x \in M$ the first-order Taylor expansion of f at x. The values of this map lie in a space called the *1-jet bundle*, which is itself a smooth manifold. Locally this looks like a product of an open piece of M locally coordinatised by (part of) $\mathbb{R}^m$, an open piece of $SE(3)$ with coordinates in $\mathbb{R}^6$, and the vector space of matrices $M(\mathbb{R}^m, \mathbb{R}^6)$—the *fibre* of the bundle—representing derivatives of a mapping $\mathbb{R}^m$ to $\mathbb{R}^6$ representing f in local coordinates. The matrices of fixed corank s form a submanifold of $M(\mathbb{R}^m, \mathbb{R}^6)$ of codimension $s(|6-m|+s)$, so there is a submanifold Σ^s of the jet bundle consisting of 1-jets of corank s having the same codimension.

The 1-jet extension $j^1 f$ is *transverse* to one of the submanifolds Σ^s, denoted $j^1 f \pitchfork \Sigma^s$, provided that whenever $x \in M$ and $j^1 f(x) \in \Sigma^s$, then the image of the derivative of $j^1 f$ at x spans a complement to the tangent space to Σ^s in the tangent space to the 1-jet bundle at $j^1 f(x)$. If $j^1 f$ is transverse to all the submanifolds Σ^s, $s = 0, 1, \ldots, \min\{m, 6\}$ then the manipulator is called *one-generic*, sometimes abbreviated to simply *generic*. This is sufficient to ensure that the inverse image $\Sigma^s f = (j^1 f)^{-1}(\Sigma^s)$ is either empty or a submanifold of M with the same codimension as Σ^s.

Transversality here depends on the second-order partial derivatives of f at x. There is an invariant part of this family of second-order derivatives called the *intrinsic second derivative*. Consider for each x_l, $l = 1, \ldots, m$, that part of the matrix $\left(\frac{\partial^2 f_j}{\partial x_i \partial x_l}\right)_{ij}$ given by restricting it to the kernel of $Df(x)$ and then projecting the result onto the cokernel, i.e. the quotient of $T_{f(x)}SE(3)$ by the image of $Df(x)$ (the column space of the Jacobian matrix). Each of these could be written as a $s \times (|6-m|+s)$ matrix with respect to some bases for the kernel and cokernel. A necessary and sufficient condition for transversality to Σ^s is that these m matrices span the vector space of all such matrices. Equivalently, the matrices obtained by restriction but not projection, together with the submatrices formed by taking s columns of the Jacobian should span the space of $6 \times s$ matrices.

A necessary condition for transversality to Σ^s is that $m \geq s(|6-m|+s)$. As a consequence, for 1-generic manipulators $\Sigma^s f$ is empty for $s \geq 2$ unless $m = 6, 7, 8$ in which case $\Sigma^2 f$ may be a a submanifold of codimension $4, 6$ or 8 respectively. In [13], it was also noted that for reasons of symmetry the resulting dimensions in the cases $m = 7, 8$ are too small to occur transversely. However, one is often interested not simply in a given manipulator but a family of manipulators in which some design parameters may be altered. Suppose the design parameters lie in a manifold B of dimension k. Then there is a family of kinematic mappings which can be thought of as a function of the joint variables in M and parameters in B, $F : M \times B \to SE(3)$. For each $b \in B$, there is an ordinary kinematic mapping $F_b : M \to SE(3)$, given by $F_b(x) = F(x, b)$. So now we have a map

$$\Phi : M \times B \to J^1(M, SE(3)), \quad \Phi(x, b) = j^1 F_b(x). \tag{1}$$

In this situation one could encounter transversely singularity strata, Σ^s, in the jet bundle up to codimension $m + k$. The Elementary Transversality Theorem (see for example [4]) asserts that if If $\Phi \pitchfork \Sigma^s$ for all $0 \leq s \leq \min\{m, 6\}$ then the set of parameter values for which the individual kinematic mappings are 1-generic

$$\{b \in B : j^1 F_b \pitchfork \Sigma^s, 0 \leq s \leq \min\{m, 6\}\}$$

is open and dense in the parameter space B.

3 One-Genericity for Serial Manipulators

For a serial manipulator, the kinematic mapping as a function of the joint variables $\theta_1, \ldots, \theta_m$ can be written as a product of exponentials of the screws $X_1, \ldots, X_m$ representing the sequence of joints:

$$f(\theta_1, \theta_2, \ldots, \theta_k) = H.e^{\theta_1 X_1}.e^{\theta_2 X_2} \cdots e^{\theta_m X_m}. \tag{2}$$

$H \in SE(3)$ determines the home configuration, $\theta_1 = \cdots = \theta_m = 0$, of the end-effector and, when considering the mapping in the neighbourhood of a given point, we may choose end-effector and base coordinates so that H is the identity. In this representation, the design parameters of the manipulator are only implicitly available via the sequence of screws $X_1, \ldots, X_m$. The exponential function, $\exp(\theta X)$ or $e^{\theta X}$, represents the position of a link attached by the joint represented by screw X to the previous link, after the joint has moved through joint variable θ (either angle of rotation for an R- or H-joint, or translation for a P-joint) as an element of $SE(3)$. It can be explicitly calculated in matrix form if screws and elements of $SE(3)$ are given matrix representations by means of the usual series expansion for the exponential. In order to simplify exposition and concentrate on the most important case, from here on we set $m = 6$.

Non-commutativity of Euclidean isometries means that the exponentials in (2) cannot be reordered. The Lie bracket of a pair of screws is another screw defined as follows: if X_1, X_2 are represented in Plücker coordinates by $(\omega_i, \mathbf{v}_i)$, $i = 1, 2$ then

$$[X_1, X_2] = (\omega_1 \wedge \omega_2, \omega_1 \wedge \mathbf{v}_2 - \omega_2 \wedge \mathbf{v}_1)$$

where $\wedge$ is the standard vector product in $\mathbb{R}^3$. One way to think of the Lie bracket is that it measures the infinitesimal difference between the effect of a pair of screws or joints if one reverses the order in which they occur. The finite, rather than infinitesimal,consequence for exponentials is expressed in the formula of Baker–Campbell–Hausdorff; see, for example, [12]. For sufficiently small values of the joint variables, multiple applications of the Baker–Campbell–Hausdorff formula convert (2) to a single exponential in terms of the Lie brackets of pairs of joint screws:

$$f(\theta_1, \ldots, \theta_6) = \exp\left(\sum_{i=1}^{6} \theta_i X_i + \frac{1}{2} \sum_{1 \leq i < j \leq 6} \theta_i \theta_j [X_i, X_j] + O(3)\right) \tag{3}$$

where the coefficients of the order 3 and higher terms in the θ_is involve nested brackets of the X_is. The kinematic mapping encompasses the full capability of the end-effector motion; to obtain elementary kinematic notions such as velocity and acceleration one considers a time-dependent path in the jointspace so that each joint variable is dependent on time t. Formulae for the velocity and acceleration that can be found in, for example, [8, 12] can be obtained by differentiating with respect to t, using the Chain Rule and the fact that $(d/dt)\exp(\theta(t)X)|_{t=0} = \dot{\theta}(0)X$. Higher-order derivatives are also considered in [9].

Since the exponential map on a Lie group provides a coordinate system on a neighbourhood of the identity, the kinematic mapping with respect to these exponential coordinates can be represented by the expression within the exponential in (3). From this we are able to derive the following expression for the 1-jet extension, see [2]:

$$j^1 f(\theta) = \Bigg(\theta, \sum_{i=1}^{6} \theta_i X_i, X_1 + \tfrac{1}{2}\sum_{j=2}^{6}\theta_j [X_1, X_j], \ldots,$$
$$X_l - \tfrac{1}{2}\sum_{j=1}^{l-1}\theta_j [X_l, X_j] + \tfrac{1}{2}\sum_{j=l+1}^{6}\theta_j [X_l, X_j], \ldots, X_6 - \tfrac{1}{2}\sum_{j=1}^{5}\theta_j [X_6, X_j]\Bigg) + O(2) \quad (4)$$

where $\theta = (\theta_1, \ldots, \theta_6)$.

From this we can deduce the following conditions for transversality to the singularity strata Σ^s. This represents a different derivation and clarification of the result of Lerbet and Hao [6], Appendix A.

Theorem 3.1 *Suppose f is a serial manipulator kinematic mapping given by (2). Necessary and sufficient conditions for $j^1 f$ to be transverse to Σ^s at $\theta_i = 0$, $i = 1, \ldots, 6$ are as follows:*

(a) If $0 \in \Sigma^1 f$, let $\mathbf{c} \in \mathbb{R}^6$ span the kernel of $Df(0)$. Then the vectors

$$X_1, \ldots, X_6, [c_1X_1, X_2],\ [c_1X_1 + c_2X_2, X_3], \ldots, \left[\textstyle\sum_{i=1}^4 c_iX_i, X_5\right]$$

span $\mathbb{R}^6$.

(b) If $0 \in \Sigma^2 f$, let $\mathbf{c}, \mathbf{d}$ span the kernel of $DF(0)$. Then the 6×2 matrices (where a $|$ separates columns)

$$(X_i \,|\, X_j), \quad (1 \le i < \cdots < j \le 6),$$
$$([c_1X_1, X_2] \,|\, [d_1X_1, X_2]),\ ([c_1X_1 + c_2X_2, X_3] \,|\, [d_1X_1 + d_2X_2, X_3]), \ldots$$
$$\ldots, \left(\left[\textstyle\sum_{i=1}^4 c_iX_i, X_5\right] \,\middle|\, \left[\textstyle\sum_{i=1}^4 d_iX_i, X_5\right]\right)$$

span $M(2,6)$.

(c) $\Sigma^3 f$ is empty for $s \ge 3$.

Proof. The matrices of second-order partial derivatives required for determining transversality, described in Section 2, are found by differentiating with respect to each θ_i, $i = 1, \ldots, k$ the components in (4) that are in the fibre of the jet bundle. They are

$$\Upsilon_l = \tfrac{1}{2}\left([X_1, X_l] \cdots [X_{l-1}, X_l]\ \mathbf{0}\ -[X_{l+1}, X_l] \cdots -[X_6, X_l]\right) \quad (5)$$

for $l = 1, \ldots, 6$. These must be restricted to the kernel of the Jacobian matrix. Suppose $\mathbf{c} \in \ker Df(0)$ then

$$Df(0)\mathbf{c} = \sum_{i=1}^{6} c_i X_i = \mathbf{0} \tag{6}$$

while

$$\Upsilon_l \mathbf{c} = \frac{1}{2}\left[c_1 X_1 + \cdots + c_{l-1} X_{l-1} - c_{l+1} X_{l+1} - \cdots - c_6 X_6, X_l\right]. \tag{7}$$

Adding $\frac{1}{2}\times$(6) to (7) and using the fact that $[X_l, X_l] = 0$ gives

$$\Upsilon_l \mathbf{c} = [c_1 X_1 + \cdots + c_{l-1} X_{l-1}, X_l] \tag{8}$$

and the theorem follows from the general condition for transversality to the singularity strata in Section 2.

Notes.

1. It is important to observe that this is only a local condition for transversality—it does not guarantee the global property of one-genericity unless it applies at every singular configuration of a serial manipulator. To apply the result in that way requires the presentation of the kinematic mapping to be re-calibrated so that the joint screws $X_1, \ldots, X_6$ are those in the current singular configuration. Although they are not the same as those at another configuration, they are related.
2. A different formulation of the theorem could be given by subtracting $\frac{1}{2}\times$(6) from (7) to obtain
$$\Upsilon_l \mathbf{c} = [-c_{l+1} X_{l+1} - \cdots - c_6 X_6, X_l]. \tag{9}$$
3. The theorem can be generalised readily to cases where $m \neq 6$.
4. The theorem can be generalised to the case of parametrised families of kinematic mappings $F : M \times B \to SE(3)$. The function Φ as in (1) can be written in the form (4), with the parameters $u_1, \ldots, u_k$ in B appearing explicitly in the formula. Then for transversality at $(0,0) \in M \times B$, in addition to the matrices in Theorem 3.1, we can also use the k matrices arising from the fibre components of $\partial\Phi/\partial u_i$, $i = 1, \ldots, k$ restricted to the kernel of $DF_0(0)$.

In [6], the authors observed (Proposition 4.4) that a necessary condition for transversality to Σ^1 follows from Theorem 3.1(a): the vector subspace Δ_1 spanned by the screws $X_1, \ldots, X_6$ together with their pairwise brackets $[X_i, X_j]$, $1 \leq i < j \leq 6$ must be the whole 6-dimensional screw space, usually denoted $\mathfrak{se}(3)$. In particular, the subalgebra Δ generated by $X_1, \ldots, X_6$ must be the whole of $\mathfrak{se}(3)$. In the case $m = 6$ these conditions are trivial since then $X_1, \ldots, X_6$ span a 5-space, but there are no subalgebras of dimension 5 so the conditions are automatically satisfied. However the conditions are not sufficient, as can be seen by considering the case where the screw dependency is defined by $c_1 X_1 + c_3 X_3 = 0$ and the bracket $[X_1, X_2] = 0$. In this case all the brackets in Theorem 3.1(a) vanish so the screws cannot span $\mathfrak{se}(3)$.

Note that Δ is an invariant of the manipulator [6] and in the case that it is a proper subalgebra of $\mathfrak{se}(3)$, it is sensible to regard its corresponding subgroup as the range of the kinematic mapping so that one considers, for example, a translational or a rotational manipulator (see for example [6, 11]). In that case of course the dimensions of the singularity strata change.

In general, there is a terminating nested sequence of vector subspaces

$$\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_r = \Delta$$

where Δ_0 is the subspace spanned by $X_1, \ldots, X_m$ and $\Delta_{i+1} = \Delta_i + [\Delta_0, \Delta_i]$ for $i \geq 0$ (see [5, 12]). It is fairly easy to show that not only is the sequence independent of the spanning set of screws for Δ_0, so it depends only on the screw system spanned by them, moreover the sequence is invariant up to conjugation (i.e. under the adjoint action of the Euclidean group) and hence the necessary condition $r = 1$ for transversality to Σ^1 can be determined from the Gibson–Hunt class of the screw system [3]. For example, if $m = 4$ and there is a singular configuration at which the 4 joint screws $X_1, \ldots, X_4$ span a 3-system of type IB_2 with pitch modulus $h_\beta = 0$, so that a spanning set of screws, in Plücker coordinates, is

$$(1,0,0,0,0,0), \quad (0,1,0,0,0,0), \quad (0,0,0,1,0,0)$$

then $\Delta_r = \mathfrak{se}(3)$ for $r = 2$ but not $r = 1$. It follows that in such a configuration the serial manipulator kinematic mapping is *not* transverse to Σ^1.

4 Conclusions

A straightforward condition for testing for manipulator genericity has been presented. The conditions in Theorem 3.1 would seem to lend themselves to further analysis of genericity for families of serial manipulators and the formula for the jet extension of a kinematic mapping can be extended to consider higher-order singularities and hence to the study of bifurcation sets such as those separating families of cuspidal and non-cuspidal regional manipulators.

References

1. Burdick, J.W. (1995) A classification of 3R regional manipulator singularities and geometries, *Mechanism and Machine Theory*, vol. 30 no. 1, pp. 71–89
2. Donelan, P.S. (2007) Singularity-theoretic methods in robot kinematics, *Robotica*, vol. 25, no. 6, pp. 641–659
3. Gibson, C.G. and Hunt, K.H. (1990) Geometry of screw systems I & II, *Mechanism and Machine Theory*, vol. 25, no. 1, pp. 1–27.
4. Golubitsky, M. and Guillemin, V. (1973) *Stable Mappings and Their Singularities*, Springer Verlag, New York
5. Hao, K. (1998) Dual number method, rank of a screw system and generation of Lie subalgebras, *Mechanism and Machine Theory*, vol. 33, no. 7, pp. 1063–1084
6. Lerbet, J. and Hao, K. (1999) Kinematics of mechanisms to the second order—application to the closed mechanisms, *Acta Applicandae Mathematicae*, vol. 59, no. 1, pp. 1–19

7. Levine H.I. (1971) Singularities of differentiable manifolds *Proc. Liverpool Singularities Symposium 1, Lecture Notes in Mathematics*, vol. 192, pp. 1–89, Berlin, Springer Verlag
8. Rico Martínez J. M. and Duffy J. (1996) An application of screw algebra to the acceleration analysis of serial chains, *Mechanism and Machine Theory*, vol. 31, no. 4, pp. 445–457.
9. Müller A. and Maißer P. (2003) A Lie-group formulation of kinematics and Dynamics of constrained MBS and its application to analytical mechanics, *Multibody System Dynamics*, vol. 9, pp. 311–352.
10. Narasimhan S. and Kumar V. (1994) A Second Order Analysis of Manipulator Kinematics in Singular Configurations *Proc. 23rd Biennial Mechanisms Conference, Minneapolis 1994*, pp. 477–484, ASME
11. Pai, D.K. and Leu, M.C. (1992) Genericity and singularities of robot manipulators, *IEEE Trans. Robotics and Automation*, vol. 8, no. 5, pp. 545–559
12. Selig, J.M. (2005), *Geometric Fundamentals of Robotics*, New York, Springer.
13. Tsai, K.Y., Arnold, J. and Kohli, D. (1993) Generic maps of mechanical manipulators, *Mechanism and Machine Theory*, vol. 28, no. 1, pp. 53–64