Transversality and its applications to kinematics

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Abstract. Transversality is a mathematical concept, widely used in singularity theory, which generalises the idea of regularity of a function. One of its uses is in providing a method for certifying that a property of a given set of functions or mappings is *generic*, which is to say that it holds for almost all members of that set. Its application in kinematics is illustrated via a prototype for showing that in a given class of parallel mechanisms, for almost all choices of design parameter the configuration space is non-singular and hence a manifold.

Keywords: Transversality, kinematic mapping, design parameter, genericity

1 Introduction

For a given robot mechanism architecture, actual models are determined by fixing a number of design parameters. For serial mechanisms, the Denavit–Hartenberg parameters are widely used while the product-of-exponentials formulation [2] provides an alternative parametrisation in terms of screw coordinates. Among parallel mechanisms, the simplest examples are planar 4-bars, for which the design parameters are the link lengths, while more complex examples such as a planar 3-RRR have higher-dimensional spaces of design parameters [1]. It is reasonable to ask then how the design parameters affect the kinematics of the mechanism. This may involve determining whether certain properties are *generic*, which is to ask whether the property holds for almost all choices of design parameter. At a more refined level, one may look for bifurcation conditions. These correspond to non-generic choices of parameter across which there may be a significant change of behaviour. The classic case is the condition of Grashof [6] which distinguishes between planar 4-bars in terms of rotatability of links and, implicitly between those whose configuration space (C-space) has one or two connected components [4, 11]. It is worth noting, however, that the requirement of symmetry or generation of special motion may be precisely what renders a choice of design parameters non-generic so that, from the engineering point of view, genericity may *not* be a desirable characteristic.

Transversality is a mathematical concept, generalising that of regularity, by which we can seek to answer such questions in general. In particular, establishing transversality for functions with respect to certain submanifolds will enable us deduce not only genericity results but also information about the dimensions of a hierarchy of singularity sets. For example, if transversality could be verified for a kinematic mapping with respect to submanifolds that encode the rank (or its complement, the corank) of derivatives—in effect, Jacobian matrices of partial derivatives—then the set of configurations at which there is a singularity of a given corank will be a submanifold of specified dimension. Although not used extensively, these ideas have appeared already in the kinematics literature in the work of Pai [9], Tchoń [10] and others. More recently, applications of transversality to the analysis of singularities of mechanisms and related notions of genericity have been explored [3, 8].

In this paper, we first give a brief introduction to transversality and some of the important theorems it gives rise to. To illustrate its application to kinematics we review the classical planar 4R mechanism. More general consideration of Grashof-type conditions, based on transversality considerations is then illustrated in both negative and positive senses by the extended Watt 7R mechanism, which has two degrees of freedom.

2 Transversality of Mappings

The appropriate mathematical language for transversality is that of differential topology (see, for example, [7]). The objects are (differentiable) manifolds—spaces that are everywhere locally parametrised by a fixed number of parameters, the *dimension* of the manifold—and differentiable functions or mappings between them. This encompasses the familiar spaces in kinematics such as Lie groups (the Euclidean group) and circles, spheres and their products including tori.

We start with the definition of regularity of a given function at a point of its domain:

Definition 1. *Given a differentiable function $f : M \to N$ between manifolds and a point $y \in N$. If for every $x \in M$ for which $f(x) = y$ the derivative df_x of f at x has maximum rank then y is called a regular value and each such x is called a regular point.*

Note that, for computational purposes, the derivative can be represented by a Jacobian matrix $Jf(x)$, which can be computed with respect to any choice of parametrisation and whose rank is invariant. Properly, we are considering the underlying linear map that best approximates f in a neighbourhood of x; the Jacobian is the matrix representation of this linear map with respect to choices of basis for the tangent spaces at x and y.

A fundamental theorem of topology states that if y is a regular value, so that the image of T_xM under the Jacobian is the whole of $T_{f(x)}N$, then $P = f^{-1}(y) \subset M$ is a submanifold (or possibly empty). Moreover, the codimension of P is the dimension of N, or equivalently $\dim(P) = \dim(M) - \dim(N)$. This generalises the familiar result of linear algebra concerning the dimension of the solution space of an equation $A\mathbf{x} = \mathbf{b}$.

A key motivation for transversality is to generalise this situation to $f^{-1}(Q)$ where $Q \subset N$ is a submanifold. The idea is that for regularity, the image of the Jacobian $Jf(x)$ must be the whole of $T_{f(x)}N$, the tangent space at the image $f(x) = y$. Since T_yQ already fills up a subspace of T_yN, we now only need the image of the Jacobian $Jf(x)$, where $f(x) \in Q$, to span a *complement* to T_yQ.

Definition 2. *Define f to be transverse to Q (written $f \pitchfork Q$) if for all $x \in M$ such that $f(x) \in Q$,*

$$df_x(T_xM) + T_{f(x)}Q = T_{f(x)}N \tag{1}$$

In this case, an extension of the previous theorem [7] states that $P = f^{-1}(Q)$ is a submanifold of M and $\operatorname{codim} P = \operatorname{codim} Q$.

A point $y \in N$ which is the image of at least one point $x \in M$ is a *critical value* of f if it is not a regular value. So for at least some point $x \in M$, with $f(x) = y$, rank of $Jf(x) < \min(\dim M, \dim N)$. Critical points are scarce in the following sense [7]:

Theorem 1 (Sard's Theorem). *The set of critical values of $f : M \to N$, which is a subset of N, has measure zero.*

Here, a subset of a manifold has *measure zero* if it can be covered via parametrisations by at most a countably infinite union of hypercubes in $\mathbb{R}^n$ whose total volume can be made arbitrarily small. The idea of the proof is that in the neighbourhood of any critical point, f reduces volumes by at least an order of magnitude—think of the image of an interval $[-r, r]$ under the function $f(x) = x^2$. A property of points in a set that is true except on a set of measure zero is said to hold *almost everywhere*. So the theorem states that for almost all $y \in N$, y is a regular value. A corollary of Sard's Theorem is the following [7]:

Theorem 2 (Parametric Transversality Theorem). *Given a family of functions $F : M \times D \to N$, where D is a manifold of parameters, then for each fixed $d \in D$ there is a function $F_d : M \to N$ where, for $x \in M$, $F_d(x) = F(x, d)$. If $Q \subset N$ is a submanifold and $F \pitchfork Q$ then for almost all $d \in D$, $F_d \pitchfork Q$.*

This has direct interpretations in kinematics. We could take M as the joint space for a family of mechanisms, D as the space of design parameters and N the workspace, for example the Euclidean group. In this case, Q may be a submanifold that characterises some desired geometry for the end-effector or platform such as that it has a fixed orientation.

Alternatively, in the terminology of kinematic constraint maps (KCMs) [1], let M be the space of pose parameters for a class of mechanisms with $\dim M = m$, D as the space of design parameters with $\dim D = \delta$ and $N = \mathbb{R}^c$ the constraint space, where c is the total number of constraint equations. Here, the role of Q is more limited as there is typically only one relevant choice of values (usually zero) for each equation so that $Q = \{\mathbf{0}\}$.

This corresponds to a special case of Theorem 2, when $Q = \{y_0\} \subset N$, that is a single point or 0-dimensional submanifold. In this case, $F \pitchfork Q$ if and only if y_0 is a regular value of F, since the second term in the defining condition 1 reduces to $\{0\}$. That is, the Jacobian has maximum rank for all $(x, d) \in M \times D$ for which $F(x, d) = y_0$. In this case, we can conclude that for almost all $d \in D$:

$$C_d := F_d^{-1}(y_0) \subset M \tag{2}$$

is a C^∞ submanifold with $\dim C_d = m - c$ (or else an empty set). In the context of KCMs, C_d is the configuration space for the choice of design parameters, $d \in D$. We illustrate this result in the following sections.

3 Planar 4R linkage

To illustrate the application of Theorem 2 we consider the property 'that the configuration space for a given class of mechanism be a manifold' is generic. The simplest interesting case is the planar 4R mechanism, illustrated in Fig. 1. We follow the approach of [1].

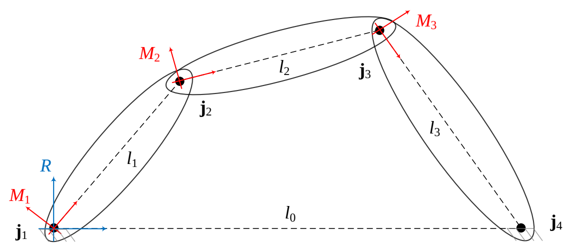

Fig. 1: 4-bar planar mechanism

There are three mobile links whose coordinates are given by moving frames $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and a fixed link whose coordinates are expressed in the reference frame $\mathcal{R}$. The displacement of each moving frame with respect to the reference frame is represented by pose parameters $(\theta_i, x_i, y_i)^T, i = 1, \ldots, 3$. There are four design parameters $l_i, i = 0, \ldots, 3$ whilst the four revolute joints each impose two constraint equations. As a result, the KCM has the form $F : SE(2)^3 \times \mathbb{R}^4 \to \mathbb{R}^8$:

$$F(\theta_i, x_i, y_i, l_j)_{i=1,\ldots 3}^{j=0,\ldots 3} = [x_1, y_1, l_1c_1 + x_1 - x_2, l_1s_1 + y_1 - y_2, \\ l_2c_2 + x_2 - x_3, l_2s_2 + y_2 - y_3, l_3c_3 + x_3 - l_0, l_3s_3 + y_3]^T \tag{3}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i, i = 1, 2, 3$. The extended Jacobian of F with respect to both pose and design parameters is an 8×13 matrix. By row reduction and deletion of rows and columns with a leading 1, this has the same rank as the reduced 6×11

matrix:

$$JF^{\mathrm{red}} = \begin{bmatrix} 1\,0\,0\,0\,0 & l_1 s_1 & 0 & 0 & -c_1 & 0 & 0 \\ 0\,1\,0\,0\,0 & -l_1 c_1 & 0 & 0 & -s_1 & 0 & 0 \\ 0\,0\,1\,0\,0 & l_1 s_1 & l_2 s_2 & 0 & -c_1 & -c_2 & 0 \\ 0\,0\,0\,1\,0 & -l_1 c_1 & -l_2 c_2 & 0 & -s_1 & -s_2 & 0 \\ 0\,0\,0\,0\,1 & l_1 s_1 & l_2 s_2 & l_3 s_3 & -c_1 & -c_2 & -c_3 \\ 0\,0\,0\,0\,0 & -l_1 c_1 & -l_2 c_2 & -l_3 c_3 & -s_1 & -s_2 & -s_3 \end{bmatrix} \tag{4}$$

This clearly has full rank since the last row cannot be identically zero since there is no solution to the simultaneous equations $\sin\theta = \cos\theta = 0$.

We have thus established that $F \pitchfork \{\mathbf{0}\}$ and thus for almost all $d \in \mathcal{D}$ the configuration space C_d has no singularity. Of course, in this case we already know that C_d is non-singular unless a certain equation relating to the Grashof condition holds for the design parameters, i.e. $l_0 \pm l_1 \pm l_2 \pm l_3 = 0$. This tells us directly that C_d is non-singular except on the closed subset of the positive octant defined by the union of these linear equations.

As an aside, we note that this Grashof-type condition can be found by solving the constraint equations $F = 0$ along with the equations that determine positive corank for its Jacobian JF, namely that all the degree 1 minors vanish simultaneously. Since all the equations are algebraic in x_i, y_i, c_i, s_i, $i = 1, 2, 3$ and subject to the additional conditions $c_i^2 + s_i^2 = 1$, we may use Gröbner bases to eliminate the pose parameters and obtain:

$$\begin{aligned} & l_0^8 - 4l_0^6 l_1^2 + 6l_0^4 l_1^4 - 4l_0^2 l_1^6 - 4l_0^6 l_2^2 + 4l_0^4 l_1^2 l_2^2 + 4l_0^2 l_1^4 l_2^2 + 6l_0^4 l_2^4 + 4l_0^2 l_1^2 l_2^4 \\ & \quad - 4l_0^2 l_2^6 - 4l_0^6 l_3^2 + 4l_0^4 l_1^2 l_3^2 + 4l_0^2 l_1^4 l_3^2 + 4l_0^4 l_2^2 l_3^2 - 40 l_0^2 l_1^2 l_2^2 l_3^2 + 4l_0^2 l_2^4 l_3^2 + 6l_0^4 l_3^4 \\ & \quad + 4l_0^2 l_1^2 l_3^4 + 4l_0^2 l_2^2 l_3^4 - 4l_0^2 l_3^6 + l_1^8 - 4l_1^6 l_2^2 + 6l_1^4 l_2^4 - 4l_1^2 l_2^6 + l_2^8 - 4l_1^6 l_3^2 + 4l_1^4 l_2^2 l_3^2 \\ & \quad + 4l_1^2 l_2^4 l_3^2 - 4l_2^6 l_3^2 + 6l_1^4 l_3^4 + 4l_1^2 l_2^2 l_3^4 + 6l_2^4 l_3^4 - 4l_1^2 l_3^6 - 4l_2^2 l_3^6 + l_3^8 = 0 \end{aligned} \tag{5}$$

which factorises as:

$$\begin{aligned} & (l_0 - l_1 - l_2 - l_3)(l_0 + l_1 - l_2 - l_3)(l_0 - l_1 + l_2 - l_3)(l_0 + l_1 + l_2 - l_3) \\ & \quad (l_0 - l_1 - l_2 + l_3)(l_0 + l_1 - l_2 + l_3)(l_0 - l_1 + l_2 + l_3)(l_0 + l_1 + l_2 + l_3) = 0 \end{aligned} \tag{6}$$

or, simply, the product of the linear Grashof-type conditions.

This union of 8 hyperplanes has codimension 1 as an algebraic variety in the design space. Furthermore, the factorisation yields non-generic mechanisms that includes some overconstrained cases. Consider the case $l_0 - l_1 - l_2 - l_3 = 0$, for instance; it is has a single shaky (infinitesimally mobile) configuration but is immobile.

4 Grashof-type conditions

It is reasonable to pose the question whether we can find Grashof-type conditions for singularity in the configuration space. We can undertake some dimensional analysis to indicate what is likely.

The set of points where the corank of the Jacobian matrix of a mapping $F : M \to N$ between manifolds is positive is called its *critical set*, denoted as follows:

$$\Sigma F := \{x \mid \operatorname{rank} JF(x) < \min(\dim M, \dim N)\} \tag{7}$$

This may be further refined by considering the sets of matrices of a fixed corank. Set $\dim M = m$, $\dim N = n$ and let us suppose that $m \geq n$; we are interested in the map

$JF : M \to L(\mathbb{R}^m, \mathbb{R}^n)$ where the codomain is the set of linear functions between $\mathbb{R}^m$ and $\mathbb{R}^n$, i.e. $n \times m$ matrices. Define: $\Sigma^r = \{A \in L(\mathbb{R}^m, \mathbb{R}^n) \mid \operatorname{corank} A = r\}$. It can be shown [5] that these sets are submanifolds satisfying:

$$\operatorname{codim}(\Sigma^r) = r(m - n + r) \tag{8}$$

In the case that $JF \pitchfork \Sigma^r$, we would have $\Sigma^r F = JF^{-1}(\Sigma^r)$ a submanifold of M of the same codimension; in the simplest case, $r = 1$, this is $m - n + 1$.

Let us apply this to the setting of kinematic constraint maps. Here $F : M \times D \to \mathbb{R}^c$. Under the assumption of transversality on F, the *total* configuration space $C = F^{-1}(0) \subset M \times D$ has codimension c. For each $d \in D$ we have $C_d = C \cap (M \times \{d\})$, with the same codimension (provided it is a submanifold). Each C_d may have singularities and the simplest singularity set $\Sigma^1 F_d$ has codimension $m - c + 1$. Since the sum of the codimensions of C_d and $\Sigma^1 F_d$ in M is $(m - c + 1) + c > m$ we would not typically expect a given C_d to have any singularities.

On the other hand, in the total configuration space, the singularity set of interest is $\Sigma_1^1 F = J_1 F^{-1}(\Sigma^1)$, where $J_1 F : M \times D \to L(\mathbb{R}^m, \mathbb{R}^c)$ is the Jacobian map which treats D as the parameter space rather than as variables to be differentiated. Transversality then gives $\operatorname{codim} \Sigma_1^1 F = m - c + 1$. Under the additional assumption that C and $\Sigma_1^1 F$ meet transversely, their intersection is a submanifold satisfying:

$$\operatorname{codim}(C \cap \Sigma_1^1 F) = \operatorname{codim} C + \operatorname{codim} \Sigma_1^1 F = m + 1. \tag{9}$$

In other words, in $M \times D$, the singularity set is defined by $m + 1$ simultaneous (non-linear) equations. If it is possible to eliminate the m pose parameters we should arrive at just one equation in the d design parameters. If such a condition can be found, we call it the *Grashof-type condition* for the class of mechanisms. Note that there should be a single condition, independent of the number of links, joints or design parameters. This, of course, concurs with what we described in Section 3.

5 Extended Watt mechanism

As shown in Fig. 2, this mechanism consists of six mobile links which can be considered as a combination of a planar 4-bar and a planar 5-bar mechanism that share common bodies 2 and 3. The frames $\mathcal{M}_i, i = 1, \ldots 6$ are assigned to the moving bodies whose poses are described by the parameters (θ_i, x_i, y_i), $i = 1, \ldots, 6$ measured in the reference frame $\mathcal{R}$ attached to fixed link. There are 11 design parameters $a_i, i = 0, \ldots, 6$, b_2, b_3, and α, β. It is assumed that all $a_i, b_i > 0$. Eight revolute joints each impose two constraint equations, and so the KCM has the form $F : SE(2)^6 \times \mathbb{R}^{11} \to \mathbb{R}^{16}$ given by:

$$\begin{aligned}
[x_1, y_1, & -a_0 + x_2, y_2, -x_1 + x_3 - a_1 c_1, -y_1 + y_3 - a_1 s_1 - x_2 + x_4 - a_2 c_2, \\
& - y_2 + y_4 - a_2 s_2, -x_3 + x_5 - b_3 c_{\beta,3}, -y_3 + y_5 - b_3 s_{\beta,3}, -x_4 + x_6 - a_4 c_4, \\
& - y_4 + y_6 - a_4 s_4, -x_5 + x_6 - a_5 c_5 + a_6 c_6, -y_5 + y_6 - a_5 s_5 + a_6 s_6, \\
& -x_2 + x_3 - b_2 c_{\alpha,2} + a_3 c_3, -y_2 + y_3 - b_2 s_{\alpha,2} + a_3 s_3]^T .
\end{aligned} \tag{10}$$

Here $c_{\alpha,2} = \cos(\alpha + \theta_2)$, $c_{\beta,3} = \cos(\beta + \theta_3)$ and similarly for sines. For a fixed choice d of design parameters, we would anticipate the configuration space $C_d = F_d^{-1}(0)$ to have dimension 2, corresponding to its mobility.

In order to determine whether this family of mechanisms is generic in the sense that its configuration space is a 2-dimensional manifold for almost all choices of design

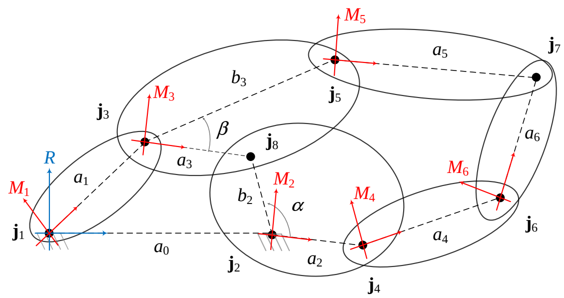

Fig. 2: Watt 7-bar mechanism

parameters, it is necessary to check whether the 16×29 Jacobian matrix of F (with respect to *all* parameters) drops rank. After some relevant row and column reductions, the matrix can be reduced to 3×16 as follows:

$$\begin{bmatrix} a_1c_1 & -a_2c_2 & b_3c_{\beta,3} & -a_4c_4 & a_5c_5 & -a_6c_6 & s_1 & s_{\beta,3} & b_3c_{\beta,3} & 0 & 0 & -s_2 & -s_4 & s_5 & 0 & -s_6 \\ 0 & a_2s_2 - b_2s_{\alpha,2} & a_3s_3 - b_3s_{\beta,3} & a_4s_4 & -a_5s_5 & a_6s_6 & 0 & c_{\beta,3} & -b_3s_{\beta,3} & c_{\alpha,2} & -b_2s_{\alpha,2} & -c_2 & -c_4 & c_5 & -c_3 & -c_6 \\ -a_1c_1 & b_2c_{\alpha,2} & -a_3c_3 & 0 & 0 & 0 & -s_1 & 0 & 0 & s_{\alpha,2} & b_2c_{\alpha,2} & 0 & 0 & 0 & -s_3 & 0 \end{bmatrix} \tag{11}$$

Given the assumptions on design parameters, this matrix is never rank deficient so that $F \pitchfork \{\mathbf{0}\}$, so that the total configuration space C is a submanifold of codimension 16.

Turning to the singularities, the Jacobian J_1F, after some relevant row/column reductions, reduces to a 4×6 matrix whose rank is the same:

$$\begin{bmatrix} a_1s_1 & -a_2s_2 & b_3s_{\beta,3} & -a_4s_4 & a_5s_5 & -a_6s_6 \\ -a_1c_1 & a_2c_2 & -b_3c_{\beta,3} & a_4c_4 & -a_5c_5 & a_6c_6 \\ -a_1s_1 & b_2s_{\alpha,2} & -a_3s_3 & 0 & 0 & 0 \\ a_1c_1 & -b_2c_{\alpha,2} & a_3c_3 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

Of the 15 4×4 minors, 12 are non-trivial and can be solved simultaneously using Mathematica or Maple to give distinct solutions determining the set $\Sigma_1^1 F$. From Eq. 8, it is expected that this set have codimension 3. However, one of the possible solutions is given by:

$$\theta_2 = \theta_1 - \alpha + n_2\pi, \qquad \theta_3 = \theta_1 + n_3\pi; \qquad n_2, n_3 \in \mathbb{Z} \tag{13}$$

defining submanifolds of codimension 2 in the space of pose parameters. Although there are other singular branches with various codimensions, this solution set relates simply to the base 4-bar linkage visible in Fig. 2. This tells us that there is a branch of singularities along which the Jacobian J_1F is not transverse to Σ^1. Nevertheless, since the conditions only relate to the 4-bar, we may still obtain the Grashof-type conditions:

$$a_0 \pm a_1 \pm b_2 \pm a_3 = 0. \tag{14}$$

as illustrated in Fig. 3. As mentioned, there are other singular branches, the most relevant one being of the form:

$$\theta_1 = g(\theta_2, \theta_3, \theta_4), \qquad \theta_5 = \theta_4 + n_5\pi, \qquad \theta_6 = \theta_4 + n_6\pi, \tag{15}$$

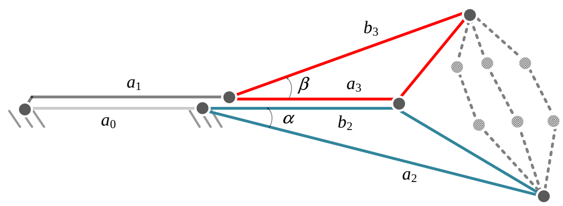

Fig. 3: Watt 7-bar $\mathcal{C}$-space singularity associated to the 4-bar at a flat configuration

where g is specific function. Given this branch has the correct codimension, it is likely that it will give rise to a corresponding Grashof-type condition but the form of this has not yet been determined. One special case of Eq. 15 is given by adding conditions on the remaining pose parameters:

$$\theta_i = \theta_1 + \eta_i \pi, \quad i \in \{2,4,5,6\}, \quad \eta_i \in \mathbb{Z}; \qquad \theta_3 = \theta_1 - \beta + n\pi. \tag{16}$$

These correspond to the configuration where all the mobile links (i.e. all the x-axes of the moving frames) are parallel. Eliminating pose parameters leads to a collection of three Grashof-type conditions on the design parameters as follows:

$$a_0 \pm a_1 \pm a_2 \pm a_4 \pm a_5 \pm a_6 \pm b_3 = 0; \; a_0 \pm a_1 \pm b_2 c_\alpha \pm a_3 c_\beta = 0; \; b_2 s_\alpha \pm a_3 s_\beta = 0 \tag{17}$$

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