Parisian Ruin and Resolvent Density of Terminating Spectrally Negative Lévy Process Before First-Passage Above a Level

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Abstract

In this paper, we consider Parisian ruin problem concerning excursion below zero for a fixed consecutive duration of spectrally negative Lévy process. The results are two folds. First, we derive joint Laplace transform of ruin-time and ruin-position of the Lévy process killed at the first-passage time above a level. Secondly, based on this Laplace transform, we derive Parisian ruin and resolvent density of terminating the Lévy process at the first-passage time. Parisian ruin density extends the recent result of Baurdoux et al [1] to Parisian excursion with fixed duration, whereas the resolvent density generalizes the result obtained by Suprun [25] and Bertoin [2] for ruin problem under Parisian fixed delay. The results have semi-explicit expressions in terms of the scale function of spectrally negative Lévy process. We show in the limit as excursion duration goes to zero that the joint Laplace transform leads to Emery's fluctuation identity [13] whereas the resolvent density leads to the result given in [25] and [3].

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1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$. That is to say that X is a stochastic process starting from zero, having stationary and independent increments with

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cádlág sample paths with no positive jumps. To avoid degenerate case we exclude the case where X has monotone paths. As a strong Markov process, we shall endow X with probabilities $\{\mathbb{P}_x, x\in\mathbb{R}\}$, such that $\mathbb{P}_x\{X_0=x\}=1$. Furthermore, we denote by $\mathbb{E}_x$ expectation with respect to $\mathbb{P}_x$. Recall that $\mathbb{P}=\mathbb{P}_0$ and $\mathbb{E}=\mathbb{E}_0$. The Lévy-Itô sample paths decomposition of the process is given by

$$\begin{aligned} X_t = \mu t + \sigma B_t &+ \int_0^t \int_{\{x<-1\}} x\nu(dx,ds) \\ &+ \int_0^t \int_{\{-1\le x<0\}} x\big(\nu(dx,ds)-\Pi(dx)ds\big), \end{aligned} \tag{1.1}$$

where $\mu\in\mathbb{R}$, $\sigma\ge 0$ and $(B_t)_{t\ge0}$ is standard Brownian motion, whilst $\nu(dx,dt)$ denotes the Poisson random measure associated with the jumps process $\Delta X_t := X_t - X_{t-}$ of X. This Poisson random measure has compensator given by $\Pi(dx)dt$, where Π is the Lévy measure satisfying the integrability condition:

$$\int_{-\infty}^0 (1\wedge x^2)\Pi(dx) < \infty. \tag{1.2}$$

We refer to Chapter 2 of [17] for more details on paths decomposition of X.

Due to the absence of positive jumps, it is therefore sensible to define

$$\psi(\lambda) = \frac{1}{t}\log \mathbb{E}\big\{e^{\lambda X_t}\big\} = \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \big(e^{\lambda y} - 1 - \lambda y \mathbf{1}_{\{y>-1\}}\big)\Pi(dy), \tag{1.3}$$

which is analytic on $(\mathfrak{Im}(\lambda)\le 0)$. It is easily shown that ψ is zero at the origin, tends to infinity at infinity and is strictly convex. We denote by $\Phi:[0,\infty)\to[0,\infty)$ the right continuous inverse of ψ so that it satisfies the following:

$$\Phi(\theta) = \sup\{p>0 : \psi(p)=\theta\} \quad \text{and} \quad \psi(\Phi(\lambda))=\lambda \quad \text{for all} \quad \lambda\ge 0.$$

Note that due to the convexity of ψ, there exit at most two roots for a given θ and precisely one root when $\theta>0$. The asymptotic behavior of X can de determined from the sign of $\psi'(0+)$, the right-derivative of ψ at zero. X drifts to $-\infty$, oscillates or drifts to $+\infty$ according to whether $\psi'(0+)$ is negative, zero or positive. See for instance Kyprianou and Palmowski [18] for more details.

It is worth mentioning that under the Esscher transform $\mathbb{P}^\nu$ defined by

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{\nu X_t - \psi(\nu)t} \quad \text{for all } \nu\ge 0, \tag{1.4}$$

the Lévy process $(X,\mathbb{P}^\nu)$ is still a spectrally negative Lévy process. The Laplace exponent of X under the new measure $\mathbb{P}^\nu$ has changed to $\psi_\nu(\lambda)$ given by

$$\psi_\nu(\lambda) \ = \psi(\lambda+\nu) - \psi(\nu), \quad \text{for } \lambda \ge -\nu. \tag{1.5}$$

To each $\nu \geq 0$, we will denote by $\mathbb{P}_x^\nu$ the translation of $\mathbb{P}^\nu$ under which $X_0 = x$. Subsequently, we define by $\Phi_\nu(\theta)$ the largest root of equation $\psi_\nu(\lambda) = \theta$ satisfying

$$\Phi_\nu(\theta) = \Phi(\theta + \psi(\nu)) - \nu.$$

Our main object of interest in this paper is the quantity τ_r representing the first time that the process X has spent $r > 0$ units of time consecutively below zero before getting back up to zero again. This stopping time is defined by

$$\tau_r = \inf\{t > r : (t - g_t) \geq r\} \;\text{ with }\; g_t := \sup\{0 \leq s \leq t : X_s \geq 0\}, \tag{1.6}$$

under $\mathbb{P}_x$, with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. The stopping time τ_r (1.6) was first introduced by Chesney et al. [8] in the context of pricing barrier options in mathematical finance. It was later introduced in actuarial risk theory by Dassios and Wu [10] under the classical surplus process and provided expression for the Parisian ruin probability $\mathbb{P}_x\{\tau_r < \infty\}$. Czarna and Palmowski [9] and Loeffen et al. [22] extended the result to spectrally negative Lévy processes.

In their another paper, Dassios and Wu [11] and [12] gave the law of finite-time Parisian ruin probability $\mathbb{P}_x\{\tau_r \leq t\}$ in terms of its Laplace transform $q^{-1}\mathbb{E}_x\big\{e^{-q\tau_r}\mathbf{1}_{\{\tau_r<\infty\}}\big\}$ under the classical surplus process and Brownian motion. The results were extended to spectrally negative Lévy processes by Landriault et al. [20]. These extensions were based on randomization of excursion duration r under exponential and Erlang distribution. However, the results in [20] are available when the sample paths of the Lévy process have bounded variation. Based on the perturbation method employed in [12], Baurdoux et al. [1] extended the results in [20] to unbounded variation case by giving semi-explicit expression for Parisian ruin density $\mathbb{E}_x\big\{e^{-q\tau_q}; -X_{\tau_q} \in dy, \tau_q < \tau_a^+\big\}$ with exponential implementation of delays, i.e., $\tau_q = \inf\{t > 0 : (t - g_t) > \mathbf{e}_q\}$ and $\mathbf{e}_q$ is independent of X exponential random time, whilst τ_a^+ is first-passage above $a > 0$ stopping time (3.1). Working with the stopping time (1.6), we consider ruin density

$$\mathbb{E}_x\big\{e^{-q\tau_r}; -X_{\tau_r} \in dy, \tau_r < \tau_a^+\big\}, \quad \text{for } q, r \geq 0 \tag{1.7}$$

for X when it starts from $x \in [0, a]$ at time zero, and the resolvent density,

$$q^{-1}\mathbb{P}_x\big\{X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \tau_r \wedge \tau_a^+\big\}, \quad \text{for } q > 0,\ r \geq 0. \tag{1.8}$$

The ruin and resolvent density (1.7) and (1.8) are derived based on the joint Laplace transform of τ_r and X_{τ_r} killed at first-passage time τ_a^+, given by

$$\mathbb{E}_x\big\{e^{-q\tau_r + \lambda X_{\tau_r}}\mathbf{1}_{\{\tau_r < \tau_a^+\}}\big\}, \quad \text{for } q, \lambda \geq 0. \tag{1.9}$$

We will show in this paper that when duration of excursion r goes to zero, our results for (1.8) and (1.9) coincides with the resolvent measure obtained by Suprun [25] and Bertoin [3], and fluctuation identity given by Emery [13].

This paper is organized as follows. Section 1 discusses the motivation and main objects of interest of this paper. The main results are presented in Section 2. Some preliminary results are given in Section 3. Section 4 discusses the proofs of main results and the limiting results of joint Laplace transform and resolvent density when duration of excursion of the Lévy process goes to zero.

2 Main results

Our result for (1.9) is obtained by extending the approach used in [22]. The results are expressed in terms of the $q-$scale function $W^{(q)}(x)$ of X:

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\psi(\lambda)-q}, \quad \text{for } \lambda > \Phi(q), \tag{2.1}$$

with $W^{(q)}(x)=0$ for $x<0$. We shall write for short $W^{(0)}=W$ and refer to $W_\nu^{(q)}$ the scale function under $\mathbb{P}^\nu$. Following (2.1), it is straightforward to check that

$$W_\nu^{(q)}(x) = e^{-\nu x} W^{(q+\psi(\nu))}(x) \quad \text{for all } \nu \geq 0 \text{ and } q \geq -\psi(\nu). \tag{2.2}$$

It is known following [7] that, for any $q\geq 0$, the $q-$scale function $W^{(q)}$ is $C^1(0,\infty)$ if the Lévy measure Π does not have atoms and is $C^2(0,\infty)$ if $\sigma>0$.

In the sequel below, we use the notation $\Lambda^{(q)}(x,r)$ defined by

$$\Lambda^{(q)}(x,r) := \int_0^\infty W^{(q)}(x+z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}. \tag{2.3}$$

We shall write $\Lambda=\Lambda^{(0)}$ and refer to $\Lambda_\nu^{(q)}$ the role of $\Lambda^{(q)}$ under measure $\mathbb{P}^\nu$, i.e.,

$$\Lambda_\nu^{(q)}(x,r) := \int_0^\infty W_\nu^{(q)}(x+z)\frac{z}{r}\mathbb{P}^\nu\{X_r\in dz\}. \tag{2.4}$$

Using Esscher transform of measure (1.4), we can rewrite it as

$$\Lambda_\nu^{(q)}(x,r) = e^{-\nu x}e^{-\psi(\nu)r}\Lambda^{(q+\psi(\nu))}(x,r). \tag{2.5}$$

For further details on spectrally negative Lévy process, we refer to Chapter VI of Bertoin [2] and Chapter 8 of Kyprianou [17]. Some examples of Lévy processes for which $W^{(q)}$ are available in explicit form are given by Kuznetzov et al. [15]. In any case, it can be computed by numerically inverting (2.1), see Surya [26].

To prevent the Parisian ruin occurs with probability one, we therefore impose throughout the remaining of this paper that $\psi'(0+)>0$ by which the Lévy process X drifts to infinity at infinity. Our main results are given below.

Theorem 2.1 *Given that X starts at $0\leq x\leq a$ at time zero,*

(i) for any $y\in[0,\infty)$ and $q,r\geq 0$, the ruin density (1.7) is given by

$$\begin{aligned}
&\mathbb{E}_x\big\{e^{-q\tau_r}; -X_{\tau_r}\in dy, \tau_r<\tau_a^+\big\} \\
&= e^{-qr}\Big[q\Big(W^{(q)}(x-y) - \frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)}W^{(q)}(a-y)\Big)\mathbf{1}_{\{y\geq 0\}} \\
&\quad +\mathbf{1}_{\{x\leq y\leq a\}}\Big(\frac{\partial_y\Lambda^{(q)}(y,r)}{\Lambda^{(q)}(a,r)} - q\Lambda^{(q)}(x,r)\int_0^r \partial_y F_r^{(q)}(y,u)du\Big)\Big]dy,
\end{aligned} \tag{2.6}$$

where ∂_y is partial derivative w.r.t y and $F_r^{(q)}(y,u)$ is defined by

$$F_r^{(q)}(y,u) = \frac{\Lambda^{(q)}(y,u)}{\Lambda^{(q)}(y,r)} \quad \textit{with} \quad F_r^{(q)}(0,u) = e^{q(u-r)}.$$

(ii) and for $q > 0$ and $r \geq 0$, the resolvent density (1.8) is given by

$$q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \tau_r \wedge \tau_a^+\}$$
$$= \begin{cases} e^{-qr}\Big[\frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)}W^{(q)}(a-y) - W^{(q)}(x-y), & y \in [0,a] \\ \qquad +\mathbf{1}_{\{y\geq x\}}\Lambda^{(q)}(x,r)\int_0^r \partial_y \overline{F}_r^{(q)}(y,u)du\Big]dy, & \\ 0, & y < 0 \end{cases} \tag{2.7}$$

where ∂_y is partial derivative w.r.t y and $\overline{F}_r^{(q)}(y,u)$ is defined by

$$\overline{F}_r^{(q)}(y,u) = \frac{(\Lambda^{(q)}(y,u) - e^{qu})}{\Lambda^{(q)}(y,r)}, \quad \textit{with} \quad \overline{F}_r^{(q)}(0,u) = 0.$$

Note that the ruin density (2.6) is different from the one given in [1], even when we apply Laplace transform to (2.6) w.r.t duration r. Furthermore, we also notice the presence of an additional term in the resolvent density (2.7). That is to say, when duration of excursion is positive, there is an extra charge of measure applied on interval $[x,a]$. However, when excursion duration r goes to zero, we have the same result given in [25] and [3], and other results when $a \to \infty$.

Corollary 2.2 *When duration r of excursion goes to zero,*

(i) given $0 \leq x \leq a < \infty$, we have

(ia) for any $y \in [0,\infty)$ and $q \geq 0$,

$$\lim_{r\downarrow 0}\mathbb{E}_x\{e^{-q\tau_r}; -X_{\tau_r} \in dy, \tau_r < \tau_a^+\} \tag{2.8}$$
$$= \Big[q\Big(W^{(q)}(x-y) - \frac{W^{(q)}(x)}{W^{(q)}(a)}W^{(q)}(a-y)\Big)\mathbf{1}_{\{y\geq 0\}} + \frac{W^{(q)\prime}(y)}{W^{(q)}(a)}\mathbf{1}_{\{x\leq y\leq a\}}\Big]dy.$$

(ib) for any $q > 0$,

$$\lim_{r\downarrow 0} q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \tau_r \wedge \tau_a^+\}$$
$$= \begin{cases} \Big[\frac{W^{(q)}(x)}{W^{(q)}(a)}W^{(q)}(a-y) - W^{(q)}(x-y)\Big]dy, & y \in [0,a] \\ 0, & y < 0 \end{cases} \tag{2.9}$$

(ii) given $x \geq 0$, we have as $a \to \infty$,

(iia) that for any $q > 0$ and $r \geq 0$,

$$q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \tau_r\}$$
$$= \begin{cases} e^{-qr}\Big[e^{-\Phi(q)y}\frac{\Lambda^{(q)}(x,r)}{\Gamma^{(q)}(r)} - W^{(q)}(x-y), & y \geq 0 \\ \qquad +\mathbf{1}_{\{y\geq x\}}\Lambda^{(q)}(x,r)\int_0^r \partial_y \overline{F}_r^{(q)}(y,u)du\Big]dy, & \\ 0, & y < 0 \end{cases} \tag{2.10}$$

where we have defined

$$\Gamma^{(q)}(r) = \int_0^\infty e^{\Phi(q)z}\frac{z}{r}\mathbb{P}\{X_r \in dz\}.$$

(iib) that for any $q>0$,

$$\lim_{r\downarrow 0} q^{-1}\mathbb{P}_x\big\{X_{\mathbf{e}_q}\in dy, \mathbf{e}_q < \tau_r \wedge \tau_a^+\big\}$$
$$= \begin{cases}\Big[e^{-\Phi(q)y}W^{(q)}(x) - W^{(q)}(x-y)\Big]dy, & y\geq 0\\ 0, & y<0.\end{cases} \tag{2.11}$$

Recall that the limiting measure (2.9) and (2.11) coincides with the $q-$potential measure of a spectrally negative Lévy process killed on exiting $[0,a]$ and without killing given in [25] and [3], respectively. See Theorem 8.7 in [17].

The theorem below gives the Laplace transform of the density (2.6) and (2.7).

Theorem 2.3 *For a given* $q,r,\theta\geq 0$ *with* $q>\psi(\theta)$, *for any* $x\leq a$

(i) the Laplace transform of (2.6) is give by

$$\begin{aligned}\mathbb{E}_x\big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\tau_a^+\}}\big\} &= e^{-qr}\Big\{e^{\theta x}e^{\psi(\theta)r} + (q-\psi(\theta))e^{\psi(\theta)r}\int_0^x e^{\theta z}W^{(q)}(x-z)dz\\ &+(q-\psi(\theta))\int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(x,u)du - \frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)}\Big[e^{\theta a}e^{\psi(\theta)r}\\ &+(q-\psi(\theta))e^{\psi(\theta)r}\int_0^a e^{\theta z}W^{(q)}(a-z)dz + (q-\psi(\theta))\int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(a,u)du\Big]\Big\}.\end{aligned} \tag{2.12}$$

(ii) the Laplace transform of (2.7) is given by

$$\begin{aligned}\mathbb{E}_x\Big\{\int_0^{\tau_r\wedge\tau_a^+} e^{-qt}e^{\theta X_t}dt\Big\} &= \frac{\Lambda^{(q)}(x,r)}{(q-\psi(\theta))}\Big[\frac{e^{\theta x}}{\Lambda^{(q)}(x,r)} - \frac{e^{\theta a}}{\Lambda^{(q)}(a,r)}\Big]\big(1-e^{-qr}e^{\psi(\theta)r}\big)\\ &-e^{-qr}\Big[e^{\psi(\theta)r}\int_0^x e^{\theta z}W^{(q)}(x-z)dz + \int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(x,u)du\\ &-\frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)}\Big(e^{\psi(\theta)r}\int_0^a e^{\theta z}W^{(q)}(a-z)dz + \int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(a,u)du\Big)\Big].\end{aligned} \tag{2.13}$$

Corollary 2.4 *When duration* r *goes to zero,* τ_r *has the same law as* τ_0^-.

Using density (2.6), we can calculate discounted penalty at Parisian ruin

$$\mathbb{E}_x\big\{e^{-q\tau_r}f(X_{\tau_r})\mathbf{1}_{\{\tau_r<\tau_a^+\}}\big\} = \int_0^\infty f(-y)\mathbb{E}_x\big\{e^{-q\tau_r}; -X_{\tau_r}\in dy, \tau_r<\tau_a^+\big\}, \tag{2.14}$$

for nonnegative Borel measurable penalty function f. However, when sending $a \to \infty$ in (2.14), the ruin density (2.6) shall be used in the limit sense, i.e.,

$$\mathbb{E}_x\big\{e^{-q\tau_r} f(X_{\tau_r})\mathbf{1}_{\{\tau_r<\infty\}}\big\} = \lim_{a\to\infty}\int_0^\infty f(-y)\mathbb{E}_x\big\{e^{-q\tau_r}; -X_{\tau_r}\in dy, \tau_r<\tau_a^+\big\}.$$

For instance, we have following (2.6) and (2.12) the result below for $f(x)=e^{\theta x}$.

Corollary 2.5 *For any $x\in\mathbb{R}$ and $q,\theta\geq 0$ with $q>\psi(\theta)$,*

$$\begin{aligned}\mathbb{E}_x\big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\infty\}}\big\} &= e^{-qr}\Big\{e^{\theta x}e^{\psi(\theta)r} + (q-\psi(\theta))e^{\psi(\theta)r}\int_0^x e^{\theta z}W^{(q)}(x-z)dz \\ &\quad +(q-\psi(\theta))\int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(x,u)du - \frac{\Lambda^{(q)}(x,r)}{\Gamma^{(q)}(r)}\Big[\frac{(q-\psi(\theta))}{(\Phi(q)-\theta)}e^{\psi(\theta)r} \\ &\quad +(q-\psi(\theta))\int_0^r e^{\psi(\theta)(r-u)}\Gamma^{(q)}(u)du\Big]\Big\}. \end{aligned} \tag{2.15}$$

Furthermore, using density (2.7), we can evaluate aggregate payoff

$$\mathbb{E}_x\Big\{\int_0^{\tau_r\wedge\tau_a^+} e^{-qt}g(X_t)dt\Big\} = q^{-1}\int_0^\infty g(y)\mathbb{P}_x\big\{X_{\mathbf{e}_q}\in dy, \mathbf{e}_q<\tau_r\wedge\tau_a^+\big\}. \tag{2.16}$$

for nonnegative Borel measurable function g. To sum up, using the ruin and resolvent density (2.6) and (2.7), we can evaluate total discounted penalty function

$$\mathbb{E}_x\Big\{\int_0^{\tau_r\wedge\tau_a^+} e^{-qt}g(X_t)dt\Big\} + \mathbb{E}_x\big\{e^{-q(\tau_r\wedge\tau_a^+)}f(X_{\tau_r\wedge\tau_a^+})\big\}.$$

This quantity concerning total discounted payoff appears in some places in finance, especially in the capital structure problem. We refer, among others, to Kyprianou and Surya [16], Francois and Morellec [14] and Broadie et al. [6].

3 Preliminaries

Before we prove the main results, we devote this section to some preliminary results required to establish (2.6)-(2.15); in particular, to prove Theorem 2.3 on getting the expression for joint Laplace transform of τ_r and X_{τ_r} on the events $\tau_r<\tau_a^+$. Then, we show that this expression coincides with the Laplace transform of (2.6) and (2.7). To begin with, we define for $a\in\mathbb{R}$ two stopping times:

$$\tau_a^+ = \inf\{t>0: X_t>a\} \quad \text{and} \quad \tau_a^- = \inf\{t>0: X_t<a\}. \tag{3.1}$$

Due to the absence of positive jumps, we have by the strong Markov property of X that τ_a^+ can equivalently be rewritten as $\tau_a^+=\inf\{t>0: X_t\geq a\}$.

It is known that $W^{(q)}(x)$ is continuously differentiable on $(0,\infty)$ whenever X has paths of unbounded variation or the Lévy measure Π has no atoms for bounded variation, see Lambert [19] and Chan et al. [7]. We denote its derivative by $W^{(q)\prime}(x)$. Furthermore, it is also known that $W^{(q)}(0)=0$ and $W^{(q)}(0)=1/\mathrm{d}$ when X has paths of unbounded and bounded variation, respectively. See Lemma 8.6 in [17] and Lemma 4.4. in Kyprianou and Surya [16]. Also, due to the absence of positive jumps in the sample paths, the Laplace transform of τ_a^+ is given by

$$\mathbb{E}_x\big\{e^{-q\tau_a^+}\big\} = e^{\Phi(q)(x-a)}, \quad x\leq a, q\geq 0. \tag{3.2}$$

see Section 8.1 in [17]. In the derivation of the main results (2.12)-(2.11), we will also frequently use Kendall's identity (Corollary VII.3 in [2]), which relates the distribution $\mathbb{P}\{X_t\in dx\}$ of a spectrally negative Lévy process X to the distribution $\mathbb{P}\{\tau_x^+\in dt\}$ of its first passage time τ_x^+. This identity is given by

$$t\mathbb{P}\{\tau_x^+\in dt\}dx = x\mathbb{P}\{X_t\in dx\}dt, \tag{3.3}$$

To establish our main results, we need to recall the following identities.

Lemma 3.1 *For any $x\leq a$ and $q,\theta\geq 0$,*

$$\begin{aligned}\mathbb{E}_x\big\{e^{-q\tau_0^-+\theta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\big\} =& 1+\big(q-\psi(\theta)\big)\int_0^x e^{-\theta z}W^{(q)}(z)dz\\ &- e^{-\theta(x-a)}\frac{W^{(q)}(x)}{W^{(q)}(a)}\Big(1+\big(q-\psi(\theta)\big)\int_0^x e^{-\theta z}W^{(q)}(z)dz\Big).\end{aligned} \tag{3.4}$$

Proof To establish the above, we need to recall the following result borrowed from Theorem 8.1 in [17], which holds for any $x\leq a$ and $q\geq 0$,

$$\mathbb{E}_x\big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\big\} = 1+q\int_0^x W^{(q)}(x-z)dz - \frac{W^{(q)}(x)}{W^{(q)}(a)}\Big[1+q\int_0^a W^{(q)}(a-z)dz\Big].$$

Let $p=q-\psi(\theta)$. As $\{\tau_0^-<\tau_a^+\}\in\mathcal{F}_{\tau_0^-}$, we arrive at (3.4) by strong Markov property and applying Esscher transform of measure after rewriting (3.4) as

$$\mathbb{E}_x\big\{e^{-q\tau_0^-+\theta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\big\} = e^{\theta x}\mathbb{E}_x^\theta\big\{e^{-p\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\big\}. \quad \square$$

Corollary 3.2 *For any $x\leq a$ and $\theta,q\geq 0$ with $q>\theta$,*

$$\begin{aligned}\mathbb{E}_x\Big\{e^{-q\tau_0^-+\Phi(\theta)X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\Big\} =& \frac{(\theta-q)}{\Phi(\theta)}\int_0^\infty e^{-\Phi(\theta)z}W^{(q)\prime}(x+z)dz\\ &- \frac{(\theta-q)}{\Phi(\theta)}\frac{W^{(q)}(x)}{W^{(q)}(a)}\int_0^\infty e^{-\Phi(\theta)z}W^{(q)\prime}(a+z)dz.\end{aligned} \tag{3.5}$$

Proof The proof follows from applying the fluctuation identity (3.4) from which we obtain after a change of variables and integration by parts that

$$
\begin{aligned}
\mathbb{E}_x\Big\{e^{-q\tau_0^- + \Phi(\theta)X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}\Big\} =& e^{\Phi(\theta)x}\Big(1+(q-\theta)\int_0^x e^{-\Phi(\theta)z}W^{(q)}(z)dz\Big) \\
&- e^{\Phi(\theta)a}\frac{W^{(q)}(x)}{W^{(q)}(a)}\Big(1+(q-\theta)\int_0^a e^{-\Phi(\theta)z}W^{(q)}(z)dz\Big) \\
=&(\theta-q)e^{\Phi(\theta)x}\int_x^\infty e^{-\Phi(\theta)z}W^{(q)}(z)dz \\
&- (\theta-q)e^{\Phi(\theta)a}\frac{W^{(q)}(x)}{W^{(q)}(a)}\int_a^\infty e^{-\Phi(\theta)z}W^{(q)}(z)dz \\
=&\frac{(\theta-q)}{\Phi(\theta)}\Big(W^{(q)}(x)+\int_0^\infty e^{-\Phi(\theta)z}W^{(q)\prime}(x+z)dz\Big) \\
&- \frac{(\theta-q)}{\Phi(\theta)}\frac{W^{(q)}(x)}{W^{(q)}(a)}\Big(W^{(q)}(a)+\int_0^\infty e^{-\Phi(\theta)z}W^{(q)\prime}(a+z)dz\Big),
\end{aligned}
$$

from which we arrive at the result in (3.5) and indeed establishes our claim. □

Along with Lemma 3.2, the two results below are used when applying inverse Laplace transforms to get the main results presented in Section 2.

Lemma 3.3 *For* $y > 0$ *and* $\theta > 0$ *such that* $0 \le \alpha < \Phi(\theta)$,

$$
\int_0^\infty e^{-\theta r}e^{-\alpha y}\int_y^\infty e^{\alpha z}\frac{z}{r}\mathbb{P}\{X_r \in dz\}dr = \frac{e^{-\Phi(\theta)y}}{\big(\Phi(\theta)-\alpha\big)}, \tag{3.6}
$$

$$
\int_0^\infty e^{-\theta r}e^{-\alpha y}\int_y^\infty e^{\alpha z}\mathbb{P}\{\tau_z^+ \le r\}dzdr = \frac{e^{-\Phi(\theta)y}}{\theta\big(\Phi(\theta)-\alpha\big)}. \tag{3.7}
$$

Proof On recalling (3.2), Kendall's identity (3.3) and Tonelly, we have

$$
\begin{aligned}
\int_0^\infty e^{-\theta r}e^{-\alpha y}\int_y^\infty e^{\alpha z}\frac{z}{r}\mathbb{P}\{X_r \in dz\}dr =& e^{-\alpha y}\int_0^\infty e^{-\theta r}\int_y^\infty e^{\alpha z}\mathbb{P}\{\tau_z^+ \in dr\}dz \\
=& e^{-\alpha y}\int_y^\infty e^{\alpha z}\int_0^\infty e^{-\theta r}\mathbb{P}\{\tau_z^+ \in dr\}dz \\
=& e^{-\alpha y}\int_y^\infty e^{(\alpha-\Phi(\theta))z}dz \\
=& \frac{e^{-\Phi(\theta)y}}{(\Phi(\theta)-\alpha)}.
\end{aligned}
$$

The equation (3.6) follows from applying Tonelli and Laplace inversion on account that both sides of the equation are right-continuous in r. □

To establish the equation (3.7), we need to recall from (3.2) that

$$\begin{aligned}
\int_0^\infty e^{-\theta r}\mathbb{E}\big\{e^{-q\tau_z^+}\mathbf{1}_{\{\tau_z^+\le r\}}\big\}dr =& \int_0^\infty e^{-\theta r}\int_0^r e^{-qy}\mathbb{P}\{\tau_z^+\in dy\}dr\\
=& \int_0^\infty\int_y^\infty e^{-\theta r}dre^{-qy}\mathbb{P}\{\tau_z^+\in dy\}\\
=& \int_0^\infty \frac{1}{\theta}e^{-(\theta+q)y}\mathbb{P}\{\tau_z^+\in dy\}\\
=& \frac{1}{\theta}e^{-\Phi(\theta+q)z}.
\end{aligned} \tag{3.8}$$

Again the equation follows after applying Tonelli and Laplace inversion for the same reason as before. Using similar arguments for the proof of (3.6) and (3.8),

$$\begin{aligned}
\int_0^\infty e^{-\theta r}e^{-\alpha y}\int_y^\infty e^{\alpha z}\mathbb{P}\{\tau_z^+\le r\}dzdr =& e^{-\alpha y}\int_y^\infty e^{\alpha z}\int_0^\infty e^{-\theta r}\mathbb{P}\{\tau_z^+\le r\}drdz\\
=& e^{-\alpha y}\int_y^\infty e^{\alpha z}\frac{1}{\theta}e^{-\Phi(\theta)z}dz\\
=& \frac{e^{-\Phi(\theta)y}}{\theta\big(\Phi(\theta)-\alpha\big)}.
\end{aligned}$$

We deduce (3.7) by Tonelli and Laplace inversion (right-continuity in r). $\square$

Lemma 3.4 *Given $q\ge 0$, then for any $x\le 0$ and $r\ge 0$,*

$$\Lambda(x,r)=\Lambda^{(q)}(x,r)-q\int_0^r \Lambda^{(q)}(x,u)du. \tag{3.9}$$

Proof Following the right-continuity in r of the expressions on both side of the equality, the proof follows from showing that the Laplace transform of the left-hand side is equal to that of the other. By Kendall's identity (3.3) and Tonelly,

$$\begin{aligned}
\int_0^\infty dre^{-\theta r}\Lambda^{(q)}(x,r) &= \int_0^\infty e^{-\theta r}\int_0^\infty W^{(q)}(x+z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}dr\\
&= \int_0^\infty e^{-\theta r}\int_0^\infty W^{(q)}(x+z)\mathbb{P}\{\tau_z^+\in dr\}dz\\
&= \int_0^\infty dzW^{(q)}(x+z)\int_0^\infty e^{-\theta r}\mathbb{P}\{\tau_z^+\in dr\}\\
&= \int_0^\infty dze^{-\Phi(\theta)z}W^{(q)}(x+z)\\
&= e^{\Phi(\theta)x}\int_x^\infty e^{-\Phi(\theta)z}W^{(q)}(z)dz\\
&= \frac{e^{\Phi(\theta)x}}{(\theta-q)},
\end{aligned} \tag{3.10}$$

where the last equality is due to $x \leq 0$. By Kendall's identity (3.3) and Tonelly, we have $\int_0^r \Lambda^{(q)}(x,u)du = \int_0^\infty dz W^{(q)}(x+z)\mathbb{P}\{\tau_z^+ \leq r\}$. By Tonelly and (3.2),

$$
\begin{aligned}
q\int_0^\infty dr e^{-\theta r}\int_0^r \Lambda^{(q)}(x,u)du &= \frac{q}{\theta}\int_0^\infty e^{-\Phi(\theta)z}W^{(q)}(x+z)dz \\
&= \frac{qe^{\Phi(\theta)x}}{\theta(\theta-q)}.
\end{aligned}
$$

Following (3.10), $\int_0^\infty dr e^{-\theta r}\Lambda(x,r) = \frac{e^{\Phi(\theta)x}}{\theta} = \frac{e^{\Phi(\theta)x}}{(\theta-q)} - \frac{qe^{\Phi(\theta)x}}{\theta(\theta-q)}$ proving (3.9). $\square$

Lemma 3.5 *For any $x \leq 0$, $q, r, \theta \geq 0$ with $q > \psi(\theta)$ and $\epsilon \geq 0$,*

$$
\begin{aligned}
\mathbb{E}_x\Big\{e^{-q\tau_\epsilon^+}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\Big\} &= e^{-qr}\Lambda^{(q)}(x-\epsilon,r), && (3.11)\\
\mathbb{E}_x\Big\{e^{-qr+\theta X_r}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\Big\} &= e^{-qr}e^{\theta\epsilon}\Big(\Lambda^{(q)}(x-\epsilon,r) && (3.12)\\
&\quad -(q-\psi(\theta))\int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(x-\epsilon,u)du\Big).
\end{aligned}
$$

Proof The proof follows from applying Laplace inversion approach in similar fashion used before. To start with, recall following identity (3.2) that

$$
\int_0^\infty dr e^{-\theta r}\mathbb{E}_x\Big\{e^{-q\tau_\epsilon^+}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\Big\} = \theta^{-1}e^{\Phi(\theta+q)(x-\epsilon)}.
$$

Then, based on (2.1) and the fact that $x \leq 0$, we can rewrite the above as:

$$
\begin{aligned}
\theta^{-1}e^{\Phi(\theta+q)(x-\epsilon)} &= e^{\Phi(\theta+q)(x-\epsilon)}\int_x^\infty W^{(q)}(z)e^{-\Phi(\theta+q)z}dz \\
&= e^{\Phi(\theta+q)(x-\epsilon)}\int_\epsilon^\infty W^{(q)}(x+z-\epsilon)e^{-\Phi(\theta+q)(x+z-\epsilon)}dz \\
&= \int_\epsilon^\infty W^{(q)}(x+z-\epsilon)e^{-\Phi(\theta+q)z}dz \\
&= \int_\epsilon^\infty W^{(q)}(x+z-\epsilon)\int_0^\infty e^{-\theta r}e^{-qr}\mathbb{P}\{\tau_z^+\in dr\}dz \\
&= \int_\epsilon^\infty W^{(q)}(x+z-\epsilon)\int_0^\infty dr e^{-\theta r}e^{-qr}\frac{z}{r}\mathbb{P}\{X_r\in dz\} \\
&= \int_0^\infty dr e^{-\theta r}e^{-qr}\int_\epsilon^\infty W^{(q)}(x+z-\epsilon)\frac{z}{r}\mathbb{P}\{X_r\in dz\}.
\end{aligned}
$$

Since $x \leq 0$ and $W^{(q)}(x) = 0$ for $x < 0$, we arrive at our claim in (3.11) by recalling that for any $\epsilon \geq 0$ we have $\int_0^\epsilon W^{(q)}(x+z-\epsilon)\frac{z}{r}\mathbb{P}\{X_r\in dz\} = 0$. $\square$

Applying Esscher transform of measure to (3.12), we have by (3.11)

$$
\mathbb{E}_x\Big\{e^{-qr+\theta X_r}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\Big\} = e^{\theta x}e^{-qr}e^{\psi(\theta)r}\Lambda_\theta(x-\epsilon,r),
$$

where $\Lambda_\theta(x,r)$ is equal to $\Lambda(x,r)$ under measure $\mathbb{P}_x^\theta$. Furthermore, since Lemma 3.4 is applicable for any $q\geq 0$ we have by replacing $q\to p:=q-\psi(\theta)$, i.e.,

$$\Lambda_\theta(x-\epsilon,r)=\Lambda_\theta^{(p)}(x-\epsilon,r)-p\int_0^r \Lambda_\theta^{(p)}(x-\epsilon,u)du.$$

Our proof is complete by recalling that $\Lambda_\theta^{(p)}(x,r)=e^{-\theta x}e^{-\psi(\theta)r}\Lambda^{(q)}(x,r)$. □

We are now in the position of having necessary tools to establish our main results presented in Section 2, which the next section is concerned with.

4 Proof of main results

4.1 Proof of Theorem 2.3 (i)

The proof is established for the case where X has paths of bounded and unbounded variation. To deal with unbounded variation case, we will use a limiting argument similar to the one employed in [22] and adjust the ruin time (1.6) accordingly. For this reason, we introduce for $\epsilon\geq 0$ the stopping time τ_r^ϵ

$$\tau_r^\epsilon=\inf\{t>r:\big(t-g_t^\epsilon\big)\geq r\}\;\text{ with }\; g_t^\epsilon:=\sup\{s<t:X_s\geq\epsilon\}.$$

This stopping time represents the first time that the process X has spent a certain $r>0$ units of time consecutively below zero ending before X gets back up again to a level $\epsilon\geq 0$. Note that $\tau_r=\tau_r^0$. Due to the absence of positive jumps and strong Markov property of X, we have for any $x<0$ that

$$\begin{aligned}\mathbb{E}_x\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big\}=&\mathbb{E}_x\big\{e^{-qr+\theta X_r}\big\}-\mathbb{E}_x\big\{e^{-qr+\theta X_r}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\big\}\\&+\mathbb{E}_x\big\{e^{-q\tau_\epsilon^+}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\big\}\mathbb{E}_\epsilon\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big\}.\end{aligned}\tag{4.1}$$

Using the above for $x\geq 0$ and applying the strong Markov property, we have

$$\begin{aligned}\mathbb{E}_x\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big\}\;&=\;\mathbb{E}_x\big\{\mathbb{E}_x\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big|\mathcal{F}_{\tau_0^-}\big\}\big\}\\&=\mathbb{E}_x\Big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\mathbb{E}_{X_{\tau_0^-}}\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big\}\Big\}.\end{aligned}\tag{4.2}$$

Incorporating the result (4.1) in (4.2), we arrive at the following.

$$\begin{aligned}\mathbb{E}_x\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big\}\;&=\;\mathbb{E}_x\Big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\mathbb{E}_{X_{\tau_0^-}}\big\{e^{-qr+\theta X_r}\big\}\\&\quad-\mathbb{E}_x\Big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\mathbb{E}_{X_{\tau_0^-}}\big\{e^{-qr+\theta X_r}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\big\}\Big\}\\&+\mathbb{E}_x\Big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\mathbb{E}_{X_{\tau_0^-}}\big\{e^{-q\tau_\epsilon^+}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\big\}\Big\}\mathbb{E}_\epsilon\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big\}.\end{aligned}\tag{4.3}$$

The first expectation in the last equality of (4.3) can be worked out in terms of the $q-$scale function $W^{(q)}(x)$ using the fluctuation identity (3.4), whereas the other two expectations are given by the following propositions.

Proposition 4.1 *For any $q \geq 0$ and $r > 0$, we have for all $x \leq a$,*

$$\mathbb{E}_x\Big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\Big(e^{-qr}\Lambda^{(q)}(X_{\tau_0^-}-\epsilon,r)\Big)\Big\} \tag{4.4}$$
$$= e^{-qr}\int_\epsilon^\infty\Big[W^{(q)}(x+z-\epsilon)-\frac{W^{(q)}(x)}{W^{(q)}(a)}W^{(q)}(a+z-\epsilon)\Big]\frac{z}{r}\mathbb{P}\{X_r\in dz\}.$$

Proof On recalling (3.11) and (3.8), we have by Tonelly and Lemma 3.2 that

$$\begin{aligned}
\int_0^\infty dre^{-\theta r}\mathbb{E}_x\Big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\mathbb{E}_{X_{\tau_0^-}}\big\{e^{-q\tau_\epsilon^+}\mathbf{1}_{\{\tau_\epsilon^+\leq r\}}\big\}\Big\}
&= \theta^{-1}\mathbb{E}_x\Big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\mathbb{E}_{X_{\tau_0^-}}\big\{e^{-(q+\theta)\tau_\epsilon^+}\big\}\Big\}\\
&= \theta^{-1}e^{-\Phi(\theta+q)\epsilon}\mathbb{E}_x\Big\{e^{-q\tau_0^-+\Phi(\theta+q)X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\Big\}.\\
&= \int_0^\infty\Big(\frac{1}{\Phi(\theta+q)}e^{-\Phi(\theta+q)(z+\epsilon)}\Big)W^{(q)\prime}(x+z)dz\\
&\quad -\frac{W^{(q)}(x)}{W^{(q)}(a)}\int_0^\infty\Big(\frac{1}{\Phi(\theta+q)}e^{-\Phi(\theta+q)(z+\epsilon)}\Big)W^{(q)\prime}(a+z)dz.
\end{aligned}$$

On account of Lemma 3.3, Tonelli and applying Laplace inversion (noting that both sides of the equation are right-continuous in r), we then deduce (4.4). □

Corollary 4.2 *For any $q \geq 0$ and $r > 0$, we have for all $x \leq a$*

$$\mathbb{E}_x\big\{e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\Lambda^{(q)}(X_{\tau_0^-},r)\big\} = \Lambda^{(q)}(x,r)-\frac{W^{(q)}(x)}{W^{(q)}(a)}\Lambda^{(q)}(a,r). \tag{4.5}$$

Proof The proof of (i) follows from taking $\epsilon = 0$ in (4.4). □

Next, we want to find a simplified expression for (4.3) for $x = \epsilon$ in terms of $W^{(q)}(x)$ and the law $\mathbb{P}\{X_t \in dx\}$. For this purpose, let us show that

Proposition 4.3 *For any $q \geq 0$ and $r > 0$, we have that*

$$\int_0^\infty W^{(q)}(z)\frac{z}{r}\mathbb{P}\{X_r\in dz\} = e^{qr}. \tag{4.6}$$

Proof Using again the Kendall's identity (3.3) and (3.8), we have

$$\begin{aligned}
\int_0^\infty e^{-\theta r}\int_0^\infty W^{(q)}(z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}dr &= \int_0^\infty e^{-\theta r}\int_0^\infty W^{(q)}(z)\mathbb{P}\{\tau_z^+\in dr\}dz\\
&= \int_0^\infty e^{-\Phi(\theta)z}W^{(q)}(z)dz\\
&= \frac{1}{(\theta-q)},
\end{aligned}$$

which indeed establish our claim. $\square$

From the above proposition, it follows that for any $q,\epsilon\geq 0$ and $a,r>0$ that

$$\int_{\epsilon}^{\infty}\Big(W^{(q)}(z)-\frac{W^{(q)}(\epsilon)}{W^{(q)}(a)}W^{(q)}(a+z-\epsilon)\Big)\frac{z}{r}\mathbb{P}\{X_r\in dz\} \tag{4.7}$$
$$=e^{qr}-\int_0^{\epsilon}W^{(q)}(z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}-\frac{W^{(q)}(\epsilon)}{W^{(q)}(a)}\int_{\epsilon}^{\infty}W^{(q)}(a+z-\epsilon)\frac{z}{r}\mathbb{P}\{X_r\in dz\}.$$

We will take $x=\epsilon$ in (4.3) and then apply (3.11)-(3.12) and (4.4) in the first step and (4.7) in the second step along with the identity (3.4), we have that

$$\begin{aligned}
\mathbb{E}_{\epsilon}\Big\{e^{-q\tau_r^{\epsilon}+\theta X_{\tau_r^{\epsilon}}}\mathbf{1}_{\{\tau_r^{\epsilon}<\tau_a^+\}}\Big\}=&e^{-qr}e^{\psi(\theta)r}e^{\theta\epsilon}\Big[1+(q-\psi(\theta))\int_0^{\epsilon}e^{-\theta z}W^{(q)}(z)dz\\
&-e^{-\theta(\epsilon-a)}\frac{W^{(q)}(\epsilon)}{W^{(q)}(a)}\Big(1+(q-\psi(\theta))\int_0^{a}e^{-\theta z}W^{(q)}(z)dz\Big)\Big]\\
&-e^{-qr}e^{\theta\epsilon}\Big[e^{qr}-\int_0^{\epsilon}W^{(q)}(z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}\\
&-\frac{W^{(q)}(\epsilon)}{W^{(q)}(a)}\int_{\epsilon}^{\infty}W^{(q)}(a+z-\epsilon)\frac{z}{r}\mathbb{P}\{X_r\in dz\}\Big]\\
+e^{-qr}e^{\theta\epsilon}(q-\psi(\theta))\int_0^{r}&due^{\psi(\theta)(r-u)}\Big[e^{qu}-\int_0^{\epsilon}W^{(q)}(z)\frac{z}{u}\mathbb{P}\{X_u\in dz\}\\
&-\frac{W^{(q)}(\epsilon)}{W^{(q)}(a)}\int_{\epsilon}^{\infty}W^{(q)}(a+z-\epsilon)\frac{z}{u}\mathbb{P}\{X_u\in dz\}\Big]\\
&+\Big[e^{-qr}\Big(e^{qr}-\int_0^{\epsilon}W^{(q)}(z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}\\
&-\frac{W^{(q)}(\epsilon)}{W^{(q)}(a)}\int_{\epsilon}^{\infty}W^{(q)}(a+z-\epsilon)\frac{z}{r}\mathbb{P}\{X_r\in dz\}\Big)\Big]\\
&\times\mathbb{E}_{\epsilon}\Big\{e^{-q\tau_r^{\epsilon}+\theta X_{\tau_r^{\epsilon}}}\mathbf{1}_{\{\tau_r^{\epsilon}<\tau_a^+\}}\Big\},
\end{aligned}$$

from which we obtain after further simplification that

$$\begin{aligned}
\mathbb{E}_{\epsilon}\Big\{e^{-q\tau_r^{\epsilon}+\theta X_{\tau_r^{\epsilon}}}\mathbf{1}_{\{\tau_r^{\epsilon}<\tau_a^+\}}\Big\}\;=\;&\Bigg\{(q-\psi(\theta))e^{\theta\epsilon}e^{\psi(\theta)r}\int_0^{\epsilon}e^{-\theta z}\frac{W^{(q)}(z)}{W^{(q)}(\epsilon)}dz\\
-\frac{e^{\theta a}e^{\psi(\theta)r}}{W^{(q)}(a)}\Big(1+(q-\psi(\theta))&\int_0^{a}e^{-\theta z}W^{(q)}(z)dz\Big)+e^{\theta\epsilon}\int_0^{\epsilon}\frac{W^{(q)}(z)}{W^{(q)}(\epsilon)}\frac{z}{r}\mathbb{P}\{X_r\in dz\}\\
+\frac{e^{\theta\epsilon}}{W^{(q)}(a)}\int_{\epsilon}^{\infty}W^{(q)}(a+z-\epsilon)&\frac{z}{r}\mathbb{P}\{X_r\in dz\}-e^{\theta\epsilon}(q-\psi(\theta))\int_0^{r}due^{\psi(\theta)(r-u)}\\
\times\Big[\int_0^{\epsilon}\frac{W^{(q)}(z)}{W^{(q)}(\epsilon)}\frac{z}{u}\mathbb{P}\{X_u\in dz\}&+\int_{\epsilon}^{\infty}\frac{W^{(q)}(a+z-\epsilon)}{W^{(q)}(a)}\frac{z}{u}\mathbb{P}\{X_u\in dz\}\Big]\Bigg\} \qquad (4.8)\\
\times\Big\{\int_0^{\epsilon}\frac{W^{(q)}(z)}{W^{(q)}(\epsilon)}\frac{z}{r}\mathbb{P}\{X_r\in dz\}&+\int_{\epsilon}^{\infty}\frac{W^{(q)}(a+z-\epsilon)}{W^{(q)}(a)}\frac{z}{r}\mathbb{P}\{X_r\in dz\}\Big\}^{-1}.
\end{aligned}$$

We now want to compute the limit as $\epsilon \downarrow 0$ to (4.8) proving (2.12) for $x = 0$. For this purpose, we introduce for $\epsilon > 0$ the following stopping time

$$\widetilde{\tau}_r^\epsilon = \inf\{t > r : (t - g_t) \geq r, X_{t-r} < -\epsilon\} \text{ with } g_t := \sup\{0 \leq s \leq t : X_s \geq 0\}.$$

We see following the above that for $0 < \epsilon' < \epsilon$, $\{\widetilde{\tau}_r^\epsilon < \infty\} \subset \{\widetilde{\tau}_r^{\epsilon'} < \infty\}$ and $\cup_{\epsilon>0}\{\widetilde{\tau}_r^\epsilon < \infty\} = \{\tau_r < \infty\}$. Hence, by spatial homogeneity, we have that

$$\lim_{\epsilon\downarrow 0} \mathbb{E}_\epsilon\big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\big\} = \lim_{\epsilon\downarrow 0}\mathbb{E}\big\{e^{-q\widetilde{\tau}_r^\epsilon+\theta X_{\widetilde{\tau}_r^\epsilon}}\mathbf{1}_{\{\widetilde{\tau}_r^\epsilon<\tau_a^+\}}\big\} = \mathbb{E}\big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\tau_a^+\}}\big\}.$$

We consider two cases: $W^{(q)}(0) > 0$ (X has paths of bounded variation) and $W^{(q)}(0) = 0$ (X has unbounded variation). For the case $W^{(q)}(0) > 0$ we have

$$\begin{aligned}\mathbb{E}\big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\tau_a^+\}}\big\} \quad = \quad & 1 - \frac{1}{\Lambda^{(q)}(a,r)}\Big[e^{\theta a}e^{\psi(\theta)r}\Big(1 + (q-\psi(\theta))\int_0^a e^{-\theta z}W^{(q)}(z)dz\Big) \\ & +(q-\psi(\theta))\int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(a,u)du\Big]. \qquad (4.9)\end{aligned}$$

For the case $W^{(q)}(0) = 0$, we have by integration by parts and l'Hôpital that

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\int_0^\epsilon \frac{W^{(q)}(z)}{W^{(q)}(\epsilon)}\frac{z}{r}\mathbb{P}\{X_r \in dz\} =& \lim_{\epsilon\downarrow 0}\Big(\int_0^\epsilon \frac{y}{r}\mathbb{P}\{X_r\in dy\} - \int_0^\epsilon \frac{W^{(q)\prime}(z)}{W^{(q)}(\epsilon)}\int_0^z \frac{y}{r}\mathbb{P}\{X_r\in dy\}dz\Big) \\ =& 0 - \lim_{\epsilon\downarrow 0}\frac{W^{(q)\prime}(\epsilon)\int_0^\epsilon \frac{y}{r}\mathbb{P}\{X_r\in dy\}}{W^{(q)\prime}(\epsilon)} = 0.\end{aligned}$$

We therefore have the limit as $\epsilon \downarrow 0$ of (4.8) is given by (4.9). Next, taking (4.3), (3.11)-(3.12), Proposition 4.1 and (4.9), we have by dominated convergence that

$$\begin{aligned}\mathbb{E}_x\Big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\tau_a^+\}}\Big\} =& \lim_{\epsilon\downarrow 0}\mathbb{E}_x\Big\{e^{-q\tau_r^\epsilon+\theta X_{\tau_r^\epsilon}}\mathbf{1}_{\{\tau_r^\epsilon<\tau_a^+\}}\Big\} \\ =& e^{-qr}\Bigg\{e^{\theta x}e^{\psi(\theta)r} + (q-\psi(\theta))e^{\psi(\theta)r}\int_0^x e^{\theta z}W^{(q)}(x-z)dz \\ & - \frac{W^{(q)}(x)}{W^{(q)}(a)}\Big(e^{\theta a}e^{\psi(\theta)r} + (q-\psi(\theta))e^{\psi(\theta)r}\int_0^a e^{\theta z}W^{(q)}(a-z)dz\Big) \\ & - \Big(\Lambda^{(q)}(x,r) - \frac{W^{(q)}(x)}{W^{(q)}(a)}\Lambda^{(q)}(a,r)\Big) + (q-\psi(\theta))\int_0^r due^{\psi(\theta)(r-u)} \\ & \times\Big(\Lambda^{(q)}(x,u) - \frac{W^{(q)}(x)}{W^{(q)}(a)}\Lambda^{(q)}(a,u)\Big) + \Big(\Lambda^{(q)}(x,r) - \frac{W^{(q)}(x)}{W^{(q)}(a)}\Lambda^{(q)}(a,r)\Big) \\ & \times\Bigg(1 - \frac{1}{\Lambda^{(q)}(a,r)}\Big[e^{\theta a}e^{\psi(\theta)r}\Big(1+(q-\psi(\theta))\int_0^a e^{-\theta z}W^{(q)}(z)dz\Big) \\ & + (q-\psi(\theta))\int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(a,u)du\Big]\Bigg)\Bigg\},\end{aligned}$$

which after further simplifications we finally arrive at our claim (2.12). $\square$

4.2 Proof of Theorem 2.3 (ii)

Denote by $\mathbf{e}_q$ independent exponential random time with mean $1/q$. Recall,

$$\mathbb{E}_x\{e^{\theta X_{\mathbf{e}_q}}\} = \frac{qe^{\theta x}}{(q-\psi(\theta))}.$$

By applying strong Markov property we have

$$\begin{aligned}\mathbb{E}_x\Big\{\int_0^{\tau_r\wedge\tau_a^+} e^{-qt}e^{\theta X_t}dt\Big\} =&\mathbb{E}_x\Big\{\int_0^{\infty} e^{-qt}e^{\theta X_t}dt\Big\} - \mathbb{E}_x\Big\{\int_{\tau_r\wedge\tau_a^+}^{\infty} e^{-qt}e^{\theta X_t}dt\Big\}\\ =&q^{-1}\mathbb{E}_x\big\{e^{\theta X_{\mathbf{e}_q}}\big\} - q^{-1}\mathbb{E}_x\Big\{e^{-q(\tau_r\wedge\tau_a^+)}\mathbb{E}_{X_{\tau_r\wedge\tau_a^+}}\big\{e^{\theta X_{\mathbf{e}_q}}\big\}\Big\}\\ =&\frac{e^{\theta x}}{(q-\psi(\theta))} - \frac{1}{(q-\psi(\theta))}\mathbb{E}_x\big\{e^{-q(\tau_r\wedge\tau_a^+)+\theta X_{\tau_r\wedge\tau_a^+}}\big\}.\end{aligned}$$

To deal with the expectation on the last equality, recall following (2.12) that $\mathbb{P}_x^{\Phi(q)}\{\tau_a^+<\tau_r\} = \frac{\Lambda_{\Phi(q)}(x,r)}{\Lambda_{\Phi(q)}(a,r)}$, where $\Lambda_{\Phi(q)}(x,r)$ plays the role of $\Lambda(x,r)$ under measure $\mathbb{P}^{\Phi(q)}$, i.e., $\Lambda_{\Phi(q)}(x,r) = e^{-\Phi(q)x}e^{-qr}\Lambda^{(q)}(x,r)$, see (2.5). By Esscher transform, we have $\mathbb{E}_x\{e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_r\}}\} = e^{-\Phi(q)(a-x)}\mathbb{P}_x^{\Phi(q)}\{\tau_a^+<\tau_r\}$. Using the former, we have $\mathbb{E}_x\{e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_r\}}\} = \frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)}$ which by (2.12) we arrive at (2.13). $\square$

4.3 Proof of Corollary 2.4

The goal is to show that the limiting result of (2.12) as $a\to\infty$ and $r\to 0$ coincides with Emery's identity on the joint Laplace transform of τ_0^- and $X_{\tau_0^-}$,

$$\begin{aligned}\lim_{a\uparrow\infty}\lim_{r\downarrow 0}\mathbb{E}_x\big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\tau_a^+\}}\big\} \;&=\; \mathbb{E}_x\big\{e^{-q\tau_0^-+\theta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\big\}\\ =e^{\theta x} - \frac{(q-\psi(\theta))}{(\Phi(q)-\theta)}W^{(q)}(x) &+ (q-\psi(\theta))e^{\theta x}\int_0^x e^{-\theta y}W^{(q)}(y)dy.\end{aligned}\tag{4.10}$$

The limiting result is the same whether we take $\lim_{r\downarrow 0}\lim_{a\uparrow\infty}$. In both cases, it is necessary to show that for a given $q\geq 0$, we have for any $x\in\mathbb{R}$ and $a>0$

$$\lim_{r\downarrow 0}\frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)} = \lim_{r\downarrow 0}\frac{\int_0^\infty W^{(q)}(x+z)z\mathbb{P}\{X_r\in dz\}}{\int_0^\infty W^{(q)}(a+z)z\mathbb{P}\{X_r\in dz\}} \;=\; \frac{W^{(q)}(x)}{W^{(q)}(a)},\tag{4.11}$$

used when taking $\lim_{a\uparrow\infty}\lim_{r\downarrow 0}$ in (2.12), whilst for $\lim_{r\downarrow 0}\lim_{a\uparrow\infty}$ we need

$$\lim_{r\downarrow 0}\frac{\Lambda^{(q)}(x,r)}{\Gamma^{(q)}(r)} = \lim_{r\downarrow 0}\frac{\int_0^\infty W^{(q)}(x+z)z\mathbb{P}\{X_r\in dz\}}{\int_0^\infty e^{\Phi(q)z}z\mathbb{P}\{X_r\in dz\}} \;=\; W^{(q)}(x).\tag{4.12}$$

By continuity of the function $z\to W^{(q)}(x+z)z$, we have by the first mean value theorem for integration that there exists, for a given $\delta>0$, $z_\delta\in[0,\delta]$

such that $\int_0^\delta W^{(q)}(x+z)z\mathbb{P}\{X_r\in dz\}=W^{(q)}(x+z_\delta)z_\delta\mathbb{P}\{X_r\in[0,\delta]\}$. By the L_1-integrability of $z\to W^{(q)}(x+z)z$ w.r.t $\mathbb{P}\{X_r\in dz\}$, we have that

$$\begin{aligned}\int_0^\infty W^{(q)}(x+z)z\mathbb{P}\{X_r\in dz\}=&W^{(q)}(x+z_\delta)z_\delta\mathbb{P}\{X_r\in[0,\delta]\}\\&+\int_\delta^\infty W^{(q)}(x+z)z\mathbb{P}\{X_r\in dz\}.\end{aligned}$$

As $\lim_{r\downarrow 0}\mathbb{P}\{X_r\in dz\}=\delta_0(dz)$, we have from the above that

$$\lim_{r\downarrow 0}\int_0^\infty W^{(q)}(x+z)z\mathbb{P}\{X_r\in dz\}=W^{(q)}(x+z_\delta)z_\delta. \tag{4.13}$$

As δ is arbitrary, the proof of (4.11) follows after sending $\delta\to 0$ on account of the continuity of $W^{(q)}$. As for a given $x\in\mathbb{R}_+$ the function $f_1: z\to e^{\Phi(q)z}z$ has the same limiting behaviour towards $z\to 0$ as $f_2: z\to W^{(q)}(x+z)z$, i.e., $f_2(z)=\mathcal{O}(f_1(z))$ as $z\to 0$, we could argue the same as before that $\int_0^\delta e^{\Phi(q)z}z\mathbb{P}\{X_r\in dz\}=e^{\Phi(q)z_\delta}z_\delta\mathbb{P}\{X_r\in[0,\delta]\}$ for a relatively small $\delta>0$. By doing so, we have

$$\int_0^\infty e^{\Phi(q)z}z\mathbb{P}\{X_r\in dz\} \;=\; e^{\Phi(q)z_\delta}z_\delta\mathbb{P}\{X_r\in[0,\delta]\}+\int_\delta^\infty e^{\Phi(q)z}z\mathbb{P}\{X_r\in dz\},$$

which in turn gives that

$$\lim_{r\downarrow 0}\int_0^\infty e^{\Phi(q)z}z\mathbb{P}\{X_r\in dz\}=e^{\Phi(q)z_\delta}z_\delta. \tag{4.14}$$

Since δ is arbitrary, the proof of (4.12) follows from (4.13) and (4.14) on account of continuity of $W^{(q)}$. For taking the second limit $\lim_{a\uparrow\infty}$, we write $W^{(q)}(a)=e^{\Phi(q)a}W_{\Phi(q)}(a)$. Recall that $\lim_{a\uparrow\infty}W_{\Phi(q)}(a)=\frac{1}{\psi'(\Phi(q))}$, or equivalently, $\lim_{a\uparrow\infty}W'_{\Phi(q)}(a)=0$. Since $q>\psi(\theta)$, we have $\lim_{a\uparrow\infty}\frac{e^{\theta a}}{W^{(q)}(a)}=0$ and by change of variable and L'Hôpital's rule that $\lim_{a\uparrow\infty}\frac{\int_0^a e^{\theta z}W^{(q)}(a-z)dz}{W^{(q)}(a)}=\frac{1}{(\Phi(q)-\theta)}$. Following (4.21), we have $0\le\int_0^r e^{\psi(\theta)(r-u)}\Lambda^{(q)}(x,u)du\le e^{\psi(\theta)r}\int_0^r\Lambda^{(q)}(x,u)du$ which vanishes as $r\to 0$. By similar approach, $\lim_{r\downarrow}\lim_{a\uparrow\infty}$ gives the same result. □

4.4 Proof of Theorem 2.1 (i)

The proof is to show that the Laplace transform of (2.6) is given by (2.12), i.e.,

$$\begin{aligned}&\int_0^\infty e^{-\theta y}\mathbb{E}_x\big\{e^{-q\tau_r};-X_{\tau_r}\in dy,\tau_r<\tau_a^+\big\}\\&\qquad=e^{\theta x}\int_0^\infty\mathbb{E}_x\big\{e^{-p\tau_r}e^{\theta(X_{\tau_r}-x)-\psi(\theta)\tau_r};-X_{\tau_r}\in dy,\tau_r<\tau_a^+\big\}\\&\qquad\qquad=e^{\theta x}\int_0^\infty\mathbb{E}_x^\theta\big\{e^{-p\tau_r};-X_{\tau_r}\in dy,\tau_r<\tau_a^+\big\},\end{aligned}$$

with $p=q-\psi(\theta)$, whilst $\mathbb{E}_x^\theta$ is the expectation operator associated with $\mathbb{P}_x^\theta$,

$$\begin{aligned}\mathbb{E}_x^\theta\big\{e^{-p\tau_r};-X_{\tau_r}\in dy,\tau_r<\tau_a^+\big\} &= e^{-pr}\Big[p\Big(W_\theta^{(p)}(x-y)-\frac{\Lambda_\theta^{(p)}(x,r)}{\Lambda_\theta^{(p)}(a,r)}W_\theta^{(p)}(a-y)\Big)\mathbf{1}_{\{y\geq 0\}}\\ &+\Big(\frac{\partial_y\Lambda_\theta^{(p)}(y,r)}{\Lambda_\theta^{(p)}(a,r)}-p\Lambda_\theta^{(p)}(x,r)\int_0^r\partial_y\widetilde{F}_r^{(p)}(y,u)du\Big)\mathbf{1}_{\{x\leq y\leq a\}}\Big]dy, \qquad (4.15)\end{aligned}$$

where $\widetilde{F}_r^{(p)}(y,u)=\frac{\Lambda_\theta^{(p)}(y,u)}{\Lambda_\theta^{(p)}(y,r)}$ with $\Lambda_\theta^{(p)}(x,r)$ playing the role of $\Lambda^{(p)}(x,r)$ under $\mathbb{P}^\theta$. On account that $\int_x^a\frac{\partial_y\Lambda_\theta^{(p)}(y,r)}{\Lambda_\theta^{(p)}(a,r)}dy=1-\frac{\Lambda_\theta^{(p)}(x,r)}{\Lambda_\theta^{(p)}(a,r)}$ and by Fubini's theorem we have

$$\begin{aligned}\int_x^a\int_0^r\partial_y\widetilde{F}_r^{(p)}(y,u)dudy &= \int_0^r\int_x^a\partial_y\widetilde{F}_r^{(p)}(y,u)dydu\\ &= \int_0^r\Big(\frac{\Lambda_\theta^{(p)}(a,u)}{\Lambda_\theta^{(p)}(a,r)}-\frac{\Lambda_\theta^{(p)}(x,u)}{\Lambda_\theta^{(p)}(x,r)}\Big)du.\end{aligned}$$

As $x\leq a$ and $W^{(q)}(x)=0$ for $x<0$, we have after integrating out (4.15) that

$$\begin{aligned}&\int_0^\infty e^{-\theta y}\mathbb{E}_x\big\{e^{-q\tau_r};-X_{\tau_r}\in dy,\tau_r<\tau_a^+\big\}\\ &=e^{\theta x}\int_0^\infty\mathbb{E}_x^\theta\big\{e^{-p\tau_r};-X_{\tau_r}\in dy,\tau_r<\tau_a^+\big\}\\ &=e^{\theta x}e^{-pr}p\int_0^\infty\Big(W_\theta^{(p)}(x-y)-\frac{\Lambda_\theta^{(p)}(x,r)}{\Lambda_\theta^{(p)}(a,r)}W_\theta^{(p)}(a-y)\Big)dy\\ &\quad+\int_x^a\Big(\frac{\partial_y\Lambda_\theta^{(p)}(y,r)}{\Lambda_\theta^{(p)}(a,r)}-p\Lambda_\theta^{(p)}(x,r)\int_0^r\partial_y\widetilde{F}_r^{(p)}(y,u)du\Big)dy\\ &=e^{\theta x}e^{-pr}\Big[1+p\int_0^x W_\theta^{(p)}(x-y)dy+p\int_0^r\Lambda_\theta^{(p)}(x,u)du\\ &\quad-\frac{\Lambda_\theta^{(p)}(x,r)}{\Lambda_\theta^{(p)}(a,r)}\Big(1+p\int_0^a W_\theta^{(p)}(a-y)dy+p\int_0^r\Lambda_\theta^{(p)}(a,u)du\Big)\Big]\\ &=e^{\theta x}e^{-qr}e^{\psi(\theta)r}\Big[1+(q-\psi(\theta))\int_0^x e^{-\theta(x-y)}W^{(q)}(x-y)dy\\ &\quad+(q-\psi(\theta))\int_0^r e^{-\theta x}e^{-\psi(\theta)u}\Lambda^{(q)}(x,u)du-e^{-\theta(x-a)}\frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)}\\ &\qquad\times\Big(1+(q-\psi(\theta))\int_0^a e^{-\theta(a-y)}W^{(q)}(a-y)dy\\ &\qquad+(q-\psi(\theta))\int_0^r e^{-\theta a}e^{-\psi(\theta)u}\Lambda^{(q)}(a,u)du\Big)\Big],\end{aligned}$$

which after further simplifications we arrive at our claim in (2.12). □

4.5 Proof of Theorem 2.1 (ii)

Recall that we can rewrite (2.13) as the Laplace transform of (2.7), i.e.,

$$\mathbb{E}_x\Big\{\int_0^{\tau_r\wedge\tau_a^+} e^{-qt}e^{\theta X_t}dt\Big\} = \int_0^\infty e^{\theta y}q^{-1}\mathbb{P}_x\big\{X_{\mathbf{e}_q}\in dy, \mathbf{e}_q<\tau_r\wedge\tau_a^+\big\}$$

Define $p=q-\psi(\theta)$. Since the event $\{t<\tau_a^+\}$ is $\mathcal{F}_t-$adapted, we deduce that $\{t<\tau_r\wedge\tau_a^+\}\in\mathcal{F}_t$. Therefore, by Esscher transform of measure we have

$$\begin{aligned}
&\int_0^\infty e^{\theta y}q^{-1}\mathbb{P}_x\big\{X_{\mathbf{e}_q}\in dy, \mathbf{e}_q<\tau_r\wedge\tau_a^+\big\}\\
&\quad=\int_0^\infty\int_0^\infty dte^{-qt}e^{\theta y}\mathbb{P}_x\big\{X_t\in dy, t<\tau_r\wedge\tau_a^+\big\}\\
&\quad=p^{-1}e^{\theta x}\int_0^\infty\int_0^\infty dtpe^{-pt}e^{\theta(X_t-x)-\psi(\theta)t}\mathbb{P}_x\big\{X_t\in dy, t<\tau_r\wedge\tau_a^+\big\}\\
&\quad=p^{-1}e^{\theta x}\int_0^\infty\int_0^\infty dtpe^{-pt}\mathbb{P}_x^\theta\big\{X_t\in dy, t<\tau_r\wedge\tau_a^+\big\}\\
&\quad=e^{\theta x}\int_0^\infty p^{-1}\mathbb{P}_x^\theta\big\{X_{\mathbf{e}_p}\in dy, \mathbf{e}_p<\tau_r\wedge\tau_a^+\big\}\\
&\quad=e^{\theta x}\Big[\int_0^a dye^{-pr}\Big(\frac{\Lambda_\theta^{(p)}(x,r)}{\Lambda_\theta^{(p)}(a,r)}W_\theta^{(p)}(a-y)-W_\theta^{(p)}(x-y)\Big)\\
&\qquad\qquad+e^{-pr}\Lambda_\theta^{(p)}(x,r)\int_0^r du\int_x^a dy\partial_y\overline{F}_r^{(\theta,p)}(y,u)\Big]\\
&\quad=e^{\theta x}\Big[\int_0^a dye^{-pr}\Big(\frac{\Lambda_\theta^{(p)}(x,r)}{\Lambda_\theta^{(p)}(a,r)}W_\theta^{(p)}(a-y)-W_\theta^{(p)}(x-y)\Big)\\
&\quad+e^{-pr}\Lambda_\theta^{(p)}(x,r)\int_0^r du\Big(\frac{(\Lambda_\theta^{(p)}(a,u)-e^{pu})}{\Lambda_\theta^{(p)}(a,r)}-\frac{(\Lambda_\theta^{(p)}(x,u)-e^{pu})}{\Lambda_\theta^{(p)}(x,r)}\Big)\Big],
\end{aligned}$$

where we have defined $\overline{F}_r^{(\theta,p)}(y,u)=\frac{\Lambda_\theta^{(p)}(y,u)-e^{pu}}{\Lambda_\theta^{(p)}(y,r)}$ with $\Lambda_\theta^{(p)}(y,r)$ defined in (2.5). Following (2.2), we have $W_\theta^{(p)}(x)=e^{-\theta x}W^{(q)}(x)$. By inserting all of these facts above, we arrive after some algebra and calculations at our claim in (2.13). $\square$

Next, we want to show that $q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q}\in dy, \mathbf{e}_q<\tau_r\wedge\tau_a^+\}=0$ for any $y<0$ for a given $q>0$, $r\geq0$ and $x\leq a$. For this purpose, we need the $q-$potential measure $\mathbb{P}\{X_{\mathbf{e}_q}\in dy\}$ without killing given following Corollary 8.9 in [17] by

$$\mathbb{P}_x\{X_{\mathbf{e}_q}\in dy\}=\big(q\Phi'(q)e^{-\Phi(q)(y-x)}-qW^{(q)}(x-y)\big)dy,\quad x,y\in\mathbb{R}.$$

Using this resolvent measure without killing, we have

$$\begin{aligned}\mathbb{P}_x\{X_{\mathbf{e}_q}\in dy,\mathbf{e}_q<\tau_r\wedge\tau_a^+\} =&\mathbb{P}_x\{X_{\mathbf{e}_q}\in dy\}-\mathbb{E}_x\big\{e^{-q(\tau_r\wedge\tau_a^+)}\mathbb{P}_{X_{\tau_r\wedge\tau_a^+}}\{X_{\mathbf{e}_q}\in dy\}\big\}\\ =&\big(q\Phi'(q)e^{-\Phi(q)(y-x)}-qW^{(q)}(x-y)\big)dy\\ &-\Big(q\Phi'(q)e^{-\Phi(q)(y-x)}\mathbb{E}_x\big\{e^{-q(\tau_r\wedge\tau_a^+)+\Phi(q)(X_{\tau_r\wedge\tau_a^+}-x)}\big\}\\ &-q\mathbb{E}_x\big\{e^{-q(\tau_r\wedge\tau_a^+)}W^{(q)}(X_{\tau_r\wedge\tau_a^+}-y)\big\}\Big)dy\\ =&\Big(q\mathbb{E}_x\big\{e^{-q(\tau_r\wedge\tau_a^+)}W^{(q)}(X_{\tau_r\wedge\tau_a^+}-y)\big\}-qW^{(q)}(x-y)\Big)dy.\end{aligned}$$

Our proof is complete once we show that the following equation holds:

$$\mathbb{E}_x\big\{e^{-q(\tau_r\wedge\tau_a^+)}W^{(q)}(X_{\tau_r\wedge\tau_a^+}-y)\big\}=W^{(q)}(x-y). \tag{4.16}$$

This turns out to be the case as $\{e^{-qt}W^{(q)}(X_t):t\geq 0\}$ is $\mathbb{P}_x-$martingale. The proof is based on Theorem 28 and 33 in [23] or Theorem 4.3 in [17]). By smoothness of $W^{(q)}$ (see [7]), we have Lévy-Itô decomposition

$$e^{-qt}W^{(q)}(X_t-y)=W^{(q)}(x-y)+\int_0^t e^{-qs}\big(\mathcal{L}-q\big)W^{(q)}(X_s-y)ds+\mathcal{M}_t^y, \tag{4.17}$$

where $\{\mathcal{M}_t^y;t\geq 0\}$ is the sum of three stochastic integrals given by

$$\begin{aligned}\mathcal{M}_t^y \;=\;& \int_0^t\int_{\{-1\leq y<0\}} e^{-qs}\Big(W^{(q)}(X_{s-}-y+z)-W^{(q)}(X_{s-}-y)\Big)\widetilde{\nu}(dz,ds)\\ &+\int_0^t\int_{\{z<-1\}} e^{-qs}\big(W^{(q)}(X_s-y+z)-W^{(q)}(X_s-y)\big)\widetilde{\nu}(dz,ds)\\ &+\sigma\int_0^t e^{-qs}W^{(q)\prime}(X_{s-}-y)dB_s,\end{aligned} \tag{4.18}$$

whilst $\widetilde{\nu}(dz,ds)=\nu(dz,ds)-\Pi(dz)ds$ and $\mathcal{L}$ is the infinitesimal generator of X. It is known that $W^{(q)}\in C^2(\mathbb{R})$ or $W^{(q)}\in C^1(\mathbb{R})$ when X has paths of unbounded variation with $\sigma>0$ or when it has paths of bounded variation with no atoms on the Lévy measure. See [7]. Furthermore, following [17] and [4] we have $(\mathcal{L}-q)W^{(q)}(x)=0$ for all $x\in\mathbb{R}_+$. Recall that the stochastic integrals with respect to standard Brownian motion $(B_t:t\geq 0)$ and the process $(\mathcal{M}_t^y:t\geq 0)$ are local martingales. Thus, by Doob's optional stopping theorem we have (4.16). Furthermore, as $\mathbb{E}_x\big\{e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_r\}}W^{(q)}(X_{\tau_a^+}-y)\big\}=W^{(q)}(a-y)\frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(a,r)}$, we deduce from the above that $q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q}\in dy,\mathbf{e}_q<\tau_r\wedge\tau_a^+\}=0$ for any $y<0$. $\square$

4.6 Proof of Corollary 2.2

The proof for the case $r=0$ goes as follows. By similar arguments for the proof of Corollary 2.4, we can show that $\lim_{r\downarrow 0}\frac{\partial_y\Lambda^{(q)}(y,r)}{\Lambda^{(q)}(x,r)}=\frac{W^{(q)\prime}(y)}{W^{(q)}(x)}$. The integral term in (2.6) vanishes $r \to 0$. To see this, recall that as $\Lambda^{(q)}(x,r) \geq 0$ for all $x \in \mathbb{R}$,

$$\begin{aligned} 0 \leq \lambda \int_0^\infty dre^{-\lambda r} \int_0^r du\Lambda^{(q)}(x,u) &= \int_0^\infty e^{-\Phi(\lambda)z} W^{(q)}(x+z)dz \\ &= e^{\Phi(q)x} \int_0^\infty e^{-(\Phi(\lambda)-\Phi(q))z} W_{\Phi(q)}(x+z)dz \\ &\leq \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} \int_0^\infty e^{-(\Phi(\lambda)-\Phi(q))z} dz \qquad (4.19) \\ &= \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))\big(\Phi(\lambda)-\Phi(q)\big)}. \end{aligned}$$

Furthermore, as $\Lambda^{(q)}(\bullet, r)$ is an increasing function, we have

$$\begin{aligned} 0 \leq \lambda \int_0^\infty dre^{-\lambda r} \int_0^r du\partial_y\Lambda^{(q)}(x,u) &= \int_0^\infty e^{-\Phi(\lambda)z} W^{(q)\prime}(x+z)dz \\ &= \Phi(q)e^{\Phi(q)x} \int_0^\infty e^{-(\Phi(\lambda)-\Phi(q))z} W_{\Phi(q)}(x+z)dz \\ &\quad + e^{\Phi(q)x} \int_0^\infty e^{-(\Phi(\lambda)-\Phi(q))z} W'_{\Phi(q)}(x+z)dz \\ &\leq \frac{\Phi(q)e^{\Phi(q)x}}{\psi'(\Phi(q))} \int_0^\infty e^{-(\Phi(\lambda)-\Phi(q))z} dz \qquad (4.20) \\ &\quad + e^{\Phi(q)x} W'_{\Phi(q)}(x) \int_0^\infty e^{-(\Phi(\lambda)-\Phi(q))z} dz \\ &= \frac{e^{\Phi(q)x}}{\big(\Phi(\lambda)-\Phi(q)\big)} \Big(\frac{\Phi(q)}{\psi'(\Phi(q))} + W'_{\Phi(q)}(x) \Big), \end{aligned}$$

where in the last equality we used the fact that $W_{\Phi(q)}$ is a decreasing function. In the two inequalities, we have taken that $\lambda > q$ and the fact that $\Phi(\bullet)$ is an increasing function. By taking $\lim_{\lambda\to\infty}$ in (4.19) and (4.20), we conclude that

$$\lim_{r\downarrow 0} \int_0^r \Lambda^{(q)}(x,u)du = 0 \quad \text{and} \quad \lim_{r\downarrow 0} \int_0^r \partial_x \Lambda^{(q)}(x,u)du = 0 \qquad (4.21)$$

Based on the limiting results (4.21), we arrive at our claim by rewriting

$$\begin{aligned} \Lambda^{(q)}(x,r) \int_0^r \partial_y F_r^{(q)}(y,u)du &= \frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(y,r)} \int_0^r \partial_y \Lambda^{(q)}(y,u)du \\ &\quad - \frac{\Lambda^{(q)}(x,r)}{\Lambda^{(q)}(y,r)} \frac{\partial_y \Lambda^{(q)}(y,r)}{\Lambda^{(q)}(y,r)} \int_0^r \Lambda^{(q)}(y,u)du. \end{aligned}$$

Remark 4.4 *Employing the same steps of proof and arguments, one can show*

$$\lim_{r\downarrow 0} \Lambda^{(q)}(x,r) \int_0^r \partial_y \overline{F}_r^{(q)}(y,u)du = 0, \qquad (4.22)$$

where the function $\overline{F}_r^{(q)}(y,u)$ is specified in the Theorem 2.1.

The limiting results for $r \to 0$ is complete on account of (4.11) and (4.12). $\square$

4.7 Proof of Corollary 2.5

The proof is complete once we establish the following limits:

$$\begin{aligned}\lim_{a\to\infty}\frac{e^{\theta a}}{\Lambda^{(q)}(a,r)} &= 0, \text{ for } q>\psi(\theta)\\ \lim_{a\to\infty}\frac{\int_0^\infty e^{\theta z}W^{(q)}(a-z)dz}{\Lambda^{(q)}(a,r)} &= \frac{1}{\Gamma^{(q)}(r)(\Phi(q)-\theta)},\\ \lim_{a\to\infty}\frac{\Lambda^{(q)}(a,u)}{\Lambda^{(q)}(a,r)} &= \frac{\Gamma^{(q)}(u)}{\Gamma^{(q)}(r)}.\end{aligned}$$

To prove these limits, we use the representation of $W^{(q)}$ under $\mathbb{P}^{\Phi(q)}$, i.e., $W^{(q)}(x) = e^{\Phi(q)x}W_{\Phi(q)}(x)$, by which we can write $\Lambda^{(q)}(a,r) = e^{\Phi(q)a}\int_0^\infty e^{\Phi(q)z}W_{\Phi(q)}(a+z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}$. Therefore, since $q>\psi(\theta)$, or equivalently, $\Phi(q)>\theta$, the first limit follows. Under such representation for $W^{(q)}$,

$$\begin{aligned}\frac{\int_0^\infty e^{\theta z}W^{(q)}(a-z)dz}{\Lambda^{(q)}(a,r)} &= \frac{\int_0^\infty e^{-(\Phi(q)-\theta)z}W_{\Phi(q)}(a-z)dz}{\int_0^\infty e^{\Phi(q)z}W_{\Phi(q)}(a+z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}},\\ \frac{\Lambda^{(q)}(a,u)}{\Lambda^{(q)}(a,r)} &= \frac{\int_0^\infty e^{\Phi(q)z}W_{\Phi(q)}(a+z)\frac{z}{u}\mathbb{P}\{X_u\in dz\}}{\int_0^\infty e^{\Phi(q)z}W_{\Phi(q)}(a+z)\frac{z}{r}\mathbb{P}\{X_r\in dz\}}.\end{aligned}$$

The proof is complete on account that $\lim_{x\to\infty}W_{\Phi(q)}(x)=1/\psi'(\Phi(q))$. □

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