Invariant Properties of the Denavit–Hartenberg Parameters

Mohammed Daher and Peter Donelan

Abstract The Denavit–Hartenberg (DH) notation for kinematic chains makes use of a set of parameters that determine the relative positions of and between successive joints. The corresponding matrix representation of a chain's kinematics is a product of two exponentials in the homogeneous representation of the Euclidean group. While the DH notation is based on sound kinematic intuition, it is not obviously natural in mathematical terms. In this paper, we use the principle of transference to determine fundamental algebraic (polynomial) invariants of the Euclidean group $SE(3)$ acting on sets of twists, elements of the group's Lie algebra, $\mathfrak{se}(3)$, representing joints, and show that the DH parameters are algebraic functions of these invariants. We make use of the fact that for a set of three twists, there is an algebraic–geometric duality with the corresponding set of Lie brackets, so that link lengths of one correspond to offsets of the other.

Key words: Euclidean group, Denavit–Hartenberg parameters, polynomial invariants, dual numbers

1 Introduction

The kinematics of serial chains are usually modelled either by the Denavit–Hartenberg notation [5] or by the product of exponentials notation. In fact, Brockett [1] showed how the two are related to one another. The key idea in both cases is that the motion of a link with respect to its neighbour in the chain is described by an exponen-

Mohammed Daher
School of Mathematics, Statistics & Operations Research, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand e-mail: Mohammed.Daher@msor.vuw.ac.nz

Peter Donelan
School of Mathematics, Statistics & Operations Research, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand e-mail: peter.donelan@vuw.ac.nz

tial matrix in terms of the joint parameter. This arises from the fact that the one degree-of-freedom joints—revolute, helical or prismatic—can be represented by a twist (or motor or screw). In mathematical terms, a twist is an element of the Lie algebra $\mathfrak{se}(3)$ of the Euclidean group $SE(3)$ of rigid-body transformations.

Twists can be represented in a variety of ways, but the most succinct is using Plücker coordinates [17]. We shall write a twist S as a pair of 3-vectors: $S = (\omega, \mathbf{v})$. These coordinates rely on a choice of spatial coordinate frame, so we would expect kinematic properties to be invariant under changes of coordinate. Mathematically, they should be invariants of the *adjoint action* Ad of the Euclidean group on its Lie algebra. The most fundamental of these is the *pitch* of a twist, which in terms of Plücker coordinates is the ratio:

$$h = \frac{\omega.\mathbf{v}}{\omega.\omega} \tag{1}$$

of the Klein form $\omega.\mathbf{v} = \omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3$ and Killing form $\omega.\omega = \omega_1^2 + \omega_2^2 + \omega_3^2$. Each of these is in fact an *invariant* polynomial $f(\omega, \mathbf{v})$ in the sense that for any $g \in SE(3)$:

$$f(Ad(g)(\omega, \mathbf{v})) = f(\omega, \mathbf{v}). \tag{2}$$

For a serial chain, we have more than one joint, and hence are interested in the invariants of a set of twists $S_1, \ldots, S_k$. Since the Euclidean group is algebraic, that is it can be represented as the zero set of polynomials, its polynomial invariants are of particular significance. Other invariant quantities, such as the pitch, may be expressed as rational or algebraic in terms of them. Our primary goal is to determine fundamental polynomial invariants for sets of twists and to show that the DH parameters can indeed be written in terms of them.

Our guide is the *principle of transference*, whose origins are in the work of Clifford and whose value was recognised by Kotelnikov [7] and, especially, Study [18]—see, for example, Chevallier [2], Rico and Duffy [12], Rooney, [13], Selig [17]. An algebraic version of the principle states that on replacing real coordinates by dual coordinates, valid statements about vectors in $\mathbb{R}^3$ become valid statements about twists, written as dual vectors $\omega + \varepsilon\mathbf{v}$, where ε is a quantity such that $\varepsilon^2 = 0$. Chevallier notes that this should not be regarded as a theorem, as there are exceptions to its application: it is a valuable generic guide.

We follow Study [18], Section 23, in making use of the principle by starting with invariants and syzygies of the rotation group $SO(3)$, acting on m copies of $\mathbb{R}^3$ [20], which we refer to as m-fold invariants. Dualisation in $SO(3)$ leads to dual invariant polynomials, whose real and dual parts are real invariants of the Euclidean group. A number of authors [6, 11, 15, 19] have explored invariants of the adjoint and co-adjoint action (the latter of importance in theoretical physics). Selig [17] also makes use of the principle of transference to derive invariants that correspond to those we obtain. In [3], the algebraic method of SAGBI bases is employed to find some of these invariants. The work of Takiff [19] has been extended by Panyushev [10], but in a theoretical algebraic–geometry setting, and he too obtains the dualised form of 2-fold invariants.

The group and dualisation are described in Sections 2, while its application to invariants is in Section 3. In Section 4, the Denavit–Hartenberg parameters are derived in terms of Plücker coordinates. Since three joints are required to define the offset, the expectation is that it can be written in terms of 3-vector polynomial invariants of the Euclidean group (using Weyl's term). To achieve such as expression we introduce in Section 5 the algebraic-geometric duality between a set of three screws and its associated set of Lie brackets.

2 The Euclidean group and dualisation

Let $\mathbb{D}$ denote the ring of *dual numbers* $a+\varepsilon b$, $a,b\in\mathbb{R}$ and $\varepsilon^2=0$ with componentwise addition, and multiplication defined in the obvious way. Note that $\mathbb{D}$ is not a field, as there are zero divisors and not every non-zero quantity has a multiplicative inverse, but is a 2-dimensional real associative algebra. In a dual number $a+\varepsilon b$, a is referred to as the *primal part* and b, the *dual part*. Modules of various sorts can be constructed by taking vectors and matrices of dual numbers; these can also be written as a sum of primal and dual parts.

The position of a link in a serial chain with respect to some reference position, in a given coordinate frame, is represented by an element of the Euclidean group $SE(3)$. The group is a (semi-direct) product of the orientation-preserving rotations $SO(3)$ and translations $\mathbb{R}^3$, and is a 6–dimensional Lie group [9, 17]. The rotation group $SO(3)$ is characterised by the following conditions on a 3×3 (real) matrix A:

$$AA^t=I,\quad \det A=1. \tag{3}$$

Replacing entries in A by dual numbers gives rise to a dual matrix $\hat{A}=A_0+\varepsilon A_1$ where A_0, A_1 are real 3×3 matrices, the primal and dual parts respectively. The same equations (3) determine a group $SO(3,\mathbb{D})$ [8]. Equating primal and dual parts we obtain:

$$A_0A_0^t=I,\quad (A_1A_0^t)^t=-A_1A_0^t,\quad \det(A_0)=1, \tag{4}$$

so that $A_0\in SO(3)$ and $A_1A_0^t$ is skew symmetric. Identifying this skew-symmetric matrix with a translation vector in $\mathbb{R}^3$ gives rise to an isomorphism between $SO(3,\mathbb{D})$ and $SE(3)$ that is at the heart of the principle of transference.

Correspondingly, the Lie algebra $\mathfrak{se}(3)$ of the Euclidean group is the sum of $\mathfrak{so}(3)$, the infinitesimal rotations and $\mathfrak{t}(3)$, infinitesimal translations. The algebra $\mathfrak{so}(3)$ can be represented by 3×3 skew-symmetric matrices but these in turn can be identified with elements of $\mathbb{R}^3$ in a standard way [9]. Dualising works here too, in that twists in$\mathfrak{se}(3)$ can be written as dual vectors $S=\omega+\varepsilon\mathbf{v}$, where the primal and dual parts $\omega,\mathbf{v}$ are the Plücker coordinates. Geometrically, a twist $S=\omega+\varepsilon\mathbf{v}$ can be interpreted as a vector field on $\mathbb{R}^3$ whose integral curves—the motion generated by the twist—are helices of pitch $h=\omega.\mathbf{v}/\omega.\omega$ about an axis with direction vector ω and moment $\omega\times\mathbf{q}=\mathbf{v}-h\omega$ about $\mathbf{0}$ (where $\mathbf{q}=(\mathbf{v}\times\omega)/\omega.\omega$ is a point on the axis [17]). Note that if $h=0$ then the the motion is revolute—the integral curves are circles centred on the axis and lying in planes orthogonal to it. If, on the other hand, $\omega=\mathbf{0}$, then the motion is translational and the integral curves are lines parallel to $\mathbf{v}$.

A change of coordinate frame corresponds to conjugation in the group and this gives rise to the Lie bracket, a bilinear, skew-symmetric product, in its Lie algebra [9]. Writing elements of $\mathfrak{so}(3)$ as 3-vectors $\omega_1, \omega_2 \in \mathbb{R}^3$, the Lie bracket is the standard vector product on $\mathbb{R}^3$, $[\omega_1, \omega_2] = \omega_1 \times \omega_2$ Dualising, so writing elements of $\mathfrak{se}(3)$ as dual vectors $S_i = \omega_i + \varepsilon \mathbf{v}_i$, $i = 1,2$, the Lie bracket is the dual vector product:

$$[S_1, S_2] = (\omega_1 + \varepsilon \mathbf{v}_1) \times (\omega_2 + \varepsilon \mathbf{v}_2) = \omega_1 \times \omega_2 + \varepsilon(\omega_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \omega_2). \tag{5}$$

The geometric interpretation of the Lie bracket $[S_1, S_2]$, in the generic case $\omega_1 \times \omega_2 \neq \mathbf{0}$, is as a twist whose axis is the common perpendicular to the axes of S_1, S_2 with twist h_{12} a function of the pitches h_i of S_i and the relative placement of the twists [16].

3 Invariants and dualisation

The standard action of the rotation group $SO(3)$ on $\mathbb{R}^3$ is the same as its adjoint action on $\mathfrak{so}(3)$. Weyl [20] gives a complete account of its vector polynomial invariants. They are of two types: given $\omega_1, \ldots, \omega_m \in \mathbb{R}^3 \cong \mathfrak{so}(3)$, every m-fold polynomial invariant is generated by (i.e. is a polynomial function of) the quadratic and cubic invariants

$$I_{ij} = \omega_i.\omega_j, \quad 1 \leq i, j \leq m; \qquad I_{ijk} = [\omega_i, \omega_j, \omega_k], \quad 1 \leq i < j < k \leq m \tag{6}$$

The latter is termed a *bracket* and is the determinant of the matrix whose columns are the three vectors. Note that these invariants are not algebraically independent but are linked by several types of *syzygies*, that is, polynomial relations. We shall only be concerned with $m \leq 3$ and in these cases the only syzygy occurs when $m = 3$. There are 7 invariants of type connected by a single syzygy

$$I_{123}^2 = \det(I_{ij}). \tag{7}$$

If $f(\omega_1, \ldots, \omega_m)$ is a polynomial in the coordinates of the m vectors, then replacing the real vectors by the dual vectors $\omega_i + \varepsilon \mathbf{v}_i$, $i = 1, \ldots, m$ and expanding by the binomial theorem gives the dual polynomial in differential form:

$$\hat{f}(\omega_1, \ldots, \omega_m, \mathbf{v}_1, \ldots, \mathbf{v}_m) = f(\omega_1, \ldots, \omega_m) + \varepsilon \cdot \sum_{r=1}^{m} \sum_{j=1}^{3} v_{rj} \frac{\partial f}{\partial \omega_{rj}}(\omega_1, \ldots, \omega_m). \tag{8}$$

It is straightforward to show that if f is an m-fold polynomial invariant of $SO(3)$, then the primal and dual parts of $\hat{f}$ are indeed m-fold invariants of the adjoint action of $SE(3)$. For the adjoint action itself, $SO(3)$ has the single generating invariant $I_{11} = \omega.\omega$ which dualises to give:

$$(\omega+\varepsilon\mathbf{v}).(\omega+\varepsilon\mathbf{v})=\omega.\omega+2\varepsilon\omega.\mathbf{v}. \tag{9}$$

Up to a multiple, the primal and dual parts I_{11} and $\hat{I}_{11}=\omega.\mathbf{v}$ are the familiar Killing and Klein forms whose ratio is the pitch of the twist $S=(\omega,\mathbf{v})$. For $m=2$, there are 6 quadratic invariants arising from dualisation of the $SO(3)$ invariants [3]. In the case $m=3$, there are 14 invariants arising from the primal and dual parts of the dualisations of the 7 generating 3-fold invariants for $SO(3)$:

$$\begin{aligned}
&I_{ii}=\omega_i\cdot\omega_i, && \hat{I}_{ii}=\omega_i\cdot\mathbf{v}_i, && i=1,2,3\\
&I_{ij}=\omega_i\cdot\omega_j && \hat{I}_{ij}=\omega_i\cdot\mathbf{v}_j+\mathbf{v}_i\cdot\omega_j, && 1\le i<j\le 3\\
&I_{123}=[\omega_1,\omega_2,\omega_3],\ \hat{I}_{123}=[\mathbf{v}_1,\omega_2,\omega_3]+[\omega_1,\mathbf{v}_2,\omega_3]+[\omega_1,\omega_2,\mathbf{v}_3]
\end{aligned} \tag{10}$$

The syzygy (7) can also be dualised and therefore gives rise to a pair of syzygies, the primal part being (7) and the dual part:

$$\begin{aligned}
I_{123}\hat{I}_{123}=&\hat{I}_{11}I_{22}I_{33}-\hat{I}_{11}I_{23}^2-\hat{I}_{22}I_{13}^2+\hat{I}_{22}I_{11}I_{33}+\hat{I}_{33}I_{11}I_{22}-\hat{I}_{33}I_{12}^2\\
&-\hat{I}_{12}I_{12}I_{33}+\hat{I}_{12}I_{13}I_{23}-\hat{I}_{13}I_{13}I_{22}+\hat{I}_{13}I_{12}I_{23}+\hat{I}_{23}I_{12}I_{13}-\hat{I}_{23}I_{11}I_{23}.
\end{aligned} \tag{11}$$

4 Link length and offset

The *link length*—the length of the common perpendicular between the axis-lines of two twists—is one of the DH parameters determined by a successive pair of joints. Let us assume they have finite pitch and the axes of screws are non-parallel. To calculate the link length d between twists S_1 and S_2 in terms of Plücker coordinates $(\omega_i,\mathbf{v}_i)$, $i=1,2$. Let plane π_i be the plane that contains the axis for S_i and is normal to the common perpendicular; the planes have common normal $\omega_1\times\omega_2$. The normal vector meets the axes at points:

$$\mathbf{x}_i=\frac{\mathbf{v}_i\times\omega_i}{\omega_i\cdot\omega_i}. \tag{12}$$

The planes have the form $(\omega_1\times\omega_2)\cdot\mathbf{x}=k_i$, $i=1,2$ and the link length is given by $a=(|k_2-k_1|)\|\omega_1\times\omega_2\|$. Substituting $\mathbf{x}_i$ from (12) into the plane equations to determine k_i and applying Lagrange's identity:

$$a=\frac{\omega_1\cdot\omega_2}{\|\omega_1\times\omega_2\|}\left(\frac{\mathbf{v}_1\cdot\omega_1}{\omega_1\cdot\omega_1}+\frac{\mathbf{v}_2\cdot\omega_2}{\omega_2\cdot\omega_2}\right)-\frac{\omega_1\cdot\mathbf{v}_2+\mathbf{v}_1\cdot\omega_2}{\|\omega_1\times\omega_2\|} \tag{13}$$

We observe from (13) that if either ω_1, ω_2, or $\omega_1\times\omega_2$ is equal to zero, then the link length will be undefined. It is a lengthy but essentially straightforward exercise to verify that the link length is invariant under the adjoint action of $SE(3)$. Indeed, clearly link length can be expressed in terms of 2-fold invariants:

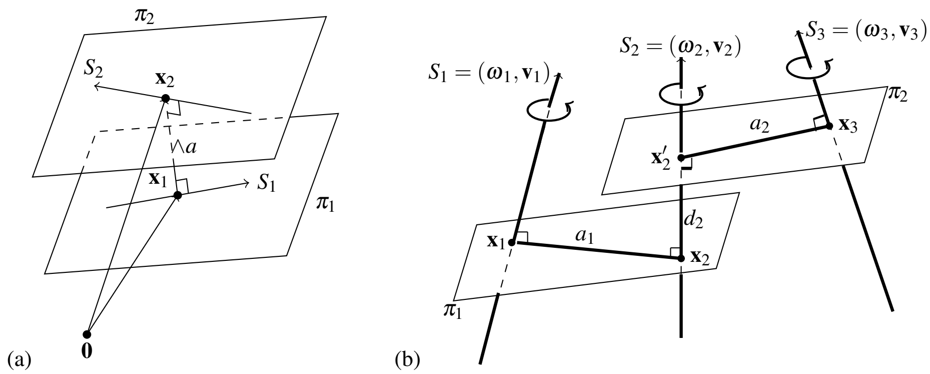

Fig. 1 (a) Link length and (b) offset in Plücker coordinates

$$a = \frac{I_{12}}{\sqrt{I_{11}I_{22} - I_{12}^2}} \left(\frac{\hat{I}_{11}}{I_{11}} + \frac{\hat{I}_{22}}{I_{22}} \right) - \frac{\hat{I}_{12}}{\sqrt{I_{11}I_{22} - I_{12}^2}} \tag{14}$$

The offset is the distance between the feet of successive common perpendiculars along the axis of the middle of three twists. In Figure (**??**), the offset d_2 is the distance between $\mathbf{x}_2$, $\mathbf{x}_2'$, the feet of successive common perpendiculars from the axes of S_1, S_3 to the axis of S_2, so:

$$d_2 = \|\mathbf{x}_2 - \mathbf{x}_2'\| = \left\| \frac{\mathbf{v}_2 \times \omega_2}{\omega_2 \cdot \omega_2} + t_2\omega_2 - \frac{\mathbf{v}_2 \times \omega_2}{\omega_2 \cdot \omega_2} + t_2'\omega_2 \right\| = |t_2 - t_2'|\|\omega_2\|. \tag{15}$$

We must determine t_2, t_2'. If $\mathbf{x}_1, \mathbf{x}_2$ are the feet of the perpendicular between axes of S_1, S_2 then $\mathbf{x}_i = \dfrac{\mathbf{v}_i \times \omega_i}{\omega_i \cdot \omega_i} + t_i\omega_i$ for some t_i, $i = 1,2$; the fact that $\mathbf{x}_2 - \mathbf{x}_1$ is perpendicular to axes of screws S_1 and S_2 gives us two equations and eliminating t_1 gives:

$$t_2 = \frac{((\mathbf{v}_2 \times \omega_2) \cdot \omega_1)(\omega_1 \cdot \omega_2) + ((\mathbf{v}_1 \times \omega_1) \cdot \omega_2)(\omega_2 \cdot \omega_2)}{\|\omega_2\|^2 \|\omega_1 \times \omega_2\|^2}. \tag{16}$$

A similar formula applies for t_2', from which we establish by (15), writing scalar triple products as 3×3 matrix determinants:

$$d_2 = \frac{\begin{pmatrix} \left(|\mathbf{v}_2\ \omega_2\ \omega_1|(\omega_1 \cdot \omega_2) + |\mathbf{v}_1\ \omega_1\ \omega_2|\,\|\omega_2\|^2\right)\|\omega_3 \times \omega_2\|^2 \\ \qquad - \left(|\mathbf{v}_2\ \omega_2\ \omega_3|(\omega_3 \cdot \omega_2) + |\mathbf{v}_3\ \omega_3\ \omega_2|\,\|\omega_2\|^2\right)\|\omega_1 \times \omega_2\|^2 \end{pmatrix}}{\|\omega_2\|\,\|\omega_3 \times \omega_2\|^2\,\|\omega_1 \times \omega_2\|^2} \tag{17}$$

Once again it is possible to show that this is invariant under the action of $SE(3)$. In practice we used Maple show that it is invariant under the adjoint action of the Lie algebra on itself. However it is not immediately clear whether the numerator Ω_2 of this expression can be written in terms of the 14 invariants (10).

5 Algebro–geometric dual of three twists

Given a set $\mathscr{S}$ of three twists S_1, S_2, S_3, there are in fact three offsets determined by the three pairwise common perpendiculars, one associated with each twist. Recall from Section 2 that associated to each pair of twists S_i, S_j, their Lie bracket $[S_i, S_j]$ in the generic situation corresponds to a twist whose axis is the common perpendicular to the given pair. Thus we can associate with the given set a new dual 3-twist set $\mathscr{S}'$ consisting of the brackets $S'_1 = [S_2, S_3], S'_2 = [S_3, S_1], S'_3 = [S_1, S_2]$, which has the property that the link lengths of $\mathscr{S}$ are the offsets of $\mathscr{S}'$ and vice versa. We can therefore determine an expression for the offset (17) in terms of the 3-fold invariants (10) by determining forms for the invariants of the dual set. By way of example, for $I_{11} = \omega_1 \cdot \omega_1$, we replace ω_1 by $\omega'_1 = \omega_2 \times \omega_3$ and then, using Lagrange's identity:

$$I'_{11} = (\omega_2 \times \omega_3) \cdot (\omega_2 \times \omega_3) = (\omega_2 \cdot \omega_2)(\omega_3 \cdot \omega_3) - (\omega_2 \cdot \omega_3)^2 = I_{22} I_{33} - I_{23}^2. \quad (18)$$

In the same way, we can determine expressions for the dual versions of all the invariants. For example, $I'_{123} = I_{123}^2$ and $\hat{I}'_{123} = 2 I_{123} \hat{I}_{123}$.

We are now able to combine (14) with these expressions for the link lengths of $\mathscr{S}'$ to obtain an expression for the offset in terms of the basic invariants (10). Explicitly:

$$d_2 = \frac{\hat{I}_{123} I_{22}(I_{12} I_{23} - I_{13} I_{22}) + I_{123}\left(\hat{I}_{22}(I_{12} I_{23} + I_{13} I_{22}) + I_{22}(\hat{I}_{13} I_{22} - I_{12}\hat{I}_{23} - \hat{I}_{12} I_{23})\right)}{\sqrt{I_{22}}(I_{11} I_{22} - I_{12}^2)(I_{22} I_{33} - I_{23}^2)}. \quad (19)$$

Finally, note that the duality between $\mathscr{S}$ and $\mathscr{S}'$ is not an exact involution as, although the axes of the twists interchange, applying the process twice affects the pitches h''_i of the double duals S''_i, $i = 1, 2, 3$. We obtain the following relation between the pitches: $h''_i = \dfrac{\hat{I}_{123}}{I_{123}} + h_i$.

6 Conclusion

By applying the principle of transference to the well-known invariants of the rotation group, we have obtained basic invariants for sets of twists under Euclidean change of coordinates. Although the question has not been explored here, it can be shown [4] that any m-fold polynomial invariant for $m \leq 3$ can be expressed as a rational function of these invariants. Ideally one would like to establish that they provide a generating set for the subalgebra of invariant polynomials. This appears to be a deep problem for $m \geq 3$. The virtue of these simple formulae is that they are defined for all sets of twists, unrestricted by special geometry, in contrast to Denavit–Hartenberg parameters. Indeed, we can find explicit algebraic expressions for the DH parameters in terms of them. A further issue is that our invariants are defined for a static set of twists. If the twists are regarded as defining joints in a serial chain, then the twists vary as the chain moves. So a second open question is whether the polynomials remain invariant under this intrinsic motion of the chain—DH parameters are invariant in this sense. A potential application for a set of polynomials, invariant in this full sense, would be to provide a firm basis for a classification of serial chains.

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References

1. Brockett, R. (1984) Robotic manipulators and the product of exponentials formula. In: Fuhrman, P. (Ed.), Proc. Math. Theory Netw. Syst., Springer, Berlin/Heidelberg, pp120–129
2. Chevallier, D. P. (1996) On the transference principle in kinematics: Its various forms and limitations. *Mech. Machine Theory,***31**, 57–76
3. Crook, D. (2009) Polynomial invariants of the Euclidean group action on multiple screws. Master's Thesis, Victoria University of Wellington
4. Daher, M. (2013) Dual numbers and invariant theory of the Euclidean group. PhD thesis, Victoria University of Wellington (under examination)
5. Denavit, J., Hartenberg, R. S. (1955) A kinematic notation for lower pair mechanisms based on matrices. *J. Appl. Mech.*, **22**, 215–221
6. Donelan, P. S. and Gibson, C. G. (1991) First–order invariants of Euclidean motions. *Acta Appl. Math.*, **24** 233–251
7. Kotelnikov, A. P. (1895) Screw calculus and some of its applications to geometry and mechanics. *Annals of Imperial University of Kazan.*
8. McCarthy, J. M., (1986) Dual orthogonal matrices in manipulator kinematics. *Int. J. Robot. Res.*, **5**, 45–51
9. Murray, R. M., Li, Z., Shastry, S. S. (1994) A Mathematical Introduction to Robotic Manipulation. CRC Press, Boca Raton.
10. Panyushev, D. I. (2007) Semi-direct products of lie algebras and their invariants. *Publ. Res. Inst. Math. Sci.*, **43**, 1199-1257
11. Perroud, M. (1983) The Fundamental Invariants of Inhomogeneous Classical Groups. *J. Math. Phys.*, **24** 1381–1391
12. Rico Martínez, J. M. and Duffy, J. (1993) The principle of transference: History, statement and proof. *Mech. Machine Theory*, **28**, 165–177
13. Rooney, J. (1975) On the principle of transference. In: J.N. Fawcett (Ed.), Proc. 4th World Congress on the Theory of Machines and Mechanisms, Institution of Mechanical Engineers, London, pp. 10881092
14. Rooney, J. (2007) William Kingdon Clifford (1845-1879). In: Ceccarelli, Marco (Ed.), Distinguished figures in mechanism and machine science: Their contributions and legacies. History of mechanism and machine science, Vol. 1. Springer, Dordrecht, Netherlands, pp. 79116
15. Rosen, J. (1968) Construction of invariants for Lie algebras of inhomogeneous pseudo-orthogonal and pseudo-unitary groups. *J. Math. Phys.*, **9** 1305–1307
16. Samuel, A. E., McAree, P. and Hunt, K. H., (1991) Unifying screw geometry and matrix transformations. *Int. J. Robot. Res.*, **10**, 454–470
17. Selig, J. (2005) Geometric Fundamentals of Robotics. Springer, New York
18. Study, E., (1903) Geometrie der Dynamen, Teubner, Leipzig
19. Takiff, S. J. (1972) Invariant polynomials on Lie algebras of inhomogeneous unitary and special orthogonal groups. *Trans. American Math. Soc.*, **170** 221–230
20. Weyl, H. (1997), The Classical Groups: Their Invariants and Representations. Princeton Univ. Press, Princeton, NY