GENERIC MUCHNIK REDUCIBILITY AND PRESENTATIONS OF FIELDS

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Abstract. We prove that if I is a countable ideal in the Turing degrees, then the field $\mathcal{R}_I$ of real numbers in I is computable from exactly the degrees that list the functions (i.e., elements of ω^ω) in I. This implies, for example, that the degree spectrum of the field of computable real numbers consists exactly of the high degrees. We also prove that if I is a countable Scott ideal, then it is strictly easier to list the sets (i.e., elements of 2^ω) in I than it is to list the functions in I. This allows us to answer a question of Knight, Montalbán, and Schweber. They introduced *generic Muchnik reducibility* to extend the idea of Muchnik reducibility between countable structures to arbitrary structures. They asked if $\mathbb{R}$ is generically Muchnik reducible to the structure that consists of all sets of natural numbers. Our result for Scott ideals shows that this is not the case.

We finish by considering generic Muchnik reducibility of a countable structure $\mathcal{A}$ to an arbitrary structure $\mathcal{B}$. We relate this to a couple of conditions asserting the ubiquity of countable elementary substructures of $\mathcal{B}$ that are Muchnik above $\mathcal{A}$; we prove that one of these conditions is strictly stronger and the other is strictly weaker than generic Muchnik reducibility.

1. Introduction

If $\mathcal{A}$ and $\mathcal{B}$ are countable structures, then $\mathcal{A}$ is *Muchnik reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant_w \mathcal{B}$) if every ω-copy of $\mathcal{B}$ computes an ω-copy of $\mathcal{A}$.[1] This can be interpreted as saying that $\mathcal{B}$ is intrinsically as complicated as $\mathcal{A}$ is. Knight, Montalbán, and Schweber [KMS] extended this reducibility to arbitrary structures: if $\mathcal{A}$ and $\mathcal{B}$ are (possibly uncountable) structures, then $\mathcal{A}$ is *generically Muchnik reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant_w^* \mathcal{B}$) if in some forcing extension of the universe in which $\mathcal{A}$ and $\mathcal{B}$ are countable, $\mathcal{A}$ is Muchnik reducible to $\mathcal{B}$. Using Shoenfield absoluteness, they showed that generic Muchnik reducibility is robust in the sense that if $\mathcal{A} \leqslant_w^* \mathcal{B}$, then $\mathcal{A} \leqslant_w \mathcal{B}$ in *every* forcing extension that makes $\mathcal{A}$ and $\mathcal{B}$ countable.

Knight, Montalbán and Schweber considered two structures that code all reals. The structure $\mathcal{R}$ is the ordered field of real numbers. The structure $\mathcal{S}$ codes the power set of ω: its universe is $\mathcal{P}(\omega)$, with predicates $P_n(x)$ that hold if and only if $n \in x$.[2] They showed that $\mathcal{S}$ is generically Muchnik reducible to $\mathcal{R}$, and asked if the structures are generic Muchink equivalent. We show that they are not.[3] Perhaps

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[1]This is a special case of Muchnik reducibility [Muč63]; it might be more precise to say that the problem of presenting the structure $\mathcal{A}$ is Muchnik reducible to the problem of presenting $\mathcal{B}$.

[2]Knight, Montalbán and Schweber [KMS] called this structure $\mathcal{W}$. We call it $\mathcal{S}$ for "sets".

[3]After obtaining our result we learned that Igusa and Knight [IK] have independently proved the same result, using completely different means.

surprisingly, what makes $\mathcal{R}$ more complicated that $\mathcal{S}$ has little to do with the field structure. Indeed, let $\mathcal{F}$ be the structure with universe ω^ω and predicates P_σ, for every finite string $\sigma \in \omega^{<\omega}$, such that $P_\sigma(f)$ holds if and only if $\sigma \preccurlyeq f$. It is not hard to see that $\mathcal{F} \leqslant_w^* \mathcal{R}$, but in fact, we show that $\mathcal{R} \equiv_w^* \mathcal{F}$. In other words, it is no harder to present the field of reals than it is to present the structure consisting of all functions on ω. On the other hand, it is strictly easier to present the structure consisting of all subsets of ω.

These results follow from analogous results where the real numbers, subsets of ω, and functions on ω are restricted to those in a countable Turing ideal, as is the case for the ground model reals, subsets, and functions in the forcing extension. Let I be an ideal in the Turing degrees. We say that a subset of ω is in I if its Turing degree is in I, and similarly for functions on ω and real numbers. Let $\mathcal{S}_I$ be the structure consisting of the sets in I along with the predicates P_n defined as above. In the same way, let $\mathcal{F}_I$ be the structure consisting of the functions in I and the predicates P_σ. Let $\mathcal{R}_I$ denote the field of reals in I.

Our main technical result characterises the degrees that compute a presentation of $\mathcal{R}_I$. A *listing of the functions in* I is a sequence $\langle f_n \rangle_{n<\omega}$ consisting of exactly the functions in I, possibly with repetition. By Lemma 2.1, such a degree computes an *injective* listing of the functions in I, i.e., one with no repetitions. From an injective listing of the sets in I, it is easy to compute a copy of $\mathcal{F}_I$. Conversely, from a copy of $\mathcal{F}_I$, we can obviously compute an (injective) listing of the functions in $\mathcal{F}_I$. This proves the equivalence of (2) and (3) below:

Theorem 1.1. *Let I be a countable Turing ideal. The following are equivalent for a Turing degree* $\mathbf{d}$*:*

(1) $\mathbf{d}$ *computes a copy of* $\mathcal{R}_I$*,*
(2) $\mathbf{d}$ *computes a copy of* $\mathcal{F}_I$*,*
(3) $\mathbf{d}$ *computes a listing of the functions in* I*.*

We prove the equivalence of (1) with the other properties in Section 4. Note that the equivalence of (1) and (2) can be restated as saying that if I is a countable Turing ideal, then $\mathcal{R}_I \equiv_w \mathcal{F}_I$. Now consider a forcing extension that makes $(2^{\aleph_0})^V$ (and hence $\mathcal{R}^V$ and $\mathcal{F}^V$) countable. In this extension, the Turing degrees from the ground model form a countable Turing ideal I, and $\mathcal{R}^V = \mathcal{R}_I \equiv_w \mathcal{F}_I = \mathcal{F}^V$. Therefore:

Corollary 1.2. $\mathcal{R} \equiv_w^* \mathcal{F}$*.*

The proof of (1) $\Longrightarrow$ (3) in Theorem 1.1 is quite easy, and in fact, in Proposition 4.1, we will show that any ω-copy of the ordered group $(\mathbb{R}_I, +, <)$ computes a listing of the functions in I. Therefore, $(\mathbb{R}_I, +, <) \equiv_w (\mathbb{R}_I, +, \times) = \mathcal{R}_I$. This gives us a result that Igusa and Knight [IK] obtained independently using model-theoretic tools:

Corollary 1.3. $\mathcal{R} \equiv_w^* (\mathbb{R}, +, <)$*.*[4]

Our third application of Theorem 1.1 takes place in the countable world. It is straightforward to show that a Turing degree computes a listing of the computable

[4]Without the order, the group $(\mathbb{R}, +)$ is much simpler, and in fact, it is generic Muchnik equivalent to the computable structures; in any forcing extension that makes the old reals countable, the group is isomorphic to the infinite-dimensional vector space over the field of rational numbers.

functions exactly if it is high. Applying Theorem 1.1 to the Turing ideal $I = \{\mathbf{0}\}$, we get a natural characterisation of the degree spectrum of the field of computable real numbers.

Corollary 1.4. *A Turing degree computes a copy of $\mathcal{R}_{\{\mathbf{0}\}}$, the field of computable real numbers, if and only if it is high.*[5]

Now let us consider the separation between $\mathcal{S}$ and $\mathcal{R}$. Let I be a countable Turing ideal. As you would expect from our discussion of $\mathcal{F}_I$, the degrees that compute a copy of $\mathcal{S}_I$ are exactly the degrees that compute a *listing of the sets in* I, i.e., a sequence $\langle x_n \rangle_{n<\omega}$ of subsets of ω consisting of exactly those in I (see Corollary 2.2). Jockusch [Joc72] showed that a degree computes a listing of the computable sets if and only if it is high. In other words, it is exactly as hard to list the computable sets as it is to list the computable functions. This implies that $\mathcal{S}_{\{\mathbf{0}\}} \equiv_w \mathcal{F}_{\{\mathbf{0}\}} (\equiv_w \mathcal{R}_{\{\mathbf{0}\}})$. Jockush's argument is tricky and strongly relies on the computable listing of partial computable functions. That his complicated argument is necessary is witnessed by the failure of the analogous results for other ideals.

Theorem 1.5. *Let I be a countable Scott ideal. Then there is a degree that computes a listing of the sets in I, but not of the functions in I.*

This is proved in Section 3. It is now straightforward to separate $\mathcal{S}$ and $\mathcal{R}$.

Corollary 1.6. $\mathcal{S} <_w^* \mathcal{R}$.

Proof. Knight, Montalbán and Schweber [KMS] showed that $\mathcal{S} \leqslant_w^* \mathcal{R}$.[6] We must prove that $\mathcal{R} \nleqslant_w^* \mathcal{S}$. We work in a forcing extension in which $(2^{\aleph_0})^V$ is countable. Let I be the countable Turing ideal consisting of the Turing degrees from the ground model. It is a jump ideal and so certainly a Scott ideal. By Theorem 1.5, let $\mathbf{d}$ be a degree that computes a listing of the sets in I, but does not compute a listing of the functions in I. By Corollary 2.2 below, $\mathbf{d}$ computes a copy of $\mathcal{S}_I = \mathcal{S}^V$. By (3) $\Longrightarrow$ (1) of Theorem 1.1—which is straightforward; see Proposition 4.1—$\mathbf{d}$ does not compute a copy of $\mathcal{R}_I = \mathcal{R}^V$. □

The final section of the paper is not directly related to the work that we have discussed to far. In that section, we ask what the countable elementary substructures of a (possibly uncountable) structure $\mathcal{B}$ tell us about its generic Muchnik degree, and in particular, which countable structures it lies above. Knight, Montalbán and Schweber showed that a countable structure is generically Muchink reducible to the linear ordering ω_1 if and only if it is Muchnik reducible to a countable orderinal. The following theorem can be seen as a generalization from ω_1 to an arbitrary structure.

Theorem 1.7. *Let $\mathcal{A}$ be countable and assume that $\mathcal{B}$ has countable signature. Consider the following properties:*

(1) *There is a countable set $D \subseteq B$ such that for every countable $\mathcal{C} \preccurlyeq \mathcal{B}$ such that $D \subseteq C$, we have $\mathcal{A} \leqslant_w \mathcal{C}$.*
(2) $\mathcal{A} \leqslant_w^* \mathcal{B}$,

[5] We have recently been advised that this result was independently proved by Korovina and Kudinov [KK].

[6] For completeness, note that this follows easily from the facts above and the observation that a listing of the functions in I can effectively be transformed into a listing of the sets in I by changing all nonzero function values to 1.

(3) *For every countable set $D \subseteq B$, there is a countable $\mathcal{C} \preccurlyeq \mathcal{B}$ such that $D \subseteq C$ and $\mathcal{A} \leqslant_w \mathcal{C}$.*

Then (1) $\implies$ (2) $\implies$ (3), *and neither implication reverses.*

2. Listings

In this section, we collect some basic facts about listing sets and functions. Fix a countable ideal I. Recall that a listing of the functions in I is *injective* if it has no repetitions. The following is well-known.

Lemma 2.1. *Every listing of the functions in I computes an injective listing of the functions in I.*

Sketch of proof. This is a finite injury construction. Let $\langle f_n \rangle$ list the functions in I, possibly with repetitions. We compute an injective listing $\langle g_n \rangle$ of the same collection of functions. At any given stage in the construction of $\langle g_n \rangle$, only finitely many values of finitely many members of the listing will have been determined. Injectivity will be a global requirement. In addition, we have requirements of the form

$$R_n \colon (\exists m)\ g_m = f_n.$$

To meet R_0, we let g_0 copy f_0 and restrain lower priority strategies from affecting g_0. For $n > 0$, the strategy for R_n is initialized with a list $g_0, \dots, g_r$ of members of the listing $\langle g_n \rangle$ that are restrained by higher priority requirements. The strategy waits for a stage at which it sees that f_n is different from how each of $g_0, \dots, g_r$ have been defined. Say that such a stage is found. The strategy for R_n acts as follows: Let m be large enough that g_m is currently undefined on all values. The strategy declares that g_m will copy f_n. It retrains $g_0, \dots, g_m$ and reinitializes all lower priority requirements (ensuring that they will respect this restraint and injuring any that have already acted). Finally, the strategy declares each of $g_{r+1}, \dots, g_{m-1}$ to be distinct functions with finite support (hence in I) different from each of $g_0, \dots, g_r$ and g_m. □

As we observed, this lemma implies that the Turing degrees that compute a listing of the functions in I are the same as the degrees that compute a copy of $\mathcal{F}_I$. In other words, we have proved the equivalence of (2) and (3) in Theorem 1.1. The analogous facts holds for sets. As in Lemma 2.1, a listing of the sets in I computes an injective listing of the sets in I. From an injective listing of these sets, we can compute a copy of $\mathcal{S}_I$, and conversely, from a copy of $\mathcal{S}_I$ we can compute an injective listing of the sets in I.

Corollary 2.2. *A degree computes a copy of $\mathcal{S}_I$ if and only if it computes a listing of the sets in I.*

An *I-dominating function* is a function that dominates all the functions in I. Unlike the situation for high degrees, computing an I-dominating function does not imply being able to list the sets or functions in I. Indeed, it does not even imply being able to compute all the elements of I. To see this, let I be an ideal of hyperimmune-free degrees (containing some noncomputable element), and let f be a Δ^0_2 function dominating all computable functions. However:

Lemma 2.3. *A Turing degree computes a listing of the functions in I if and only if it computes both a listing of the sets in I, and an I-dominating function.*

Proof. In the interesting direction, let $\langle x_n \rangle$ be a listing of the sets in I, and let f be an I-dominating function. To each set x_n, natural number m, and $\sigma \in \omega^m$, we assign a function in our new list based on the guess that x_n is the graph of a function g such that $g\restriction_m = \sigma$ and g is majorised by f from input m onwards. As long as our guess is correct, we can compute g from x_n; if we observe that our guess was wrong, we give up and start copying some fixed function in I. □

Lemma 2.4. *The following are equivalent for a Turing degree* $\mathbf{d}$*:*

(1) $\mathbf{d}$ *computes a listing of the functions in* I*.*
(2) $\mathbf{d}$ *computes a listing of the infinite sets in* I*.*
(3) $\mathbf{d}$ *computes a listing of the infinite, coinfinite sets in* I*.*

Proof. (3) $\Longrightarrow$ (2) is immediate. For (2) $\Longrightarrow$ (1) we use Lemma 2.3. Given a listing of the infinite sets in I, we obviously obtain a listing of the sets in I. By combining principal functions of the infinite sets in I we obtain an I-dominating function. For (1) $\Longrightarrow$ (3), asumme that we have a listing of all sets in I and an I-dominating function f. For each set x in I and a natural number m, we guess that x is infinite and coinfinite, and that for all $k \geqslant m$, $x\restriction_{f(k)}$ both contains and excludes at least k numbers. We copy x as long as our guess is valid. If it is shown to be false, we stop copying x and start copying the set of even numbers. □

Finally, we need to mention listing the reals in $\mathbb{R}_I$. Computably, a real number is coded by a Cauchy name, or equivalently a shrinking sequence of closed binary rational intervals containing the real. A list of reals is simply a list of Cauchy names. From a binary expansion of a real we can easily obtain a Cauchy name (but not vice-versa, unless we know that the real is not a binary rational). Hence any listing of the sets in I computes a listing of $\mathbb{R}_I$. The problem with computing a copy of the field $\mathbb{R}_I$ is that we cannot, in finite time, determine whether one number is the sum of two other given numbers; we cannot make that deceleration after seeing only approximations of the given real numbers.

3. A separation

We are ready to prove that it is ofter strictly easier to list the sets in a countable ideal I than it is to list the functions in I. In particular, this is true if I is a countable Scott ideal. Recall that a Turing ideal I is a *Scott ideal* if for every degree $\mathbf{a} \in I$, there is a $\mathbf{b} \in I$ such that $\mathbf{b}$ has PA degree relative to $\mathbf{a}$.

Proposition 3.1. *Suppose that* S *is a countable Scott ideal. Then there is a listing of the sets in* S *that does not compute an* S*-dominating function.*

Proof. We construct the required listing using a notion of forcing $\mathbb{P}_S$. The forcing conditions are finite sequences of sets in S. Each condition is intended to be an approximation of a listing of the sets in S; a condition $\mathbf{q}$ extends a condition $\mathbf{p}$ if it extends it as a sequence. If $G \subseteq \mathbb{P}_S$ is a filter, then $\bigcup G$ is a sequence of sets in S; if G is sufficiently generic, then $\bigcup G$ lists all of the sets in S. We abuse notation and write G for the sequence $\bigcup G$.

Let $\Psi\colon (2^\omega)^\omega \to \omega^\omega$ be a Turing functional. Let $\mathbf{p} \in \mathbb{P}_S$ and suppose that $\mathbf{p}$ forces that $\Psi(G)$ is total. For $n > 0$, let U_n be the set of sequences $q \in (2^\omega)^n$ for which there is some sequence $r \in (2^\omega)^{<\omega}$ that extends the sequence $\mathbf{p}\hat{\ }q$ and such that $\Psi(r, n)\downarrow$. The set U_n is a $\Sigma^0_1(\mathbf{p})$ subset of $(2^\omega)^n$ and we claim that in fact

$U_n = (2^\omega)^n$; otherwise, since I is a Scott ideal, the complement of U_n contains a sequence q of sets in S. Then $\mathbf{p}\hat{\ }q \in \mathbb{P}_S$ is a condition forcing $\Psi(G,n)\uparrow$.

Thus, by the compactness of $(2^\omega)^n$, for all $n > 0$ we can find $f(n)$ such that for all $q \in (2^\omega)^n$ there is some sequence r extending $\mathbf{p}\hat{\ }q$ such that $\Psi(r,n)\downarrow < f(n)$. Since we only need finitely many bits from such an extension r, such a function f can be computed from $\mathbf{p}$, and so we can find such a function in S. We claim that $\mathbf{p}$ forces that $\Psi(G)$ does not dominate f. For let $\mathbf{q}$ be any proper extension of $\mathbf{p}$ and let $n = |\mathbf{q}| - |\mathbf{p}|$. There is some extension r of $\mathbf{q}$ such that $\Psi(r,n)\downarrow < f(n)$. Since finitely many bits of r suffice, and the sets in S are closed under changing finitely many bits, we may take r to be a condition in $\mathbb{P}_S$. □

Proposition 3.1 together with Lemma 2.3 yields a proof of Theorem 1.5.

4. Presentations of fields

Let I be a countable ideal.

Proposition 4.1. *Any copy of the ordered group $(\mathbb{R}_I, +, <)$ computes a listing of the functions in I. Hence the same is true for any copy of $\mathcal{R}_I$.*

Proof. Let $A = (\omega, +_A, <_A)$ be a copy of $(\mathbb{R}_I, +, <)$ and let $f \colon \mathbb{R}_I \to A$ be an isomorphism. Note that the restriction of f to the rationals is A-computable. Hence if $n \in A$ is the image of a dyadic rational number, then we will eventually know its exact value. On the other hand, if $f^{-1}(n)$ is not a dyadic rational, then we can compute the binary expansion of n uniformly in A. With this in mind, we can list the infinite sets in I using A. For each $n \in A$, we make the n^{th} set in the listing be the infinite binary expansion of $f^{-1}(n)$. We begin computing this expansion under the assumption that $f^{-1}(n)$ is not dyadic rational. This process will be partial if $f^{-1}(n)$ is a dyadic rational, but in that case we will eventually know the exact value of $f^{-1}(n)$, so we can let the n^{th} set in our listing be the cofinite binary expansion of $f^{-1}(n)$. This produces an A-computable (injective) listing of all infinite sets in I.

To prove that any copy of copy $A = (\omega, +_A, \times_A)$ of $\mathcal{R}_I$ computes a listing of the functions in I, note that the natural order is computable on A. This is because $n <_A m$ if and only if there is an $x \neq 0_A$ such that $x \times_A x +_A n = m$. This is $\Sigma^0_1(A)$, so assuming that $m \neq n$, we can A-computably determine whether $n <_A m$ or $m <_A n$ with a search. □

We now work toward showing that a degree that computes a listing of the functions in I can compute a copy of the field $\mathcal{R}_I = (\mathbb{R}_I, +, \times)$. We use Tarski's theorem on the decidability of the theory of real closed fields.

Example 4.2. We illustrate the technique by showing the well-known fact that there is a computable copy of the field of algebraic real numbers. The elements of this copy are equivalence classes of *names* for algebraic reals. A name for an algebraic real r is a pair (f, I) where $f \in \mathbb{Q}[x]$, I is a rational open interval (an open interval with rational endpoints), and r is the unique root of f in the interval I. Two names are equivalent if they are the names of the same algebraic real. Tarski's theorem shows that the set of names for algebraic numbers is computable, and that equivalence of names is computable as well. Further, it shows that the graphs of addition and multiplication on names are computable.

In our proof, we will need to generalise this construction to work over algebraically independent tuples. For a sequence $\bar{r}$ of real numbers (finite or infinite), let $\mathbb{R}_{\bar{r}}$ be the field of reals algebraic over $\bar{r}$. We will build, uniformly in an algebraically independent tuple $\bar{r}$, an $\bar{r}$-computable copy $F(\bar{r})$ of $\mathbb{R}_{\bar{r}}$. Generalising Example 4.2, we will work with $\bar{r}$-names. Let $\bar{r}$ be a tuple of real numbers.

- An $\bar{r}$-*name* is a pair (f, I) such that I is a rational open interval, $f \in \mathbb{Q}[\bar{y}, x]$, and $f(\bar{r}, x)$ has a unique root in the interval I. This unique root is denoted by $a_{\bar{r}}(f, I)$.

However, it will actually be useful to know that all of the elements of $\bar{r}$ that appear in the name are necessary. So we define:

- A *frugal* $\bar{r}$-*name* is an $\bar{r}$-name $(f(y_1, \ldots, y_k, x), I)$ for a real a such that for every $i \leqslant k$, if y_i appears in f, then the polynomial
$$f(r_1, \ldots, r_{i-1}, y_i, r_{i+1}, \ldots, r_k, a)$$
is nonzero. (This implies that r_i is an isolated root of that polynomial).

Lemma 4.3. *Let $\bar{r}$ be a tuple of reals.*

(1) *Every element of $\mathbb{R}_{\bar{r}}$ has a frugal $\bar{r}$-name.*
(2) *Suppose that $\bar{r}$ is algebraically independent. If $(f(y_1, \ldots, y_k, x), I)$ and $(g(y_1, \ldots, y_k, x), J)$ are two frugal $\bar{r}$-names for the same real, then the same variables appear in f and g.*

Proof. For (1), let (f, I) be an $\bar{r}$-name for a. If (f, I) is not frugal, then
$$(f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_k, x), I)$$
is also an $\bar{r}$-name for a and mentions fewer variables.

For (2), we use Tarski's quantifier elimination for the theory of ordered real-closed fields. Let (f, I) and (g, J) be two frugal $\bar{r}$-names for a. Suppose that y_i appears in f but not in g. Then r_i is an isolated point satisfying the formula "there is an a in $I \cap J$ that is a root of both $f(r_1, \ldots, y_i, \ldots, r_k, x)$ and $g(r_1, \ldots, r_k, x)$." By quantifier elimination, this shows that r_i is algebraic in the rest of $\bar{r}$. □

Below we construt the computable operator F that takes a finite tuple of reals $\bar{r}$ to the atomic diagram of the structure $F(\bar{r})$. The universe of $F(\bar{r})$ will be an $\bar{r}$-computable set; we treat the language as relational. The functional is partial: if $\bar{r}$ is not algebraically independent, then it is possible that the computation of $F(\bar{r})$ will output only a finite structure and then be forever stuck, waiting to extend the structure.

We will ensure that F has various properties, including:

(1) If $\bar{r}$ is algebraically independent, then $F(\bar{r})$ is total and is isomorphic to $\mathbb{R}_{\bar{r}}$.

Suppose that $\bar{r} \preccurlyeq \bar{t}$ and that $\bar{t}$ is algebraically independent. There is a unique embedding of $F(\bar{r})$ into $F(\bar{t})$. The domains of the $F(\bar{r})$ are not required to be all of ω for notational convenience. This will allows us to ensure:

(2) If $\bar{r} \preccurlyeq \bar{t}$ and $\bar{t}$ is algebraically independent, then $F(\bar{r})$ is a substructure of $F(\bar{t})$.

This allows us to define $F(\bar{r})$ for infinite algebraically independent sequences; for these sequences too we get $F(\bar{r}) \cong \mathbb{R}_{\bar{r}}$.

If $\bar{r}$ is algebraically independent and $\bar{q}$ is a tuple of rationals, then of course $\mathbb{R}_{\bar{r},\bar{q}} = \mathbb{R}_{\bar{r}}$. However $F(\bar{r}, \bar{q})$ may be partial. Nonetheless,

(3) If $\bar{r}$ is algebraically independent and $\bar{q}$ is a tuple of rationals, then there is an embedding of $F(\bar{r}, \bar{q})$ into $F(\bar{r})$.

For $s \in \omega$ and $\bar{r} \in \mathbb{R}^{<\omega}$, let $F_s(\bar{r})$ be the finite substructure of $F(\bar{r})$ decided up to stage s. We will ensure the following:

(4) If $\bar{r}$ is algebraically independent, $\bar{t} \in \mathbb{R}^{<\omega}$ and $s \in \omega$, then there is a tuple $\bar{q}$ (of length $|\bar{t}|$) of rationals such that $F_s(\bar{r}, \bar{t}) = F_s(\bar{r}, \bar{q})$.

Construction of F: Again we use Tarski's effective quantifier elimination for the theory of real closed fields. Every $(\mathbb{R}, +, \times)$-definable subset of $\mathbb{R}^k$ (known as a *semi-algebraic set*) is a positive Boolean combination of sets definable by formulas of the form $p(\bar{x}) = 0$ and $p(\bar{x}) > 0$, where $p \in \mathbb{Q}[\bar{x}]$ is a polynomial. The interior of the set is a computably enumerable open set, uniformly given the defining formula. The key point is that if an algebraically independent tuple belongs to a semi-algebraic set, it must belong to its interior. This shows that the type of an algebraically independent tuple $\bar{r}$ is $\bar{r}$-computable.

In general, for a formula $\varphi(\bar{y})$, we say that a tuple $\bar{r} \in \mathbb{R}^k$ *strongly satisfies* φ if $\bar{r}$ is in the interior of the semi-algebraic set defined by φ. So an algebraically independent tuple satisfies a formula if and only if it strongly satisfies it. We say that $\bar{r}$ *strongly decides* a formula φ if it strongly satisfies φ or it strongly satisfies its negation.

Above we have defined the notions of $\bar{r}$-names and frugal $\bar{r}$-names. Two $\bar{r}$-names (f, I) and (g, J) are *$\bar{r}$-equivalent* if they name the same real number: $a_{\bar{r}}(f, I) = a_{\bar{r}}(g, J)$; we write $(f, I) \sim_{\bar{r}} (g, J)$. If (f, I), (g, J) and (h, K) are $\bar{r}$-names then we write $(f, I) +_{\bar{r}} (g, J) \sim (h, K)$ if $a_{\bar{r}}(f, I) + a_{\bar{r}}(g, J) = a_{\bar{r}}(h, K)$. We similarly treat multiplication.

Each of these notions (including being an $\bar{r}$-name and a frugal $\bar{r}$-name) can be translated into a statement about $\bar{r}$ in the language of ordered fields. For example, $(f, I) \sim_{\bar{r}} (g, J)$ if and only if there is a number $a \in I \cap J$ such that $f(\bar{r}, a) = g(\bar{r}, a) = 0$. Thus we can make sense of strong satisfaction of these notions.

We fix an ω-ordering $\leqslant$ on the set of pairs (f, I) where I is a rational interval and $f \in \mathbb{Q}[y_1, y_2, \dots, x]$. We say that $F(\bar{r})$ is *decided up to k* if: (a) for every pair $(f, I) \leqslant k$, $\bar{r}$ strongly decides if (f, I) is a frugal $\bar{r}$-name; (b) for every pair $(f, I), (g, J) \leqslant k$ of frugal $\bar{r}$-names, $\bar{r}$ strongly decides if $(f, I) \sim_{\bar{r}} (g, J)$; and (c) for every triple $(f, I), (g, J), (h, K) \leqslant k$ of frugal $\bar{r}$-names, $\bar{r}$ strongly decides if $(f, I) +_{\bar{r}} (g, J) \sim (h, K)$ and if $(f, I) \times_{\bar{r}} (g, J) \sim (h, K)$. We let $F(\bar{r})$ be the structure whose domain is the collection of $\sim_{\bar{r}}$-minimal frugal names (f, I) such that $F(\bar{r})$ is decided up to (f, I), and define $+_{\bar{r}}$ and $\times_{\bar{r}}$ on the domain as expected. We use an enumeration of the interiors of semi-algebraic sets and so can evaluate strong satisfaction at any stage s. This tells us how to compute $F_s(\bar{r})$.

Property (1) follows from the fact that satisfaction and strong satisfaction are identical for algebraically independent tuples. For (2), we first note that every $\bar{r}$-name is a $\bar{t}$-name. Let (f, I) be a minimal frugal $\bar{r}$-name and let (g, J) be a frugal $\bar{t}$-name. If $(g, J) \sim_{\bar{t}} (f, I)$ then by Lemma 4.3(2), (g, J) is actually a frugal $\bar{r}$-name; by minimality, $(f, I) \leqslant (g, J)$, so $(f, I) \in F(\bar{t})$ as well.

For (3), we of course map a name $(f(\bar{y}, \bar{z}, x), I)$ to the minimal $\bar{r}$-name equivalent to $(f(\bar{y}, \bar{q}, x), I)$. For (4), we simply choose a tuple $\bar{q}$ sufficiently close to $\bar{t}$ so that for every c.e. open subset W of $\mathbb{R}^{|\bar{t}|}$, $\bar{q} \in W_s$ if and only if $\bar{t} \in W_s$. □

We fix a Turing degree $\mathbf{d}$ that computes a listing of the functions in I; and a $\mathbf{d}$-computable listing $\langle b_n \rangle$ of the reals in I.

Lemma 4.4. *Let $\bar{r}$ be an algebraically independent tuple of elements of $\mathbb{R}_I$. Then $\mathbf{d}$ computes a listing of $\mathbb{R}_I \backslash \mathbb{R}_{\bar{r}}$. This can be done uniformly in $\bar{r}$.*

Proof. Let $\psi\colon \mathbb{R}_{\bar{r}} \to F(\bar{r})$ be the unique isomorphism. The restriction of ψ to the rational numbers is $\bar{r}$-computable.

Let S be the collection of nonempty, finite subsets of ω. Identifying sets with their characteristic functions, we order S lexicographically. Note that S is dense under this ordering. We let $g\colon S \to F(\bar{r})$ be an order-isomorphism, which by the back-and-forth construction we can pick to be $\bar{r}$-computable.

The map $\psi^{-1} \circ g$ can be extended continuously and uniquely to an $\bar{r}$-computable injective function $h\colon X \to \mathbb{R}$, where X is the collection of nonempty, coinfinite subsets of ω. Since h is injective, if $x \in X \backslash S$ (that is, if x is infinite and coinfinite), then $h(x) \in \mathbb{R} \backslash \mathbb{R}_{\bar{r}}$, i.e., $h(x)$ is transcendental over $\bar{r}$.

Further, h is onto $\mathbb{R}$ and restricted to $\mathbb{R} \backslash \mathbb{R}_{\bar{r}}$, h^{-1} is also $\bar{r}$-computable. This means that if $\langle x_n \rangle$ is a listing of the infinite, coinfinite sets in I, then $\langle h(x_n) \rangle$ is a listing of $\mathbb{R}_I \backslash \mathbb{R}_{\bar{r}}$. □

Lemma 4.5. *There is a $\mathbf{d}'$-computable function f such that the set $\{b_{f(n)} : n \in \omega\}$ is algebraically independent over $\mathbb{Q}$, and such that $\mathbb{R}_I = \mathbb{R}_{\langle b_{f(n)} \rangle}$.*

Proof. The function f is defined by recursion; given $\bar{r} = \big(b_{f(0)}, b_{f(1)}, \dots, b_{f(k-1)}\big)$, $f(k)$ is the least n such that b_n is transcendental over $\mathbb{R}_{\bar{r}}$. To find $f(k)$, by Lemma 4.4 we can find a $\mathbf{d}$-computable listing $\langle t_n \rangle$ of the elements of $\mathbb{R}_I \backslash \mathbb{R}_{\bar{r}}$. Using $F(\bar{r})$ we also have a $\mathbf{d}$-computable listing $\langle u_n \rangle$ of the elements of $\mathbb{R}_{\bar{r}}$. So each b_n is in one of the lists but not on both. Using $\mathbf{d}'$ we can find $f(k)$, which is the least n such that b_n is one of the t_m's. □

Let $\langle f_s \rangle$ be a $\mathbf{d}$-computable approximation of the function f given by Lemma 4.5. We let $r_{n,s} = b_{f_s(n)}$ and $\bar{r}_{k,s} = (r_{0,s}, r_{1,s}, \dots, r_{k-1,s})$. We speed up the approximation $\langle f_s \rangle$ and the enumeration of interiors of semi-algebraic sets so that for all $s \in \omega$, the tuple $\bar{r}_{s,s}$ appears to be algebraically independent at stage s in that for all $k < s$, $F_s(\bar{r}_{k,s})$ is a substructure of $F_s(\bar{r}_{k+1,s})$.

Using $\mathbf{d}$ as an oracle, we define an increasing sequence $\langle \mathcal{B}_s \rangle$ of finite structures (in the signature of fields) and isomorphisms $h_s\colon \mathcal{B}_s \to F_s(\bar{r}_{s,s})$. Suppose that $\mathcal{B}_s$ and h_s have been defined. Temporarily, for $t > s$ let k_t be the least such that $r_{k,s} \neq r_{k,t}$. If t is sufficiently late, then $\bar{r}_{k_t,s}$ is algebraically independent. And so by (3) and (4) above we can find some $t > s$ for which there is an embedding $j_s\colon F_s(\bar{r}_{s,s}) \to F_t(\bar{r}_{k_t,s})$. For simplicity of notation (i.e., by a speed-up) we assume that $t = s + 1$.

We let $h_{s+1} \restriction_{\mathcal{B}_s} = j_s \circ h_s$. We then add elements to $\mathcal{B}_s$ to obtain $\mathcal{B}_{s+1}$ and extend $h_{s+1} \restriction_{\mathcal{B}_s}$ to an isomorphism $h_{s+1}\colon \mathcal{B}_{s+1} \to F_{s+1}(\bar{r}_{s+1,s+1})$. We let $\mathcal{B} = \bigcup_s \mathcal{B}_s$. This is a $\mathbf{d}$-computable structure.

The key observation is that if $\bar{r}_{k,s} = \bar{r}_{k,s+1}$, then h_s and h_{s+1} agree on $h_s^{-1}[F_s(\bar{r}_{k,s})]$. This show that the maps h_s stabilise to a limit $h\colon \mathcal{B} \to F(\bar{r})$ which is an isomorphism, where of course $\bar{r} = \big\langle b_{f(n)} \big\rangle$. Since $F(\bar{r}) \cong \mathbb{R}_I$, this proves Theorem 1.1.

5. Countable fragments

In this section, we prove Theorem 1.7. Let $\mathcal{A}$ be a countable structure, and let $\mathcal{B}$ be a (possibly uncountable) structure with countable signature. First, we prove that (2) implies (3) in Theorem 1.7.

Theorem 5.1. *Suppose that $\mathcal{A} \leqslant_w^* \mathcal{B}$. Then for every countable set $D \subseteq B$, there is a countable $\mathcal{C} \preccurlyeq \mathcal{B}$ such that $D \subseteq C$ and $\mathcal{A} \leqslant_w \mathcal{C}$.*

Proof. By adding the elements of D to the language of $\mathcal{B}$, we can ensure that every elementary substructure of $\mathcal{B}$ is a superset of D. So we only have to prove that there a countable $\mathcal{C} \preccurlyeq \mathcal{B}$ such that $\mathcal{A} \leqslant_w \mathcal{C}$. We mimic the proof of Shoenfield absoluteness.

The set of countable structures $\mathcal{C}$ such that $\mathcal{A} \leqslant_w \mathcal{C}$ is $\mathbf{\Pi}^1_1$ (it is Π^1_1 in a Scott sentence for $\mathcal{A}$.) Let S be a tree on $\omega \times \omega_1$ such that the projection $p[S]$ of the set of paths of S onto first coordinates is this set. (A $\mathbf{\Pi}^1_1$ predicate holds iff an associated tree in $\omega^{<w}$ is well-founded, which is true iff that tree has a valid labeling with countable ordinals that witnesses its well-foundedness; the ω_1-coordinates in any path in $[S]$ give a valid labeling of the tree associated to the structure coded by the ω-coordinates.)

Since the language of $\mathcal{B}$ is countable, there is a countable sequence of Skolem functions $\langle h_n \rangle$, each h_n from $\mathcal{B}^n$ to $\mathcal{B}$, such that for every countable set $D = \{d_1, d_2, \dots\} \subseteq \mathcal{B}$, $\bigcup_n h_n(d_1, \dots, d_n)$ is the domain of an elementary substructure of $\mathcal{B}$.

This gives us a tree T on $\omega \times \mathcal{B}$ such that the projection $p[S]$ of the set of paths of S onto first coordinates is the collection of structures with universe ω that are elementarily embeddable into $\mathcal{B}$. For an injective function $f \colon \omega \to \mathcal{B}$ let $\bar{f}$ be a canonical extension of f to an injective function from ω to $\bigcup_n h_n(f(0), \dots, f(n-1))$; and let $\mathcal{C}_f$ be the structure on ω that is the pullback by $\bar{f}$ of its range. The set of paths $[S]$ is the collection of pairs $(\mathcal{C}_f, f)$, where $f \colon \omega \to \mathcal{B}$ is injective.

Merging the trees S and T together, we get a tree $S * T$ which is ill-founded if and only if $\mathcal{A} \leqslant_w \mathcal{C}$ for some countable $\mathcal{C} \preccurlyeq \mathcal{B}$. The point is that this analysis works in both V and in a forcing extension in which $\mathcal{B}$ is countable. However if $\mathcal{A} \leqslant_w^* \mathcal{B}$ then in such a forcing extension, $\mathcal{B}$ is a witness to $S * T$ being ill-founded. Ill-foundedness of trees is absolute, so $S * T$ is ill-founded in V as well. $\square$

If $\mathcal{B}$ is countable, then by taking $D = \mathcal{B}$, the conclusion of Theorem 5.1 is equivalent to $\mathcal{A} \leqslant_w \mathcal{B}$. However, it is not a characterisation of $\mathcal{A} \leqslant_w^* \mathcal{B}$ for for arbitrary $\mathcal{B}$.

Claim 5.2. (3) does not imply (2) in Theorem 1.7.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be countable structures such that $\mathcal{C} \prec \mathcal{D}$, $\mathcal{C}$ has no computable copy, but $\mathcal{D}$ has a computable copy. For example, we can take $\mathcal{C}$ to be the prime model and $\mathcal{D}$ to be the saturated model of an uncountably categorical theory with the appropriate "Baldwin–Lachlan spectrum". Let $\mathcal{B}$ be the disjoint union of $\aleph_1$ many copies of $\mathcal{D}$; let $\mathcal{A}$ be Wehner's graph, whose degree spectrum is the collection of nonzero Turing degrees. Note that this property of $\mathcal{A}$ is absolute. Every countable subset of $\mathcal{B}$ can be extended to be isomorphic to a countable disjoint union of copies of $\mathcal{D}$, together with one copy of $\mathcal{C}$. This gives us a countable elementary substructure of $\mathcal{B}$ that has no computable copy, and so is Muchnik above $\mathcal{A}$. However, in an extension in which $\aleph_1$ is made countable, $\mathcal{B}$ is just a countable disjoint union of copies of $\mathcal{D}$, and so has a computable copy, whence $\mathcal{A} \nleqslant_w^* \mathcal{B}$. □

Next we prove that (1) implies (2) in Theorem 1.7.

Theorem 5.3. *Suppose that there is a countable $D \subseteq \mathcal{B}$ such that for every countable $\mathcal{C} \preccurlyeq \mathcal{B}$ such that $D \subseteq \mathcal{C}$, we have $\mathcal{A} \leqslant_w \mathcal{C}$. Then $\mathcal{A} \leqslant_w^* \mathcal{B}$.*

Proof. The proof is similar to that of Theorem 5.1. This time we start with a tree S on $\omega \times \omega$ whose projection is the set of countable structures that are *not* Muchnik above $\mathcal{A}$. We absorb D into the language of $\mathcal{B}$. Let T be the tree on $\omega \times |\mathcal{B}|$ that guesses countable elementary substructures of $\mathcal{B}$. The assumption implies that the tree $S * T$ is well-founded, and so is well-founded in any generic extension in which $\mathcal{B}$ is countable. If $\mathcal{A} \leqslant_w \mathcal{B}$ in that extension, then $\mathcal{B}$ itself gives us a path on T, contradicting the well-foundedness of $S * T$. □

Again the condition of Theorem 5.3 gives us a trivial characterisation of being Muchnik above $\mathcal{A}$ in the case that $\mathcal{B}$ is countable. However, again we do not get full equivalence with $\mathcal{A} \nleqslant_w^* \mathcal{B}$.

Claim 5.4. (2) does not imply (1) in Theorem 1.7.

Proof. In this case, $\mathcal{B}$ will consist of disjoint copies of linear orderings, where each component is marked by a natural number (say using designated unary predicates). Let x be a subset of ω. Let L be an $\aleph_1$-dense linear ordering (i.e., between any two countable subsets we can find more points). Recall that ζ is the order type of the integers. The n^{th} component of $\mathcal{B} = \mathcal{B}(x)$ is:

$$\begin{cases} \zeta \cdot L, & \text{if } n \in x, \\ (\zeta + \zeta) \cdot L, & \text{if } n \notin x. \end{cases}$$

In a universe in which L, and hence $\mathcal{B}$, is made countable, the third Turing jump of any copy of $\mathcal{B}$ can compute x. So if x is sufficiently complicated, then $\mathcal{B}$ is generically Muchnik above Wehner's graph $\mathcal{A}$. On the other hand, in V, every countable subset of $\mathcal{B}$ can be extended to a countable $\mathcal{C} \subseteq \mathcal{B}$ such that every component in $\mathcal{C}$ isomorphic to $(\zeta + \zeta) \cdot \mathbb{Q}$ and the ζ-chains in each component are ζ-chains in $\mathcal{B}$. First note that $\mathcal{C}$ is an elementary substructure of $\mathcal{B}$; this follows from the fact that any two infinite discrete linear orderings with endpoints are elementary equivalent. But $\mathcal{C}$ has a computable copy, so condition (1) in Theorem 1.7 fails. □

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