THE NUMBER OF LINES IN A MATROID WITH NO $U_{2,n}$-MINOR

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This paper is dedicated to the memory of Michel Las Vergnas.

Abstract. We show that, if q is a prime power at most 5, then every rank-r matroid with no $U_{2,q+2}$-minor has no more lines than a rank-r projective geometry over $\mathrm{GF}(q)$. We also give examples showing that for every other prime power this bound does not hold.

1. Introduction

This paper is motivated by the following special case of a conjecture due to Bonin; see Oxley [4, p. 582].

Conjecture 1.1. *For each prime power q and positive integer r, every rank-r matroid with no $U_{2,q+2}$-minor has at most $\begin{bmatrix} r \\ 2 \end{bmatrix}_q$ lines.*

Here $\begin{bmatrix} r \\ 2 \end{bmatrix}_q = \frac{(q^r-1)(q^{r-1}-1)}{(q-1)(q^2-1)}$ is a q-binomial coefficient. The projective geometry $\mathrm{PG}(r-1,q)$ has $\begin{bmatrix} r \\ 2 \end{bmatrix}_q$ lines, so the conjectured bound is attained. Blokhuis gave examples refuting Conjecture 1.1 for all $q \geq 13$; see Nelson [3]. Our main result is the following.

Theorem 1.2. *Conjecture 1.1 holds if and only if $q \leq 5$.*

All known counterexamples to Conjecture 1.1 have rank 3 and it is quite plausible that the conjecture holds whenever $r \geq 4$; this is supported by a result of Nelson [3] that the conjecture holds when r is sufficiently large relative to q.

The proof of Conjecture 1.1 is straightforward for $q \in \{2,3,4\}$. For $q = 5$ we solve the problem partly by computer search. In all four cases we devote most of our attention to the rank 3 case, to which the general case is easily reduced.

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2. Preliminaries

We follow the notation of Oxley [4]. We write $\mathcal{U}(\ell)$ for the class of matroids with no $U_{2,\ell+2}$-minor. If $e \in E(M)$ then we write $W_1(M)$ for the number of points of M, $W_2(M)$ for the number of lines of M, $W_2^e(M)$ for the number of lines of M *not* containing e, and $\delta_M(e)$ for the number of lines of M containing e. For a simple rank-3 matroid M, we have $M \in \mathcal{U}(\ell)$ iff $\delta_M(e) \leq \ell + 1$ for all $e \in E(M)$. W_1 and W_2 are the first two *Whitney numbers of the second kind*.

The following theorem was proved by Kung [2].

Theorem 2.1. *If $\ell \geq 2$ is an integer and $M \in \mathcal{U}(\ell)$ has rank r, then $W_1(M) \leq \begin{bmatrix} r \\ 1 \end{bmatrix}_q = \frac{q^r-1}{q-1}$.*

Surprisingly, we require a small graph theory result. A 1-*factorisation* of a graph is a partition of its edge set into perfect matchings.

Lemma 2.2. *Any two 1-factorisations of the graph K_6 have an element in common.*

Proof. A 1-factorisation of K_6 is a 5-edge-colouring. The union of any two colour classes is a 2-regular bipartite graph on 6 vertices and edges, so is a 6-cycle, and it is easy to check that for any 6-cycle C there is a unique 5-edge-colouring having C as the union of two of its colour classes. Each 5-edge-colouring has 10 pairs of colour classes and K_6 has 60 6-cycles, so K_6 has six 1-factorisations.

Suppose that there exist disjoint 1-factorisations F_1 and F_2. Each edge is in exactly three perfect matchings, so the set F_3 of perfect matchings not in F_1 or F_2 is also a 1-factorisation. Let F be a 1-factorisation that is not F_1, F_2 or F_3. Since $|F| = 5$ there is some i such that $|F \cap F_i| \geq 2$, but now F and F_i share two colour classes and are thus equal by our above observation. This is a contradiction. $\square$

Our next lemma, invoked twice in Section 5, was proved by a computer search whose structure we briefly sketch.

Lemma 2.3. *Let A be a twelve-element set. There do not exist partitions $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_5$ of A satisfying the following conditions:*

1. *$\mathcal{L}_0$ has exactly six blocks, each of size 2,*
2. *for each $i \in \{1, \ldots, 5\}$, the partition $\mathcal{L}_i$ has at most 5 blocks and each has size at most 4,*
3. *for every distinct $x, y \in A$, there is exactly one $i \in \{0, \ldots, 6\}$ such that $\mathcal{L}_i$ has a block containing x and y,*
4. *for each $i \in \{1, \ldots, 5\}$, if $\mathcal{L}_i$ has exactly five blocks then it has a block of size 1.*

Sketch of computational proof: Fix $\mathcal{L}_0$ arbitrarily and suppose that partitions $\mathcal{L}_1, \ldots, \mathcal{L}_5$ exist. For convenience we assume they each have exactly five parts and allow parts to be empty. The block sizes of each $\mathcal{L}_i : i \in \{1, \ldots, 5\}$ gives an integer partition $(n_{i,1}, \ldots, n_{i,5})$ of 12 so that $4 \geq n_{i,1} \geq n_2 \geq \ldots \geq n_{i,5} \geq 0$ and $n_{i,5} \leq 1$. Moreover, there are 66 unordered pairs of distinct elements of A and six of these pairs are contained in blocks of $\mathcal{L}_0$, so $\sum_{i=1}^{5} \sum_{j=1}^{5} \binom{n_{i,j}}{2} = 60$.

We say two set partitions P, P' are *compatible* if each block of P intersects each block of P' in at most one element. For each integer partition p of 12 into nonnegative parts, let $C(p)$ denote the set of partitions of A that are compatible with $\mathcal{L}_0$ and whose block sizes are the integers in p. Let $C'(p)$ denote the set of orbits of $C(p)$ under the action of the group of the $6! \cdot 2^6$ permutations of A that fix $\mathcal{L}_0$. The following table shows the nine possible p that satisfy our constraints and their associated parameters.

p	$\|C(p)\|$	$\|C'(p)\|$	$\sum_{j=1}^{5} \binom{p_j}{2}$
$(3, 3, 3, 2, 1)$	71040	5	10
$(3, 3, 3, 3, 0)$	4960	3	12
$(4, 3, 2, 2, 1)$	136320	9	11
$(4, 3, 3, 1, 1)$	41280	5	12
$(4, 3, 3, 2, 0)$	38400	4	13
$(4, 4, 2, 1, 1)$	27360	5	13
$(4, 4, 2, 2, 0)$	12720	4	14
$(4, 4, 3, 1, 0)$	15360	2	15
$(4, 4, 4, 0, 0)$	960	1	18

The tuple $(\mathcal{L}_1, \ldots, \mathcal{L}_5)$ must belong to $\mathcal{C} = C(p_1) \times C(p_1) \times \ldots \times C(p_5)$, where $p_1, \ldots, p_5$ are drawn from rows of the table above whose last column sums to 60; there are 68 such (unordered) 5-tuples $p_1, \ldots, p_5$. Moreover, the partitions $\mathcal{L}_0, \ldots, \mathcal{L}_5$ must be pairwise compatible. For each of the 68 possible $\mathcal{C}$, a backtracking search shows this cannot occur; by considering our choice for $\mathcal{L}_1$ up to a permutation of A that preserves $\mathcal{L}_0$, we need only consider one choice of $\mathcal{L}_1$ from each orbit in $C'(p_1)$. Our search was performed with a Python program that runs in under two hours on a single CPU.

□

3. Counterexamples

In this section we construct counterexamples to Conjecture 1.1. They are more elaborate versions of the aforementioned construction of Blokhuis.

Lemma 3.1. *Let q be a prime power and t be an integer with $3 \leq t \leq q$. There is a rank-3 matroid $M(q,t)$ with no $U_{2,q+t}$-minor such that $W_2(M(q,t)) = q^2 + (q+1)\binom{t}{2}$.*

Proof. Let $N \cong \mathrm{PG}(2,q)$. Let $e \in E(N)$ and let L_1, L_2, L_3 be distinct lines of N not containing e and so that $L_1 \cap L_2 \cap L_3$ is empty. Note that every line of M other than L_1, L_2 and L_3 intersects $L_1 \cup L_2 \cup L_3$ in at least 2 and at most 3 elements.

Let $\mathcal{L}$ be the set of lines of N and $\mathcal{L}_e$ be the set of lines of N containing e. For each $L \in \mathcal{L}_e$, let $T(L)$ be a t-element subset of $L - \{e\}$ containing $L \cap (L_1 \cup L_2 \cup L_3)$. Observe that the $T(L)$ are pairwise disjoint. Let $X = \cup_{L \in \mathcal{L}_e} T(L)$, noting that $L_1 \cup L_2 \cup L_3 \subseteq X$ and so each line in $\mathcal{L}$ intersects X in at least two elements. Let $M(q,t)$ be the simple rank-3 matroid with ground set X whose set of lines is $\mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1 = \{L \cap X : L \in \mathcal{L} - \mathcal{L}_e\}$, and $\mathcal{L}_2$ is the collection of two-element subsets of the sets $T(L) : L \in \mathcal{L}_e$. Note that $\mathcal{L}_1$ and $\mathcal{L}_2$ are disjoint. Every $f \in X$ lies in q lines in $\mathcal{L}_1$ and in $(t-1)$ lines in $\mathcal{L}_2$, so $M(q,t)$ has no $U_{2,q+t}$-minor. Moreover, we have $\mathcal{L}_1 = |\mathcal{L} - \mathcal{L}_e| = q^2$ and $\mathcal{L}_2 = |\mathcal{L}_e|\binom{t}{2} = (q+1)\binom{t}{2}$. This gives the lemma. □

This next theorem refutes Conjecture 1.1 for all $q \geq 7$.

Theorem 3.2. *If ℓ is an integer with $\ell \geq 7$, then there exists $M \in \mathcal{U}(\ell)$ such that $r(M) = 3$ and $W_2(M) > \ell^2 + \ell + 1$.*

Proof. If $\ell \geq 127$, let q be a power of 2 such that $\frac{1}{4}(\ell+2) < q \leq \frac{1}{2}(\ell+2)$. We have $W_2(M(q,q)) = q^2 + \binom{q}{2}(q+1) > \frac{1}{2}q^3 \geq \frac{1}{128}(\ell+1)^3 \geq (\ell+1)^2 > \ell^2 + \ell + 1$.

If $7 \leq \ell < 127$, then it is easy to check that there is some prime power $q \in \{5, 7, 9, 13, 19, 32, 59, 113\}$ such that $\frac{1}{2}(\ell+2) \leq q \leq \ell - 2$. Note that $3 < \ell+2-q \leq q$. Let $f_q(x) = q^2 + (q+1)\binom{x+2-q}{2} - (x^2+x+1)$. This function $f_q(x)$ is quadratic in x with positive leading coefficient and $f_q(q) = f_q(q+1) = 0$; it follows that $f(x) > 0$ for every integer $x \notin \{q, q+1\}$. Now the matroid $M = M(q, \ell+2-q)$ satisfies $M \in \mathcal{U}(\ell)$ and $W_2(M) - (\ell^2 + \ell + 1) = f_q(\ell) > 0$. □

We conjecture that, for large ℓ, the matroids $M(q,q)$ give the correct upper bound for the number of lines in a rank-3 matroid in $\mathcal{U}(\ell)$.

Conjecture 3.3. *If ℓ is a sufficiently large integer and $M \in \mathcal{U}(\ell)$ has rank 3, then $W_2(M) \leq W_2(M(q,q)) = q^2 + \binom{q}{2}(q+1)$, where q is the largest prime power such that $2q \leq \ell + 2$.*

4. Small q

Lemma 4.1. *Let $q \geq 2$ be an integer. If $M \in \mathcal{U}(q)$ has rank 3 and has a $U_{2,q+1}$-restriction, then $W_2(M) \leq q^2+q+1$ and $W_2^e(M) \leq q^2$ for each nonloop e of M.*

Proof. We may assume that M is simple; let $M|L$ be a $U_{2,q+1}$-restriction of M. If some line L' of M does not intersect L then contracting a point of L' yields a $U_{2,q+2}$-minor, so every line of M intersects L. Therefore $W_2(M) = \sum_{x\in L}(\delta_M(x)-1)+1 \leq (q+1)((q+1)-1)+1 = q^2+q+1$. For each $e \in E(M)-L$ we clearly have $\delta_M(e) = q+1$ so $W_2^e(M) \leq (q^2+q+1)-(q+1) = q^2$. For each $e \in L$ we have $W_2^e(M) = \sum_{x\in L-\{e\}}(\delta_M(e)-1) \leq q(q+1-1) = q^2$. □

Lemma 4.2. *If $q \in \{2,3,4\}$ and $M \in \mathcal{U}(q)$ is a rank-3 matroid with a $U_{2,q}$-restriction L and no $U_{2,q+1}$-restriction, then at most q lines of M are disjoint from L.*

Proof. We may assume that M is simple. Suppose that there is a set $\mathcal{L}$ of lines disjoint from L such that $|\mathcal{L}| = q+1$. Since each $x \in E(M)-L$ lies on q lines intersecting L it lies on at most one line in $\mathcal{L}$, so the lines in $\mathcal{L}$ are pairwise disjoint. Let X be a set formed by choosing two points from each line in $\mathcal{L}$; note that $|X| = 2(q+1)$ and $X \cap L = \varnothing$.

Since each X lies on at most one line disjoint from L, at most $(q+1)$ pairs of elements of X span lines disjoint from L, so at least $\binom{2(q+1)}{2} - (q+1) = 2q(q+1)$ pairs of elements of X span a line intersecting L. Since $|L| = q$, there is some $y \in L$ such that at least $2(q+1)$ pairs of elements of X span y. Let $\mathcal{L}_y$ be the set of lines of $M|(\{y\}\cup X)$ that contain y. Every line in $\mathcal{L}_y$ spans a line of M containing y and none spans L itself, so $|\mathcal{L}_y| \leq q$. We also have $\sum_{L\in\mathcal{L}_y}(|L|-1) = |X| = 2(q+1)$ and $\sum_{L\in\mathcal{L}_y}\binom{|L|-1}{2} \geq 2(q+1)$ by choice of y. Since M has no $U_{2,q+1}$-restriction, we also have $|L|-1 \leq q-1$ for each $L \in \mathcal{L}_y$. It remains to check that, for $q \in \{2,3,4\}$ there are no solutions to the system $n_1+n_2+\ldots+n_q = 2(q+1)$, $\binom{n_1}{2}+\ldots+\binom{n_q}{2} \geq 2(q+1)$ subject to $n_i \in \{0,\ldots,q-1\}$ for each i. This is easy.

□

Lemma 4.3. *Let $q \in \{2,3,4\}$. If $M \in \mathcal{U}(q)$ has rank 3 and has a $U_{2,q}$-restriction, then $W_2(M) \leq q^2+q+1$ and $W_2^e(M) \leq q^2$ for each nonloop e of M.*

Proof. We may assume that M is simple and, by Lemma 4.1, that M has no $U_{2,q+1}$-restriction; let $M|L$ be a $U_{2,q}$-restriction of M and let $f \in L$. If $W_2^f(M) \geq q^2+1$ then, since each $x \in L-\{f\}$ is on at most q lines not containing f, there are at most $(|L|-1)q = q^2 - q$ lines that intersect L but not f. Therefore there are at least $(q^2+1) - (q^2-q) = q+1$ lines that do not intersect L. This is a contradiction by Lemma 4.2. So $W_2^f(M) \le q^2$ for each e in a $U_{2,q}$-restriction of M; since $W^2(M) = W_2^f(M) + \delta_M(f) \le W_2^f(M) + q + 1$ for every f this resolves the first part of the lemma, as well as the second part if e is in a $U_{2,q}$-restriction.

It remains to bound $W_2^e(M)$ if e is in no $U_{2,q}$-restriction. If $\delta_M(e) \ge q+1$ then we have $W_2^e(M) = W_2(M) - \delta_e(M) \le q^2$ as required, so we may assume that $\delta_M(e) \le q$. Therefore e is in at most q lines containing at most $q-2$ other points each, so $|E(M) - e| \le q(q-2)$. Each $x \in E(M) - e$ is in at most q lines not containing e and each such line contains at least 2 points of $E(M)-e$, so $W_2^e(M) \le \frac{1}{2}q|E(M)-e| = \frac{1}{2}q^2(q-2) \le q^2$, since $\frac{1}{2}(q-2) \le 1$.

□

Lemma 4.4. *If $q \in \{2,3,4\}$ and $M \in \mathcal{U}(q)$ has rank 3 and has no $U_{2,q}$-restriction, then $W_2(M) \le q^2 + q + 1$ and $W_2^e(M) \le q^2$ for each nonloop e of M.*

Proof. We may assume that M is simple; let $n = |M|$. If $q = 2$ then the result is vacuous and if $q = 3$ then M has no $U_{2,3}$-restriction so $M \cong U_{3,n}$ and $n \le 5$ so both conclusions are clear. It remains to resolve the $q = 4$ case.

Suppose that $W_2(M) \ge 4^2 + 4 + 2 = 22$. Every line of M contains either two or three points; for each $f \in E(M)$ let ℓ_f be the number of 3-point lines of M containing f. Let ℓ be the total number of 3-point lines of M. Each 3-point line of M contains 3 pairs of points of M, so $22 \le W_2(M) = \binom{n}{2} - 2\ell$. Moreover, every $e \in E(M)$ is in at most 5 lines so $n \le 1 + 2\ell_f + (5 - \ell_f) = 6 + \ell_f$. Summing this expression over all $f \in E(M)$ gives $n^2 \le 6n + 3\ell$. Therefore $2(6n+3\ell) + 3(\binom{n}{2} - 2\ell) \ge 2n^2 + 66$, giving $0 \ge n^2 - 21n + 132 = (n - \frac{21}{2})^2 + \frac{87}{4}$, a contradiction; therefore $W_2(M) \le 4^2 + 4 + 1$. From here, it is also easy to obtain a contradiction to $W_2^e(M) > 4^2$ in a manner similar to the proof of Lemma 4.3. □

5. Five

We now consider the number of lines in rank-3 matroids in $\mathcal{U}(5)$, first dealing with those that have no $U_{2,5}$-restriction.

Lemma 5.1. *If $M \in \mathcal{U}(5)$ has rank 3 and has no $U_{2,5}$-restriction, then $W_2(M) \le 5^2 + 5 + 1$.*

Proof. We may assume that M is simple. Let $n = |M|$ and for each $i \in \{2,3,4\}$, let ℓ_i be the number of lines of length i in M, noting that every line of M has length $2, 3$ or 4. Suppose for a contradiction that $\ell_2 + \ell_3 + \ell_4 \geq 32$. Let P be the set of pairs (e, L) where $e \in L$. We have $2\ell_2 + 3\ell_3 + 4\ell_4 = |P| = \sum_{e \in E(M)} \delta_M(e) \leq 6n$. There are $\binom{n}{2}$ pairs of elements of M, each of which is contained in exactly one line of M, and an i-element line contains $\binom{i}{2}$ such pairs. We therefore have $\ell_2 + 3\ell_3 + 6\ell_4 = \binom{n}{2}$. Now

$$\begin{aligned} \ell_4 &= (\ell_2 + 3\ell_3 + 6\ell_4) + 3(\ell_2 + \ell_3 + \ell_4) - 2(2\ell_2 + 3\ell_3 + 4\ell_4) \\ &\geq \binom{n}{2} + 3 \cdot 32 - 2 \cdot 6n \end{aligned}$$

nd $\ell_1 + 3\ell_3 = \binom{n}{2} - 6\ell_4 \leq 72n - 18 \cdot 32 - 5\binom{n}{2} = \frac{-5}{2}\left(n - \frac{149}{10}\right)^2 - \frac{839}{40} < 0$, a contradiction. □

Lemma 5.2. *If $M \in \mathcal{U}(5)$ is a rank-3 matroid with no $U_{2,5}$-restriction and e is a nonloop of M, then $W_2^e(M) \leq 5^2$.*

Proof. We may assume that M is simple. If $\delta_M(e) = 6$ then $W_2^e(M) \leq 31 - 6 = 25$ by the previous lemma, so we may assume that $\delta_M(e) \leq 5$. Let $n = |M|$ and let ℓ_2^e, ℓ_3^e and ℓ_4^e be the number of lines of length 2, 3 and 4 respectively that do not contain e. Suppose for a contradiction that $\ell_2^e + \ell_3^e + \ell_4^e \geq 26$. Let P be the set of pairs (f, L), where L is a line not containing e and $f \in L$. Clearly $|P| = 2\ell_2^e + 3\ell_3^e + 4\ell_4^e$, but also, since every $f \neq e$ is on at most 5 lines not containing e, we have $|P| \leq 5(n-1)$, so $2\ell_2^e + 3\ell_3^e + 4\ell_4^e \leq 5(n-1)$. Finally, let Q be the set of two-element sets $\{f_1, f_2\} \subset E(M)$ that span a line not containing e. As before, we have $|Q| = \ell_2^e + 3\ell_3^e + 6\ell_4^e$. On the other hand, there are at most 5 lines of M through e and each contains at most 3 other points, so there are at most $5\binom{3}{2} = 15$ two-element subsets of $E(M) - \{e\}$ that are not in Q. Therefore $|Q| = \binom{n-1}{2} - s$ for some $s \in \{0, \ldots, 15\}$, and $\ell_2^e + 3\ell_3^e + 6\ell_4^e = \binom{n-1}{2} - s$. Now

$$\begin{aligned} \ell_4^e &= (\ell_2^e + 3\ell_3^e + 6\ell_4^e) + 3(\ell_2^e + \ell_3^e + \ell_4^e) - 2(2\ell_2^e + 3\ell_3^e + 4\ell_4^e) \\ &\geq \binom{n-1}{2} - s + 3 \cdot 26 - 2(5(n-1)) \\ &= \binom{n-1}{2} - 10n + 88 - s. \end{aligned}$$

Therefore, using $s \leq 15$ we have $\ell_2^e + 3\ell_3^e = |Q| - 6\ell_4 \leq \binom{n-1}{2} - s - 6(\binom{n-1}{2} - 10n + 88 - s) = 60n - 528 - 5\binom{n-1}{2} + 5s \leq 60n - 453 - 5\binom{n-1}{2} = \frac{-5}{2}\left(n - \frac{27}{2}\right)^2 - \frac{19}{8} < 0$, a contradiction. □

Lemma 5.3. *If $M \in \mathcal{U}(5)$ has rank 3 and has a $U_{2,5}$-restriction, then $W_2(M) \leq 5^2 + 5 + 1$.*

Proof. Let M be a counterexample for which $|M|$ is minimized. Note that M is simple, that $W_2(M) \geq 32$, and that, by Lemma 4.1, M has no $U_{2,6}$-restriction.

Let $L = \{x_1, x_2, x_3, x_4, x_5\}$. Each element of L lies on at most five other lines, so there are at least $32 - 5 \cdot 5 - 1 = 6$ lines $L_{0,1}, L_{0,2}, \ldots, L_{0,6}$ of M that do not intersect L. For each $i \in \{1, \ldots, 6\}$ let a_{2i-1} and a_{2i} be distinct elements of $L_{0,i}$. Note that each $e \in E(M) - L$ lies on five lines meeting L so lies on at most one other line; it follows that the set $A = \{a_1, a_2, \ldots, a_{12}\}$ has twelve elements and that $\mathcal{L}_0 = \{L_{0,0}, \ldots, L_{0,6}\}$ is a partition of A into pairs.

For each $i \in \{1, \ldots, 5\}$ let $\mathcal{L}_i'$ be the set of lines of M containing x_i other than L and let $\mathcal{L}_i = \{L' - \{x_i\} : L' \in \mathcal{L}_i'\}$. We have $|\mathcal{L}_i| \leq 5$ and clearly $\mathcal{L}_i$ is a partition of A. If there are six lines through x_i each containing at least two other points, then $W_2(M \setminus x_i) = W_2(M)$, contradicting minimality of $|M|$. Therefore $|L'| \leq 1$ for some $L' \in \mathcal{L}_i$. Since M has no $U_{2,6}$-restriction we also have $|L'| \leq 4$ for each $L \in \mathcal{L}_i$. Finally, since each two-element subset of A either spans a line in $\mathcal{L}_0$ or a line in $\mathcal{L}_i'$ for a unique i, each such pair is contained in a block of exactly one of the partitions $\mathcal{L}_0, \ldots, \mathcal{L}_5$. By Lemma 2.3 this is impossible. □

Lemma 5.4. *If $M \in \mathcal{U}(5)$ has rank 3 then $W_2^e(M) \leq 5^2$ for each nonloop e of M.*

Proof. Let (M, e) be a counterexample for which $|M|$ is minimized. Note that M simple and that, by Lemma 5.2, M has a $U_{2,5}$-restriction $M|L$. If $\delta_M(e) \geq 6$ then $W_2^e(M) \leq 5^2 + 5 + 1 - 6 = 25$ by Lemma 5.3, so $\delta_M(e) \leq 5$. If there is some $f \in E(M) - \{e\}$ on six lines each containing at least two other points, then $W_2^e(M \setminus f) = W_2^e(M)$, contradicting minimality. Therefore every $x \in E(M)$ is on at most five lines that contain two other points (note that e also has this property).

If $e \in L$ then observe that each $f \in L - \{e\}$ is on at most 5 other lines not containing e, so there are at least $26 - 20 = 6$ lines of M disjoint from e. Let B be a set formed by choosing of a pair of elements from each of these lines. In a similar manner to the previous lemma, we obtain six partitions of B that contradict Lemma 2.3. We thus assume that $e \notin L$.

Let $L = \{x_1, \ldots, x_5\}$. Each $x \in L$ lies on at most four lines other than L not containing e, so there exist $26 - 1 - 20 = 5$ lines $L_{0,1}, \ldots, L_{0,5}$ of M disjoint from $L \cup \{e\}$. If there are six such disjoint lines, then we again obtain a contradiction with Lemma 2.3; we therefore assume that every x_i in L lies on exactly four other lines of M disjoint from L, so $\delta_M(x_i) = 6$ for each $i \in \{1, \ldots, 5\}$.

For each $j \in \{1, \ldots, 5\}$ let a_{2j-1}, a_{2j} be distinct elements of $L_{0,j}$. Let $A = \{a_1, \ldots, a_{10}\}$ and let $N = M|(L \cup A \cup \{e\})$. As in the proof of the previous lemma the lines $L_{0,j}$ partition A into pairs, and so $|N| = 16$. Since e lies on at most 5 lines of N each containing at most three other points, the elements of $E(N) - \{e\}$ partition into three-element sets $L_{1,e}, \ldots, L_{5,e}$ such that $L_{j,e} \cup \{e\}$ is a four-element line of N for each j.

As before we consider the lines through each element of L, and for each $x_i \in L$ we obtain a partition $\mathcal{L}_i = \{L_{i,1}, \ldots, L_{i,5}\}$ of $A \cup \{e\}$ into five blocks corresponding to the lines of N through x_i other than L. Again we have $4 \geq |L_{i,1}| \geq |L_{i,2}| \geq \ldots \geq |L_{i,5}| = 1$, (we have $|L_{i,5}| = 1$ here by minimality of M and the fact that $\delta_M(x_i) = 6$) and $\sum_{j=1}^{5} |L_{i,j}| = 11$ for each i. Moreover, for each i the point x_i is on the four-element line $L_{i,e}$, so for some j we have $|L_{i,j}| = 3$. Finally, there are $\binom{11}{2} - 5 = 50$ pairs of elements in $A \cup \{e\}$ that do not span one of the lines $L_{0,i}$, so $\sum_{i=1}^{5} \sum_{j=1}^{5} \binom{|L_{i,j}|}{2} = 50$.

If $4 \geq n_1 \geq \ldots \geq n_5 = 1$ are integers summing to 11 such that some n_i is 3, then $\binom{n_1}{2} + \ldots + \binom{n_5}{2} \leq 10$ with equality only if $(n_1, n_2, \ldots, n_5) = (4, 3, 2, 1, 1)$. Therefore $(|L_{i,1}|, |L_{i,2}|, \ldots, |L_{i,5}|) = (4, 3, 2, 1, 1)$ for each i; note that $L_{i,e} \cup \{x_i\} = L_{i,2}$. Therefore, in the fifteen-element matroid $N \setminus e$, each $x_i \in L$ lies on two five-element lines; two three-element lines and two two-element lines. For each integer k Let $\mathcal{J}_k$ be the set of k-element lines of $N \setminus e$.

Let Y be the union of the lines in $\mathcal{J}_5$. By the above reasoning each $y \in Y$ lies on exactly two lines in $\mathcal{J}_5$, so it follows that $5|\mathcal{J}_5| = 2|Y|$ and so $|Y| \equiv 0 \pmod 5$. Since three 5-point lines account for at least 13 points, it is clear that $|Y| > 10$ and so we must have $|Y| = 15$ and $|Y| = E(N \setminus e)$. Therefore every element of $N \setminus e$ lies on exactly two lines in $\mathcal{J}_5$, $|\mathcal{J}_5| = \frac{2}{5}|Y| = 6$, and the elements of $N \setminus e$ are exactly the intersections of the $\binom{6}{2}$ pairs of lines in $\mathcal{J}_5$. There is now a natural mapping of $E(N \setminus e)$ to the edge set of the complete graph K_6 with vertex set $\mathcal{J}_5$, where the elements of each $J \in \mathcal{J}_5$ are the edges incident with the vertex J. The lines in $\mathcal{J}_3$ map to three-edge matchings. We know the lines $\mathcal{L}_{i,e} - \{e\}$ are in $\mathcal{J}_3$ and partition $E(N \setminus e)$, and each $f \in E(N \setminus e)$ is contained in exactly two lines in $\mathcal{J}_3$, so $\mathcal{J}_3$ is the union of two disjoint partitions of $E(N \setminus e)$. This gives two disjoint 1-factorisations of K_6, a contradiction by Lemma 2.2. $\square$

6. Higher Rank

Combining all lemmas in the last two sections gives the following:

Theorem 6.1. *If $q \in \{2,3,4,5\}$ and M is a rank-3 matroid in $\mathcal{U}(q)$, then $W_2(M) \le q^2+q+1$ and $W_2^e(M) \le q^2$ for each nonloop e of M.*

We now generalise this to arbitrary rank. For a matroid M and a nonloop $e \in E(M)$, let $\mathcal{P}_M(e)$ denote the set of planes of M containing e. Note that $|\mathcal{P}_M(e)| = W_2(M/e)$. When we contract a nonloop e in a matroid M, every line through e becomes a point and every set of lines not containing e that span a plane in $\mathcal{P}_M(e)$ are identified into a single line. This gives the following lemma:

Lemma 6.2. *If M is a matroid and $e \in E(M)$ is a nonloop, then $W_2(M) = W_1(M/e) + \sum_{P \in \mathcal{P}_M(e)} W_2^e(M|P)$.*

From here we can easily verify Conjecture 1.2 for all $q \le 5$.

Theorem 6.3. *If $q \in \{2,3,4,5\}$ and $M \in \mathcal{U}(q)$ then $W_2(M) \le \left[{r(M) \atop 2}\right]_q$.*

Proof. If $r \le 2$ then the result is obvious. Suppose inductively that $r \ge 3$ and that the result holds for smaller r, and let e be a nonloop of M. By Theorem 2.1 we have $W_1(M/e) \le \frac{q^{r-1}-1}{q-1}$ and by Theorem 6.1 we have $W_2^e(M|P) \le q^2$ for each $P \in \mathcal{P}_M(e)$. Therefore, by Lemma 6.2 and the inductive hypothesis,

$$\begin{aligned} W_2(M) &= W_1(M/e) + \sum_{P \in \mathcal{P}_M(e)} W_2^e(M|P) \\ &\le \tfrac{q^{r-1}-1}{q-1} + q^2|\mathcal{P}_M(e)| \\ &= \tfrac{q^{r-1}-1}{q-1} + q^2 W_2(M/e) \\ &\le \left[{r-1 \atop 1}\right]_q + q^2\left[{r-1 \atop 2}\right]_q \\ &= \left[{r \atop 2}\right]_q, \end{aligned}$$

as required.

□

7. References

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