ON THE REGULARITY OF THE INVERSE JACOBIAN OF PARALLEL ROBOTS

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Abstract Checking the regularity of the inverse jacobian matrix of a parallel robot is an essential element for the safe use of this type of mechanism. Ideally such a check should be made for all poses of the useful workspace of the robot, or for any pose along a given trajectory, and should take into account the uncertainties in the robot modeling and control. We propose various methods that facilitate this check. We exhibit especially a sufficient condition for the regularity that is directly related to the extreme poses that can be reached by the robot.

Keywords: inverse jacobian, singularity, parallel robots

1. Introduction

Determining if a parallel robot may be in a singular configuration during its motion is a problem of high practical interest. Many papers have addressed first the determination of the inverse kinematic jacobian[1], denoted **J**, of such robots and then the analysis of the condition that can be deduced from the singularity of this matrix. **J** relates the joint velocities to the twist of the end-effector and is usually pose dependent. At a singularity the end-effector will admit non-zero velocity for some motion although the actuators are instantaneously fixed, or may even undergo motion with the actuators locked down. The determinant of **J** is in general complicated but for most parallel robots **J** has as rows the Plücker vectors of well-defined lines. Consequently Grassmann geometry may be used to characterize the geometry of the singularity and to deduce simplified singularity conditions [Monsarrat 01; Merlet 89; Wolf 04]. It must be noted that even for robot with less than 6 d.o.f. it is necessary to consider the *full jacobian matrix* **J**, i.e. the matrix that involves the full twist of the end-effector by inclusion of relevant constraint equations. Indeed for a robot with n d.o.f. the jacobian that relates the n d.o.f. velocities to the n actuated joint velocities may not be singular while $\mathbf{J}$ itself is singular [Bonev 01].

A singularity detection algorithm should be able to determine the presence of a singularity within a motion variety with dimension 1 to n for a n d.o.f. robot. An important point is that the singularity detection should be *certified* i.e. the algorithm should provide a safe answer even if numerical round-off errors occur. This certification constraint usually rules out the use of an optimization procedure.

2. A singularity detection scheme

This singularity detection problem has been addressed in [Merlet 01] where an efficient algorithm was exhibited. This algorithm proceeds along the following steps: symbolic computation is used to determine an analytical form of the determinant of $\mathbf{J}$and its sign at a particular pose $\mathbf{X_1}$. Then an interval analysis based method [Jaulin 01; Moore 79], that takes round-off errors into account, allows one to determine if the motion variety includes a set of poses in which the determinant has a sign opposite to the one found at $\mathbf{X_1}$.

The main difficulty with this algorithm (aside from efficiently using interval analysis) is the calculation of the closed form of the determinant, as will be illustrated on a difficult example, the Gough platform.

2.1 The inverse jacobian of a Gough platform

We define a reference frame $(O, \mathbf{x}, \mathbf{y}, \mathbf{z})$. The attachment points of the leg i on the base will be denoted by A_i. The attachment points on the platform will be denoted by B_i and it is well known that the coordinates of B_i in the reference frame can be obtained as a function of the pose parameters. The inverse jacobian matrix is then constituted of the normalized Plücker vectors of the line associated to each leg:

$$\mathbf{J} = \left(\left(\frac{\mathbf{A_i B_i}}{||\mathbf{A_i B_i}||} \quad \frac{\mathbf{OA_i} \times \mathbf{OB_i}}{||\mathbf{A_i B_i}||} \right) \right) \tag{1}$$

Note that we may use the non-normalized Plücker vector to define another matrix $\mathbf{M} = ((\mathbf{A_i B_i} \quad \mathbf{OA_i} \times \mathbf{OB_i}))$ with the property that the sign and, in particular, the zeroes of $\mathbf{J}$ are the same as those of $|\mathbf{M}|$. As $\mathbf{M}$ is simpler than $\mathbf{J}$ it will be used for the singularity detection.

2.2 Evaluation of the determinant

Being given a motion variety the pose parameters are functions of the variety parameters and thus the components of the inverse jacobian may be obtained as functions of the variety parameters. As mentioned earlier a closed form of the determinant is obtained by symbolic computation. It should be noted that this is not strictly necessary. Indeed, being given ranges for the variety parameters, interval arithmetic may used to determine ranges for each component of the inverse jacobian. We get then an *interval matrix* $\mathbf{J}_I$, i.e. a matrix whose components are intervals. Classical methods for the calculation of the determinant may then be used to obtain an interval evaluation of the determinant but with a large overestimation of the minimum and maximum of the determinant. Indeed interval arithmetic is very sensitive to multiple occurrences of the same variable. Consider for example the matrix $\mathbf{A}$ whose closed-form determinant is xy, and its interval version $\mathbf{A}_I$, when x and y lie in the range [1,2]:

$$\mathbf{A} = \begin{pmatrix} x & x \\ y & 2y \end{pmatrix} \quad \mathbf{A}_I = \begin{pmatrix} [1,2] & [1,2] \\ [1,2] & [2,4] \end{pmatrix} \qquad (2)$$

The direct interval evaluation of $|\mathbf{A}_I|$ may be calculated as $[-2, 7]$, whereas the closed form of the determinant allows one to show that $|\mathbf{A}|$ will always be positive for any value of x, y in $[1, 2]$. We have put an emphasis on interval matrices that will be justified by the influence of uncertainties.

2.3 The influence of uncertainties

Uncertainties are inherent part of a real system such as a robot. They occur at the modeling level: the geometry of the real robot differs from its theoretical model due to the manufacturing tolerances. For example, for the Gough platform the locations of the A_i, B_i are known only up to a certain accuracy. Uncertainties also arise due to control: there will be a deviation of the robot motion from the theoretical motion variety.

An ideal singularity detection scheme should be able to determine if the robot may be in a singular pose in spite of these uncertainties. Although we may add the uncertainties as additional unknowns in the components of $\mathbf{J}$, a drawback is that the calculation of the closed-form of the determinant may become difficult. For example, for the Gough platform `Maple` is no longer able to calculate the determinant as soon as we add uncertainties on the A_i, B_i. In that case we have to resort to a numerical interval evaluation of the determinant based on the interval version of $\mathbf{J}$, but we have seen that this leads to a large overestimation of the determinant that will result in increased computation time for the singularity detection scheme. It is thus necessary to develop methods that check the regularity of the set of matrices defined by an interval matrix, without calculating its determinant. These methods should take into account that **J** is a *parametric matrix*, i.e. that its components are not independent.

3. Various methods for regularity check

3.1 A classical regularity check

Checking the regularity of all the matrices in a set defined by an interval matrix is a classical problem in interval analysis and is known to be NP-hard. Among possible approaches, the one having shown the greatest efficiency in our case has been a method proposed by Rohn [Kreinovich 00], refining the method of Baumann [Baumann 84; Hansen 05]. Baumann showed that if the determinants of all the matrices, formed by every choice of lower and upper bound for every entry of the matrix, have the same sign then the interval matrix is regular. There are $2^{(n^2)}$ such matrices. Rohn's refinement is as follows. We define the set H as the set of all n-dimensional vectors $\mathbf{h}$ whose components are either 1 or -1. For a given box we denote by $[\underline{a_{ij}}, \overline{a_{ij}}]$ the interval evaluation of the component J_{ij} of $\mathbf{J}$ at the i-th row and j-th column. Given two vectors $\mathbf{u}, \mathbf{v}$ of H, we then define the set of matrices $\mathbf{A}^{\mathbf{uv}}$ whose elements $A_{ij}^{\mathbf{uv}}$ are

$$A_{ij}^{\mathbf{uv}} = \overline{a_{ij}} \text{ if } u_i.v_j = -1, \underline{a_{ij}} \text{ if } u_i.v_j = 1$$

These matrices have thus fixed numerical components corresponding to lower or upper bound of the interval J_{ij}. There are 2^{2n-1} such matrices since $\mathbf{A}^{\mathbf{uv}} = \mathbf{A}^{-\mathbf{u},-\mathbf{v}}$. If the determinant of all these matrices have the same sign, then all the matrices $\mathbf{A}'$ whose components have a value within the interval evaluation of J_{ij} are regular. Hence, for the 6×6 $\mathbf{J}$ of a Gough platform, if the determinant of the 2048 matrices of $\mathbf{A}^{\mathbf{uv}}$ have the same sign, then all matrices in the set are regular.

But $\mathbf{A}^{\mathbf{uv}}$ includes matrices that are not actual inverse jacobians, as the dependency of the components of the matrix are not taken into account. This may be seen, for example, for the interval matrix $\mathbf{A}_I$ (2) that includes the following matrices

$$\mathbf{A_1} = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{A_2} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad \mathbf{A_3} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad (3)$$

The matrices $\mathbf{A_1}, \mathbf{A_2}$ belong to the set $\mathbf{A}^{\mathbf{uv}}$ and have determinants with opposite signs. Consequently the test proposed by Rohn fails, which is quite consistent as the matrix $\mathbf{A_3}$, that belongs to $\mathbf{A}_I$, is singular, though it does not correspond to a choice of parameters x, y.

For the Gough platform the first column of $\mathbf{J}$ can be written as $x+F_i$, x being a coordinate of the center of the platform; if the range for x is $[\underline{x}, \overline{x}]$ while the range for F_i is $[a_i b_i]$, then $\mathbf{A^{uv}}$ includes matrices with elements $\underline{x}+a_i$ and $\overline{x}+b_i$ that do not belong to the set of inverse jacobian matrices since x is fixed for them.

3.2 Pre-conditioning

A classical approach in interval analysis for regularity checking is to pre-condition the matrix by multiplying it by a real matrix $\mathbf{K}$, usually the inverse of the *mid-matrix*, i.e. the matrix whose components are the mid-point of each range of the components. The purpose of this strategy is to get $\mathbf{S} = \mathbf{KJ}$ close to the identity matrix so that its determinant $|\mathbf{S}| = |\mathbf{K}||\mathbf{J}|$ may be interval evaluated with a lower overestimation. If we apply this strategy to the matrix (2) the inverse of the mid-matrix and the interval matrix $\mathbf{KA}_I$ are:

$$\mathbf{K} = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 2/3 \end{pmatrix}, \mathbf{S} = \mathbf{KA}_I = \begin{pmatrix} [0,2] & [-4/3, 4/3] \\ [-2/3, 2/3] & [0,2] \end{pmatrix} \quad (4)$$

The interval evaluation of $|\mathbf{S}|$ is $[-8/9, 44/9] \approx [-0.8889, 4.8889]$ while $|\mathbf{K}|$ is positive. In term of sign determination this interval evaluation is indeed sharper than the one obtained with a direct evaluation of $|\mathbf{A}|$, but is still not satisfactory, as we know in fact that there are no singular matrices in terms of the parametrization.

We propose another method which consists of first computing *symbolically* the matrix $\mathbf{S}$, using k_{ij} as components of $\mathbf{K}$ and then plugging in the numerical values. The symbolic matrix $\mathbf{S}_s = \mathbf{AK}$ and its evaluation $\mathbf{S}_K$ for the numerical $\mathbf{K}$ are

$$\mathbf{S}_s = \begin{pmatrix} x(k_{11}+k_{21}) & x(k_{12}+k_{22}) \\ y(k_{11}+2\,k_{21}) & y(k_{12}+2\,k_{22}) \end{pmatrix}, \ \mathbf{S}_K = \begin{pmatrix} 2x/3 & 0 \\ 0 & 2y/3 \end{pmatrix} \quad (5)$$

If we use now the range [1,2] for x, y the interval evaluation of $|\mathbf{S}|$ is [4/3,8/3] that shows that all matrices have a positive determinant. Note that we have used $\mathbf{AK}$ instead of $\mathbf{KA}$, which is preferable as it allows us to reduce the multiple occurrences of the variables. However for the Gough platform, as $\mathbf{J}$ exhibits the same variables in its columns, it is better to pre-multiply it by the conditioning matrix.

3.3 A regularity test for parametric matrices

Where the entries of a matrix are related to some underlying interval parameters, we may evaluate the matrix as an interval matrix by taking the interval hull of its values, but by doing so we may considerably weaken the estimates of associated functions such as the determinant (see, for example, [Popova 04]). In some cases we may make more progress by working directly with the parameter intervals.

Assume that some or all components of some or all rows (denoted the *linear rows*) of a parametric matrix $\mathbf{A} = (a_{ij})$ can be written as affine-linear combinations, with real or interval coefficients, of a set of parameters $\{x_1, x_2, \ldots, x_n\}$, the remaining entries of $\mathbf{A}$ being independent of the parameters. We denote by $\mathbf{A}'$ the set of real or interval matrices that can be derived from $\mathbf{A}$ by assigning independently to each linear row either a lower or upper bound to each unknown x_i that appears in the linear combination. For example for matrix $\mathbf{A}$ the set $\mathbf{A}'$ is

$$\mathbf{A}' = \{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}\} \tag{6}$$

The following theorem holds:

Theorem 1: If the determinants of all matrices in the set $\mathbf{A}'$ have the same sign, then all matrices in the set $\mathbf{A}$ are regular

Proof Assume that there is a singular matrix $\mathbf{A_0}$ in the set $\mathbf{A}$ corresponding to parameters $x_1^0, \ldots, x_n^0$.

Consider the first row of $\mathbf{A_0}$. Without lack of generality we will assume that it is linear (otherwise proceed to the second row). In this row alone, replace x_1^0 by x_1, whose value lies in $[\underline{x_1}, \overline{x_1}]$, to get a parametric matrix $\mathbf{A}_{11}$ with parameter x_1. The j-th component of the first row of $\mathbf{A}$ may be written as $\lambda_{1j}x_1 + c_{1j}$ where for some non-empty subset $k \in \{j_1, \ldots, j_m\}$, $\lambda_{1k} \neq 0$, and for all j, c_{1j} may involve $x_2^0, \ldots, x_n^0$. Using row expansion, the determinant of the matrix may be written as

$$|\mathbf{A}_{11}| = \sum_{k \in \{j_1, \ldots, j_m\}} (-1)^{k+1}(\lambda_{1k}x_1 + c_{1k})M_{1k} + \sum_{l \notin \{j_1, \ldots, j_m\}} (-1)^{l+1} c_{1l} M_{1l} \tag{7}$$

where M_{1j} denotes the minor associated to the first row and column j of $\mathbf{A}_{11}$.

For $x_1 = x_1^0$ this expression will vanish. If we assume now that $x_1 = x_1^0 + dx_1$ we get

$$|\mathbf{A}_{11}| = dx_1 \left(\sum_{k=j_1, \ldots, j_m} (-1)^{k+1} \lambda_{1k} M_{1k} \right) = dx_1 K_1 \tag{8}$$

where K_1 is either a real number or an interval. Unless x_1^0 is one of the bounds or K_1 or one of its bounds is zero, we may assign dx_1 to either $\overline{x_1} - x_1^0$ or $\underline{x_1} - x_1^0$ so that $|\mathbf{A}|$ is positive or has a positive upper bound.

Thus, by assigning $\underline{x_1}$ or $\overline{x_1}$ to x_1, we can construct a matrix $\mathbf{A}^+_{11}$ whose determinant will be ≥ 0 or has upper bound ≥ 0. Assigning the other bound gives a matrix $\mathbf{A}^-_{11}$ whose determinant will be ≤ 0 or has lower bound ≤ 0.

Starting from these matrices we may now, in a similar way, assign x_2 to $\underline{x_2}$ or $\overline{x_2}$ in the first row to get a matrix $\mathbf{A}^+_{12}$ whose determinant is $|\mathbf{A}^+_{11}|$ plus a non-negative quantity (i.e. still ≥ 0) and a matrix $\mathbf{A}^-_{12}$ whose determinant will be less than or equal to the determinant of $|\mathbf{A}^-_{11}|$, i.e. still ≤ 0. The process is repeated for each unknown in the row to obtain $\mathbf{A}^{\pm}_{1n}$.

As soon as all unknowns in a row have a fixed value as one or other of their bounds, the process is repeated for the next linear row. When all linear rows have been processed then we will have matrices $\mathbf{A}^+$, $\mathbf{A}^-$ belonging to $\mathbf{A}'$ whose derivatives are ≥ 0 or ≤ 0 respectively. Note however that the assignment of the unknowns in a row to ensure that $|\mathbf{A}^+| \geq 0$ may differ between any two linear rows. Hence if there is a singular matrix in $\mathbf{A}$, then we are able to determine matrices whose determinants have opposite signs (or one or both are zero), or whose lower bound is ≤ 0 and upper bound is ≥ 0, which concludes the proof.

For example, as all matrices in $\mathbf{A}'$ defined by (6) have the same determinant sign, then the set $\mathbf{A}$ contains only regular matrices. Another theorem may be derived for the full inverse jacobian matrices that have Plücker vectors as rows. Let us define $A_i(a^1_i, a^2_i, a^3_i)$ and $B_i(b^1_i, b^2_i, b^3_i)$ as two points that belong to the line associated to the Plücker vector i. A row of $\mathbf{J}$ may be written as

$$((b^1_i - a^1_i, b^2_i - a^2_i, b^3_i - a^3_i, a^2_i b^3_i - a^3_i b^2_i, a^3_i b^1_i - a^1_i b^3_i, a^1_i b^2_i - a^2_i b^1_i)) \quad (9)$$

so that i-th row is linear in the b^k_i. Assume now that the locations of the A_i are fixed, while the locations of the B_i are functions of the end-effector motion. Using interval analysis (or an optimization method), for given ranges for the motion parameters we may find a bounding box $\mathcal{B}_i$ for the location of each B_i. Let $\mathbf{J}^\star$ be the set of inverse jacobians that may be obtained for the motion parameter ranges. Theorem 1 allows one to state the following theorem:

Theorem 2: Let $\mathbf{J}'$ be the set of matrices obtained by choosing as location of B_i all possible combinations of the corners of $\mathcal{B}_i$ (there will be 8^6 such matrices). If the determinants of all matrices in $\mathbf{J}'$ have the same sign, then all matrices in $\mathbf{J}^\star$ are regular.

The number of matrices in $\mathbf{J}'$ may even be reduced in some cases, using the property that we may choose as B_i any point on the line.

Assume that the bounding box $\mathcal{B}_i$ is defined by the set of ranges $[\underline{b_i^j}, \overline{b_i^j}]$, $j \in [1, 3]$ for each b_i. The following cases may occur:

- $a_k \in [\underline{b_i^k}, \overline{b_i^k}]$ for two indices $k \in \{1, 2, 3\}$, while $a_l < \underline{b_i^l}$ or $a_l > \overline{b_i^l}$ for one index l. The line always enters the bounding box $\mathcal{B}_i$ by the face defined by $b_l = \underline{b_i^l}$ or $b_l = \overline{b_i^l}$. We may thus choose as B_i the intersection point of the line with this face i.e. fix the value of b_l. Hence only 4 corners will have to be checked.

- $a_k \in [\underline{b_i^k}, \overline{b_i^k}]$ for only one index. The line may enter the bounding box by 2 faces and we have to check 6 corners.

- $a_k \notin [\underline{b_{ik}}, \overline{b_{ik}}]$ for all index. The line may enter the bounding box by 3 faces and we have 7 corners to check.

- $a_k \in [\underline{b_{ik}}, \overline{b_{ik}}]$ for all index. In that case the corresponding row of the jacobian may include a row of 0 and the ranges for the motion parameters must be bisected.

In practice we will have between 4^6 and 7^6 matrices in $\mathbf{J}'$. Uncertainties in the locations of the A_i may also be dealt with by considering that the matrices in $\mathbf{J}'$ are interval matrices.

Theorem 2 shows that checking the extreme poses of the B_i may be sufficient to check the regularity of $\mathbf{J}$ over the whole workspace.

4. Examples

The proposed regularity check has been implemented in the singularity detection scheme and has been extensively tested. It appears that among the three regularity checks the most efficient combination is to use first the pre-conditioning and then to apply Rohn test on the resulting matrix. A 6D workspace $\mathcal{W}$ is defined with the ranges x, y in $[-15, 15]$, z in $[45, 50]$ and the three Euler angles having the ranges $[-15, 15]$ degrees. The computation time on a Dell D400 laptop (1.7 Ghz) is established as follows:

- 6D workspace without uncertainty: for $\mathcal{W}$ no singularity detected in 3.12s. If the orientation ranges of $\mathcal{W}$ is extended to [-40,40] degree a singularity is detected in 9.46s.

- 6D workspace with uncertainties: for a $\pm$ 0.05 uncertainty on each coordinates of the A_i, B_i points no singularity is detected in $\mathcal{W}$ in 43mn on a cluster of 15 PC's without the regularity checks and only in 263s on a laptop if they are incorporated in the detection scheme. For an uncertainty of ± 0.1 the computation time establishes respectively at 10h 22mn and 1176s.

5. Conclusion

We have proposed regularity checks for the inverse jacobian of parallel robots that may be used to determine if such matrix may be singular over a motion variety. They allow to deal with uncertainties in the robot modeling and control and have been proved to be very efficient. One of these regularity checks, that is sufficient but not necessary, is related to the extremal poses that can be reached by the end-effector: if the determinant of a finite number of real matrices that are related to these extremal poses have all the same sign, then the inverse jacobian matrix is regular.

Notes

1. Sometimes the notation $\mathbf{J}^{-1}$ is used [Merlet 01] since this is the jacobian of the inverse kinematics. However it is not the same as the inverse of the jacobian of the forward kinematics which may not exist, so the use of the inverse notation may be confusing in some circumstances.

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