THE NUMBER OF RANK-k FLATS IN A MATROID WITH NO $U_{2,n}$-MINOR

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Abstract. We show that, if k and ℓ are positive integers and r is sufficiently large, then the number of rank-k flats in a rank-r matroid M with no $U_{2,\ell+2}$-minor is less than or equal to number of rank-k flats in a rank-r projective geometry over $\mathrm{GF}(q)$, where q is the largest prime power not exceeding ℓ.

1. Introduction

Let $W_k(M)$ denote the number of rank-k flats in a matroid M. For example, we have $W_k(\mathrm{PG}(r-1,q)) = \left[{r \atop k}\right]_q$, the q-binomial coefficient for r and k. The following conjecture appears in Oxley [4 p. 582], attributed to Bonin:

Conjecture 1.1. *If q is a prime power, $k \geq 0$ is an integer and M is a rank-r matroid with no $U_{2,q+2}$-minor, then $W_k(M) \leq \left[{r \atop k}\right]_q$.*

Unfortunately for $k = 2$, $r = 3$ this conjecture is false for all $q \geq 13$; we discuss counterexamples due to Blokhuis (private communication) soon. Our main theorem, on the other hand, resolves the conjecture whenever r is large compared to q and k. In fact we show more, obtaining an eventually best-possible bound on W_k when excluding an arbitrary rank-2 uniform minor:

Theorem 1.2. *Let $\ell \geq 2$ and $k \geq 0$ be integers. If r is sufficiently large and M is a rank-r matroid with no $U_{2,\ell+2}$-minor, then $W_k(M) \leq \left[{r \atop k}\right]_q$, where q is the largest prime power so that $q \leq \ell$.*

This was shown for $k = 1$ in [1]. The bound is attained by projective geometries over $\mathrm{GF}(q)$, so cannot be improved.

Our theorem does not resolve Conjecture 1.1 in the case where r is not too large compared to k; in particular, the conjecture remains open in the interesting case when $k = r-1$ (that is, where W_k is the number of hyperplanes).

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We now discuss the counterexamples for $r = 3$ and $k = 2$, first giving the construction of Blokhuis. For each simple rank-3 matroid M, let $\mathcal{L}^+(M)$ be the set of lines of M containing at least 3 points. Note that $\mathcal{L}^+(M)$ determines M.

Lemma 1.3. *If q is a prime power then there is a rank-3 matroid $M(q)$ with no $U_{2,2q}$-minor such that $W_2(M(q)) = \frac{1}{2}q^2(q+1)$.*

Proof. Let $N \cong \mathrm{AG}(2, q)$. Let $\mathcal{L}$ be a set of q pairwise disjoint lines of N. If $M(q)$ is the simple rank-3 matroid with $E(M(q)) = E(N)$ and $\mathcal{L}^+(M(q)) = \mathcal{L}^+(N) - \mathcal{L}$, then $M(q)$ has $q^2 + q\binom{q}{2} = \frac{1}{2}q^2(q+1)$ lines and each element of $M(q)$ lies on $2q-1$ lines of $M(q)$, so $M(q)$ has no $U_{2,2q}$-minor. □

We now verify that, when $r = 3$ and $k = 2$, Conjecture 1.1 is false for nearly all q:

Corollary 1.4. *Let $q > 125$ be a prime power. There is a rank-3 matroid M with no $U_{2,q+2}$-minor such that $W_2(M) > \begin{bmatrix}3\\2\end{bmatrix}_q$.*

Proof. Let q' be a power of 2 so that $\frac{1}{4}(q+2) < q' \le \frac{1}{2}(q+2)$. Now $M(q')$ of Lemma 1.3 has no $U_{2,2q'}$-minor so has no $U_{2,q+2}$-minor, and

$$W_2(M(q')) = \tfrac{1}{2}(q')^2(q'+1) > \tfrac{1}{128}(q+2)^3 \ge (q+2)^2 > q^2+q+1 = \begin{bmatrix}3\\2\end{bmatrix}_q.$$

□

If more care is taken, then the same construction can in fact be shown to provide counterexamples for all $q \ge 13$. Smaller values of q will be considered in detail in a future paper.

Despite these examples, it is likely that the rank-3 case is sporadic and that Conjecture 1.1 holds unconditionally for all $r \ge 4$. We also conjecture a strengthened version of Theorem 1.2, in which r is not required to be large compared to k:

Conjecture 1.5. *Let $\ell \ge 2$ be an integer. If r is sufficiently large and M is a rank-r matroid with no $U_{2,\ell+2}$-minor, then $W_k(M) \le \begin{bmatrix}r\\k\end{bmatrix}_q$ for all integers $k \ge 0$, where q is the largest prime power such that $q \le \ell$.*

2. Preliminaries

We follow the notation of Oxley [4]. In particular for each integer $\ell \ge 0$ we write $\mathcal{U}(\ell)$ for the class of matroids with no $U_{2,\ell+2}$-minor.

The first theorem we need gives a bound on W_1 for all matroids in $\mathcal{U}(\ell)$, and was proved by Kung [3]. Note that this resolves the $k = 1$ case of Conjecture 1.1.

Theorem 2.1. *If $\ell \geq 2$ and $M \in \mathcal{U}(\ell)$, then $W_1(M) \leq \frac{\ell^{r(M)}-1}{\ell-1}$.*

We often use the cruder bound $W_1(M) < \ell^{r(M)}$. The next result, which provides a large affine geometry restriction in a dense matroid in $\mathcal{U}(\ell)$ of very large rank, appears in [2].

Theorem 2.2. *There is a function $f : \mathbb{Z}^3 \times \mathbb{R} \to \mathbb{Z}$ such that, for all $n, \ell \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}^+$ and prime powers q, if $M \in \mathcal{U}(\ell)$ satisfies $W_1(M) \geq \alpha q^{r(M)}$ and $r(M) \geq f(\ell, n, q, \alpha)$, then M has either an $\mathrm{AG}(n, q)$-restriction or a $\mathrm{PG}(n, q')$-minor for some $q' > q$.*

We now consider the parameter $W_k(M)$, known as the *k-th Whitney number of M of the second kind*, and its value on projective geometries. It is well-known (see [4 p.162], for example) that $\mathrm{PG}(r-1, q)$ has exactly $\begin{bmatrix} r \\ k \end{bmatrix}_q$ rank-k flats, where $\begin{bmatrix} r \\ k \end{bmatrix}_q$ is the 'q-binomial coefficient' defined recursively by $\begin{bmatrix} r \\ 0 \end{bmatrix}_q = \begin{bmatrix} r \\ r \end{bmatrix}_q = 1$ and $\begin{bmatrix} r \\ k \end{bmatrix}_q = q^k \begin{bmatrix} r-1 \\ k \end{bmatrix}_q + \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q$ for $0 < k < r$. An equivalent definition is given by

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = \frac{(q^r - 1)(q^{r-1} - 1) \dots (q^{r-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \dots (q - 1)}.$$

Using these definitions, it is not hard to show that $\begin{bmatrix} r \\ k \end{bmatrix}_q$ satisfies a few basic properties, which we will use freely:

Lemma 2.3. *For every prime power q and all integers $0 < k < r$, the following hold:*

(1) $\begin{bmatrix} r \\ k \end{bmatrix}_q \geq q^{ki} \begin{bmatrix} r-i \\ k \end{bmatrix}_q$ for all $i \in \{0, \dots, r\}$.
(2) $q^{k(r-k)} \leq \begin{bmatrix} r \\ k \end{bmatrix}_q \leq q^{rk}$.
(3) $\begin{bmatrix} r \\ k \end{bmatrix}_q = \begin{bmatrix} r-1 \\ k \end{bmatrix}_q + q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q$.

We now consider $W_k(M)$ for a general matroid M. For each $e \in E(M)$ let $\mathcal{F}_k(M; e)$ denote the set of rank-k flats of M containing e, and let $W_k^e(M) = W_k(M) - |\mathcal{F}_k(M; e)|$ denote the number of rank-k flats of M *not* containing e. We will also freely use some basic properties of W_k:

Lemma 2.4. *If $k \geq 1$ and $\ell \geq 2$ are integers, M is a matroid, and e is a nonloop of M then the following hold:*

(1) $W_k(M) \leq W_1(M)^k$.
(2) $W_k(M) < \ell^{kr(M)}$ if $M \in \mathcal{U}(\ell)$.
(3) $|\mathcal{F}_k(M; e)| = W_{k-1}(M/e)$.
(4) $W_k(M) = W_{k-1}(M/e) + \sum_{F \in \mathcal{F}_{k+1}(M;e)} W_k^e(M|F)$.

Proof. (1) follows from the fact that every rank-k flat is spanned by k points, and (2) follows from (1) and Theorem 2.1. (3) is easy. Now by (3), there are $W_{k-1}(M/e)$ rank-k flats of M containing e. For each other rank-k flat F' of M, the set $F = \mathrm{cl}_M(F' \cup \{e\})$ is the unique rank-$(k+1)$ flat of M containing e and F', and each such F corresponds to $W_k^e(M|F)$ different F'. Combining these statements gives (4). □

3. Geometry

In this section, we deal with projective and affine geometries over $\mathrm{GF}(q)$, using them to provide a U_{2,q^2+1}-minor in various situations. We repeatedly use the fact that, if M has an $\mathrm{AG}(r(M)-1, q)$-restriction R and $e \in E(R)$, then M/e has a $\mathrm{PG}(r(M/e)-1, q)$-restriction contained in $E(R)$. The first lemma we need was also essentially proved in [1].

Lemma 3.1. *If q is a prime power and M is a simple matroid of rank at least 3 with a proper $\mathrm{PG}(r(M)-1, q)$-restriction, then M has a U_{2,q^2+1}-minor.*

Proof. Let R be a $\mathrm{PG}(r(M)-1, q)$-restriction of M. We may assume that $E(M) = E(R) \cup \{e\}$ for some $e \notin E(R)$. The point e is spanned by at most one line of R; by repeatedly contracting points not on such a line and simplifying we obtain a simple rank-3 minor of M' such that $E(M') = E(R') \cup \{e\}$ and $R' \cong \mathrm{PG}(2, q)$. Now e is spanned by at most one line of R' and such a line contains $q+1$ elements of $E(R')$, so $W_1(M'/e) \geq |E(R')| - q = q^2 + 1$, and so M'/e has a U_{2,q^2+1}-restriction. □

In particular, if M has rank at least 3, has a $\mathrm{PG}(r(M)-1, q)$-restriction and is not $\mathrm{GF}(q)$-representable then M has a U_{2,q^2+1}-minor; we use this idea in the next two lemmas.

Lemma 3.2. *Let q be a prime power and $m \geq 2$ and $b \geq 1$ be integers. If M is a matroid with an $\mathrm{AG}(m+b, q)$-restriction R, a rank-m restriction S that is not $\mathrm{GF}(q)$-representable, and every cocircuit of M has rank at least $r(M) - b$, then M has a U_{2,q^2+1}-minor.*

Proof. We may assume that no minor of M satisfies the hypotheses. Note that contracting elements of M preserves the cocircuit property, so $E(M) = \mathrm{cl}_M(E(R)) \cup \mathrm{cl}_M(E(S))$. If $r(M) > r(R)$ then $E(M) - \mathrm{cl}_M(E(R))$ contains a cocircuit of M of rank at most $r(S) = m < r(M) - b$, a contradiction. Therefore R is spanning in M. Let $f \in E(R) - \mathrm{cl}_M(E(S))$; the matroid M/f has a $\mathrm{PG}(r(M/f)-1, q)$-restriction, has rank at least 3 and is not $\mathrm{GF}(q)$-representable, so has a U_{2,q^2+1}-minor by Lemma 3.1. □

Lemma 3.3. *Let q be a prime power and $k \geq 1$ be an integer. If M is a matroid such that $r(M) \geq k+3$, M has an* $\mathrm{AG}(r(M)-1,q)$*-restriction and M has no U_{2,q^2+1}-minor, then* $W_k(M) \leq \left[{r(M) \atop k}\right]_q$.

Proof. Let R be an $\mathrm{AG}(r(M)-1,q)$-restriction of M. We may assume that M is simple. We make two claims, considering two different types of rank-k flat.

3.3.1. *If F is a flat of M with $F \cap E(R) \neq \varnothing$, then F has a basis contained in $E(R)$.*

Proof of claim: For each $e \in E(R)$, the matroid M/e has rank at least 3 and has a $\mathrm{PG}(r(M)-2,q)$-restriction contained in $E(R)-\{e\}$, so it follows from Lemma 3.1 that, for every $e \in E(R)$, each nonloop of M/e is parallel in M/e to some element of $E(R)-\{e\}$. Therefore every $x \in E(M)$ is in some line of M containing e and another element y of $E(R)$. Thus, if F is a flat of M and $e \in F \cap E(R)$, then F has a basis contained in $E(R)$, as we can include e, and then can exchange each $x \in F - E(R)$ with its corresponding $y \in E(R)$. □

3.3.2. *If F is a rank-k flat of M such that $F \cap E(R) = \varnothing$, then F is a rank-k flat of $M/e \setminus (E(R)-\{e\})$ for all $e \in E(R)$.*

Proof of claim: Let F be a rank-k flat of M that is disjoint from $E(R)$ and let $e \in E(R)$. Let $F' = \mathrm{cl}_M(F \cup \{e\})$. By the first claim, F' contains a rank-$(k+1)$ flat G of R; note that $R|G \cong \mathrm{AG}(k,q)$. If $F' = F \cup G$ then the claim holds. Otherwise, $F' \neq F \cup G$ and F' is the disjoint union of a rank-$(k+1)$ affine geometry, a rank-k flat, and at least one other point, so $M|F'$ is not $\mathrm{GF}(q)$-representable. Let $f \in E(R) - F'$. The matroid M/f has rank at least 3, has a $\mathrm{PG}(r(M/f)-1,q)$-restriction contained in $E(R)$ and has $M|F'$ as a restriction, so Lemma 3.1 gives a contradiction. □

Let $e \in E(R)$. By 3.3.1, the number of rank-k flats of M that intersect $E(R)$ is $W_k(R)$. By 3.3.2, the number of other rank-k flats of M is at most $W_k(M/e \setminus E(R))$. Now M/e has rank at least 3 and has a $\mathrm{PG}(r(M)-2,q)$-restriction, so we may assume by Lemma 3.1 that $\mathrm{si}(M/e) \cong \mathrm{PG}(r(M)-2,q)$ and so $M/e \setminus E(R)$ is $\mathrm{GF}(q)$-representable. Therefore

$$\begin{aligned} W_k(M) &\leq W_k(R) + W_k(M/e \setminus E(R)) \\ &\leq W_k(\mathrm{AG}(r(M)-1,q)) + W_k(\mathrm{PG}(r(M)-2),q). \end{aligned}$$

This upper bound is clearly equal to $W_k(\mathrm{PG}(r(M)-1,q)) = \left[{r \atop k}\right]_q$. □

4. The Main Theorem

We now restate and prove Theorem 1.2.

Theorem 4.1. *There is a function $g : \mathbb{Z}^2 \to \mathbb{Z}$ so that, for all integers $\ell \geq 2$ and $k \geq 0$, if $M \in \mathcal{U}(\ell)$ satisfies $r(M) \geq g(\ell, k)$ then $W_k(M) \leq \left[{r(M) \atop k}\right]_q$, where q is the largest prime power not exceeding ℓ.*

Proof. Set $g(\ell, 0) = 0$ for all ℓ; note that this trivially satisfies the conditions of the theorem. Let $\ell \geq 2$ and $k > 0$ be integers, and q be the largest prime power such that $q \leq \ell$. If $\ell = 2$ then M is binary and the bound is obvious; we may therefore assume that $\ell \geq q \geq 3$. Suppose recursively that $g(\ell, i)$ has been defined for each $i \in \{0, \dots, k-1\}$. Let $r_0 = \max(k + 3, \max_{0 \leq i \leq k-1} g(\ell, i))$. Note that $2q^{-k} \leq \frac{2}{3}$; let b be a positive integer so that $kq^{k^2-b} + (2q^{-k})^{b+1} \leq \frac{1}{6}\ell^{-k(k+1)}$. Recall that the function f was defined in Theorem 2.2; set $g(\ell, k)$ to be an integer n such that $q^{-k^2}2^n > \ell^{kf(\ell, r_0+b, q, q^{-k})}$.

Suppose inductively that $g(\ell, k-1)$ satisfies the theorem statement. If $g(\ell, k)$ does not, then there exists $M_0 \in \mathcal{U}(\ell)$ such that $r(M_0) \geq n$ and $W_k(M_0) > \left[{r(M_0) \atop k}\right]_q$. We will obtain a contradiction by finding a $U_{2,\ell+2}$-minor of M; since $q^2 + 1 \geq \ell + 2$ it is also enough to find a U_{2,q^2+1}-minor.

Let M be minor-minimal such that M is a minor of M_0 and $W_k(M) > 2^{r(M_0)-r(M)}\left[{r(M) \atop k}\right]_q$. Note that M is simple; let $r = r(M)$. We often use the fact that $W_k(M') < (2q^{-k})^{r-r(M')}W_k(M)$ for each proper minor M' of M, which follows from minimality and (1) of Lemma 2.3.

4.1.1. *M has an* $\mathrm{AG}(r_0 + b, q)$*-restriction.*

Proof of claim: Observe that

$$W_k(M) > 2^{r(M_0)-r}\left[{r \atop k}\right]_q \geq 2^{n-r}q^{k(r-k)} = q^{-k^2}2^n(q^k/2)^r > \ell^{kf(\ell, r_0+b, q, q^{-k})},$$

so $r > f(\ell, r_0 + b, q, q^{-k})$. By choice of M and Lemmas 2.3 and 2.4 we have $W_1(M)^k \geq W_k(M) > \left[{r \atop k}\right]_q \geq q^{k(r-k)}$, so $W_1(M) \geq q^{-k}q^r$. The required restriction exists by Theorem 2.2, since $\mathrm{PG}(r_0 + b, q')$ has a $U_{2,\ell+2}$-minor for all $q' > q$. □

4.1.2. *Every cocircuit of M has rank at least $r - b$.*

Proof of claim: Suppose not; let C be a cocircuit of M of rank less than $r - b$, let H be the hyperplane $E(M) - C$, and let B be a rank-$(r - b)$ set containing C. Note that $E(M) = H \cup B$.

Let $e \in C$; note that the matroid M/e has no loops and that $r((M/e)|(B - e)) = r - (b + 1) \geq r_0$. Let $\mathcal{F}_B$ be the collection of rank-k flats of M/e that intersect B. Each $F \in \mathcal{F}_B$ is the closure of the union of a rank-i flat of $(M/e)|(B-\{e\})$ and a rank-$(k-i)$ flat of $(M/e)|H$ for some $i \in \{1,\dots,k\}$, so

$$\begin{aligned}
|\mathcal{F}_B| &\le \sum_{i=1}^{k-1} W_i((M/e)|(B-e))W_{k-i}((M/e)|H) + W_k((M/e)|(B-e)) \\
&\le \sum_{i=1}^{k-1} \begin{bmatrix} r-b-1 \\ i \end{bmatrix}_q \begin{bmatrix} r-1 \\ k-i \end{bmatrix}_q + (2q^{-k})^{b+1} W_k(M) \\
&\le \sum_{i=1}^{k-1} q^{i(r-b-1)+(k-i)(r-1)} + (2q^{-k})^{b+1} W_k(M) \\
&\le kq^{-b}q^{k(r-1)} + (2q^{-k})^{b+1} W_k(M) \\
&\le kq^{k^2-b} \begin{bmatrix} r \\ k \end{bmatrix}_q + (2q^{-k})^{b+1} W_k(M) \\
&< \left(kq^{k^2-b} + (2q^{-k})^{b+1}\right) W_k(M) \\
&\le \tfrac{1}{6}\ell^{-k(k+1)} W_k(M).
\end{aligned}$$

For each rank-k flat F_0 of M/e that is not in $\mathcal{F}_B$, we have $F_0 \subseteq H$ so $(M/e)|F_0 = M|F_0$. The closure in M of $F = F_0 \cup \{e\}$ contains no elements of $B - \{e\}$, so $F \in \mathcal{F}_{k+1}(M;e)$ and $W_k^e(M|F) = 1$. For each other $F \in \mathcal{F}_{k+1}$ we have $W_k^e(F) < \ell^{k(k+1)}$ by Lemma 2.4. Therefore

$$\begin{aligned}
\sum_{F \in \mathcal{F}_{k+1}(M;e)} W_k^e(M|F) &\le \ell^{k(k+1)}|\mathcal{F}_B| + (W_k(M/e) - |\mathcal{F}_B|) \\
&< \ell^{k(k+1)}|\mathcal{F}_B| + 2q^{-k}W_k(M) \\
&\le \ell^{k(k+1)}\left(\tfrac{1}{6}\ell^{-k(k+1)}W_k(M)\right) + \tfrac{2}{3}W_k(M) \\
&= \tfrac{5}{6}W_k(M).
\end{aligned}$$

Now, since $r(M/e) \ge r_0$, by what is above we have

$$\begin{aligned}
W_k(M) &= W_{k-1}(M/e) + \sum_{F \in \mathcal{F}_{k+1}(M;e)} W_k^e(M|F) \\
&< \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q + \tfrac{5}{6}W_k(M) \\
&< q^{k-r}\begin{bmatrix} r \\ k \end{bmatrix}_q + \tfrac{5}{6}W_k(M),
\end{aligned}$$

a contradiction, as $\begin{bmatrix} r \\ k \end{bmatrix}_q < W_k(M)$ and $q^{k-r} \le q^{k-r_0} \le q^{-3} < \frac{1}{6}$. $\square$

Let N be a minor-minimal minor of M such that

(1) N has an $\mathrm{AG}(r_0 + b, q)$-restriction,
(2) every cocircuit of N has rank at least $r(N) - b$, and
(3) $W_k(N) > \begin{bmatrix} r(N) \\ k \end{bmatrix}_q$.

Let R be an $\mathrm{AG}(r_0 + b, q)$-restriction of N. Since $r_0 \geq k+1$, we may assume by 4.1.1, 4.1.2 and Lemma 3.2 that every rank-$(k+1)$-restriction of N is $\mathrm{GF}(q)$-representable. Note that N has no loops.

4.1.3. $W_k(N/e) > \begin{bmatrix} r(N)-1 \\ k \end{bmatrix}_q$ *for all* $e \in E(N)$.

Proof of claim: Since every rank-$(k+1)$ restriction of N is $\mathrm{GF}(q)$-representable, the value of $W_k^e(N|F)$ for each rank-$(k+1)$ flat F does not exceed q^k, its value on $\mathrm{PG}(k,q)$. Therefore $\sum_{F \in \mathcal{F}_{k+1}(e)} W_k^e(N|F) \leq q^k|\mathcal{F}_{k+1}(N;e)| = q^k W_k(N/e)$, and so by (4) of Lemma 2.4 we get $W_k(N) \leq W_{k-1}(N/e) + q^k W_k(N/e)$. Now $r(N/e) \geq r_0$ so $W_{k-1}(N/e) \leq \begin{bmatrix} r(N)-1 \\ k-1 \end{bmatrix}_q$ by the inductive hypothesis, and $W_k(N) > \begin{bmatrix} r(N) \\ k \end{bmatrix}_q$, which implies that $W_k(N/e) > q^{-k}\left(\begin{bmatrix} r(N) \\ k \end{bmatrix}_q - \begin{bmatrix} r(N)-1 \\ k-1 \end{bmatrix}_q\right) = \begin{bmatrix} r(N)-1 \\ k \end{bmatrix}_q$. □

Thus, properties (1) and (2) and (3) are all preserved by contracting elements of $E(N) - \mathrm{cl}_N(E(R))$, so it follows from minimality that R is spanning in N. We now obtain a contradiction from Lemma 3.3. □

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6. References

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