PROJECTIVE GEOMETRIES IN EXPONENTIALLY DENSE MATROIDS. II

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Abstract. We show for each positive integer a that, if $\mathcal{M}$ is a minor-closed class of matroids not containing all rank-$(a+1)$ uniform matroids, then there exists an integer c such that either every rank-r matroid in $\mathcal{M}$ can be covered by at most r^c rank-a sets, or $\mathcal{M}$ contains the GF(q)-representable matroids for some prime power q, and every rank-r matroid in $\mathcal{M}$ can be covered by at most cq^r rank-a sets. In the latter case, this determines the maximum density of matroids in $\mathcal{M}$ up to a constant factor.

1. Introduction

If M is a matroid and a is a positive integer, then $\tau_a(M)$ denotes the *a-covering number* of M, the minimum number of sets of rank at most a in M required to cover $E(M)$. We will prove the following theorem:

Theorem 1.1. *Let $a \geq 1$ be an integer. If $\mathcal{M}$ is a minor-closed class of matroids, then there is an integer $c > 0$ such that either*

(1) $\tau_a(M) \leq r(M)^c$ for all $M \in \mathcal{M}$,
(2) there is a prime power q so that $\tau_a(M) \leq cq^{r(M)}$ for all $M \in \mathcal{M}$ and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids, or
(3) $\mathcal{M}$ contains all rank-$(a+1)$ uniform matroids.

This theorem also appears in [10], and a weaker version, where the upper bound in (2) is replaced by $r(M)^c q^{r(M)}$, was proved in [5]; our proof is built with this weaker result as a starting point. $\tau_1(M)$ is just the number of points in M, and the above theorem was shown in this case by Geelen and Kabell [2].

Theorem 1.1 resolves the 'polynomial-exponential' part of the following conjecture of Geelen [1]:

Conjecture 1.2 (Growth Rate Conjecture)**.** *Let $a \geq 1$ be an integer. If $\mathcal{M}$ is a minor-closed class of matroids, then there is an integer $c > 0$ so that either*

(1) $\tau_a(M) \leq cr(M)$ for all $M \in \mathcal{M}$,

(2) $\tau_a(M) \le cr(M)^2$ *for all* $M \in \mathcal{M}$ *and* $\mathcal{M}$ *contains all graphic matroids or all bicircular matroids,*
(3) *there is a prime power* q *so that* $\tau_a(M) \le cq^{r(M)}$ *for all* $M \in \mathcal{M}$ *and* $\mathcal{M}$ *contains all* $\mathrm{GF}(q)$*-representable matroids, or*
(4) $\mathcal{M}$ *contains all rank-*$(a+1)$ *uniform matroids.*

This conjecture was proved for $a = 1$ by Geelen, Kabell, Kung and Whittle [2,4,7] and is known as the 'Growth Rate Theorem'.

If (4) holds, then $\tau_a(M)$ is not bounded by any function of $r(M)$ for all $M \in \mathcal{M}$, as a rank-$(a+1)$ uniform matroid (and consequently any matroid with such a minor) can require arbitrarily many rank-a sets to cover. Our bounds on τ_a are thus given with respect to some particular rank-$(a+1)$ uniform minor that is excluded. We prove Theorem 1.1 as a consequence of the two theorems below; the first is proved in [5], and the second is the main technical result of this paper.

Theorem 1.3. *For all integers* a, b, n *with* $n \ge 1$ *and* $1 \le a < b$*, there is an integer* m *such that, if* M *is a matroid of rank at least* 2 *with no* $U_{a+1,b}$*-minor and* $\tau_a(M) \ge r(M)^m$*, then* M *has a rank-*n *projective geometry minor.*

Theorem 1.4. *For all integers* a, b, n, q *with* $n \ge 1$*,* $q \ge 2$ *and* $1 \le a < b$*, there is an integer* c *such that, if* M *is a matroid with no* $U_{a+1,b}$*-minor and* $\tau_a(M) \ge cq^{r(M)}$*, then* M *has a rank-*n *projective geometry minor over a finite field with more than* q *elements.*

2. Preliminaries

We use the notation of Oxley [11]. A rank-1 flat is a *point*, and a rank-2 flat is a *line*. If M is a matroid, and $X, Y \subseteq E(M)$, then $\sqcap_M(X, Y)$ denotes the *local connectivity* between X and Y in M, defined by $\sqcap_M(X, Y) = r_M(X) + r_M(Y) - r_M(X \cup Y)$. If $\sqcap_M(X, Y) = 0$, then X and Y are *skew* in M. Additionally, we write $\epsilon(M)$ for $\tau_1(M)$, the number of points in a matroid M.

For integers a and b with $1 \le a < b$, we write $\mathcal{U}(a, b)$ for the class of matroids with no $U_{a+1,b}$-minor. The first tool in our proof is a theorem of Geelen and Kabell [3] which shows that τ_a is bounded as a function of rank across $\mathcal{U}(a, b)$.

Theorem 2.1. *Let* a *and* b *be integers with* $1 \le a < b$*. If* $M \in \mathcal{U}(a, b)$ *satisfies* $r(M) > a$*, then* $\tau_a(M) \le \binom{b-1}{a}^{r(M)-a}$*.*

Proof. We first prove the result when $r(M) = a + 1$, then proceed by induction. If $r(M) = a + 1$, then observe that $M|B \cong U_{a+1,a+1}$ for any basis B of M; let $X \subseteq E(M)$ be maximal such that $M|X \cong$

$U_{a+1,|X|}$. We may assume that $|X| < b$, and by maximality of X, every $e \in E(M) - X$ is spanned by a rank-a set of X. Therefore, $\tau_a(M) \le \binom{|X|}{a} \le \binom{b-1}{a}$.

Suppose that $r(M) > a+1$, and inductively assume that the result holds for matroids of smaller rank. Let $e \in E(M)$. We have $\tau_{a+1}(M) \le \tau_a(M/e) \le \binom{b-1}{a}^{r(M)-a-1}$ by induction, and by the base case each rank-$(a+1)$ set in M admits a cover with at most $\binom{b-1}{a}$ sets of rank at most a. Therefore $\tau_a(M) \le \binom{b-1}{a}\tau_{a+1}(M) \le \binom{b-1}{a}^{r(M)-a}$, as required. $\square$

The base case of this theorem gives $\tau_a(M) \le \binom{b-1}{a}\tau_a(M/e)$ for all $M \in \mathcal{U}(a,b)$ and $e \in E(M)$; an inductive argument yields the following:

Corollary 2.2. *Let a and b be integers with $1 \le a < b$. If $M \in \mathcal{U}(a,b)$ and $C \subseteq E(M)$, then $\tau_a(M/C) \ge \binom{b-1}{a}^{-r_M(C)}\tau_a(M)$.*

Our starting point in our proof is the main technical result of [5]. Note that this theorem gives Theorem 1.3 when $q = 1$.

Theorem 2.3. *There is an integer-valued function $f_{2.3}(a,b,n,q)$ so that, for any integers $1 \le a < b$, $q \ge 1$ and $n \ge 1$, if $M \in \mathcal{U}(a,b)$ satisfies $r(M) > 1$ and $\tau_a(M) \ge r(M)^{f_{2.3}(a,b,n,q)}q^{r(M)}$, then M has a $\mathrm{PG}(n-1,q')$-minor for some prime power $q' > q$.*

3. Stacks

We now define an obstruction to $\mathrm{GF}(q)$-representability. If q is a prime power and h and t are nonnegative integers, then a matroid S is a (q,h,t)-*stack* if there are pairwise disjoint subsets $F_1, F_2, \ldots, F_h$ of $E(S)$ such that the union of the F_i is spanning in S, and for each $i \in \{1, \ldots, h\}$ the matroid $(S/(F_1 \cup \ldots \cup F_{i-1}))|F_i$ has rank at most t and is not $\mathrm{GF}(q)$-representable. We write $F_i(S)$ for F_i, and when the value of t is unimportant, we refer simply to a (q,h)-*stack*.

Note that a stack has rank between $2h$ and th, and that contracting or restricting to the sets in some initial segment of $F_1, \ldots, F_h$ yields a smaller stack; we use these facts freely.

We now show that the structure of a stack cannot be completely destroyed by a small projection. The following two lemmas are similar; the first does not control rank, and the second does.

Lemma 3.1. *Let q be a prime power, and $k \ge 0$ be an integer. If M is a matroid, $C \subseteq E(M)$, and M has a $(k(r_M(C)+1),q)$-stack restriction, then $(M/C)|E(S)$ has a (k,q)-stack restriction.*

Proof. Let S be a $(k(r_M(C)+1),q)$-stack in M, with $F_i = F_i(S)$ for each i. By adding parallel extensions if needed, we may assume that $C \cap E(S) = \varnothing$. If $r_M(C) = 0$ then the result is trivial; suppose that $r_M(C) > 0$ and that the lemma holds for sets C of smaller rank. Let $F = F_1 \cup \ldots \cup F_k$. If C is skew to F in M, then $(M/C)|F$ is a (k,q)-stack, giving the lemma. Otherwise M/F has a $(kr_M(C),q)$-stack restriction, and $r_M(C) > r_{M/F}(C)$. By the inductive hypothesis, $M/(F \cup C)$ has a (k,q)-stack restriction S'; therefore $F \cup F_1(S'), F_2(S'), \ldots, F_k(S')$ give a (k,q)-stack restriction of M/C. □

Lemma 3.2. *Let q be a prime power, and a, h and t be integers with $a \geq 0$, $h \geq 1$ and $t \geq 2$. If M is a matroid with an $((a+1)h,q,t)$-stack restriction S, and $X \subseteq E(M)$ is a set satisfying $\sqcap_M(X,E(S)) \leq a$, then there exists $C \subseteq E(S)$ so that $(M/C)|E(S)$ has an (h,q,t)-stack restriction S', and X and $E(S')$ are skew in M/C.*

Proof. Let $F = F_1(S) \cup \ldots \cup F_h(S)$. If F is skew to X in M, then F contains an (h,q,t)-stack S' satisfying the lemma with $C = \varnothing$. Otherwise, M/F has an (ah,q,t)-stack restriction S_0 contained in $E(S)$, and $\sqcap_{M/F}(X-F,E(S_0)) < \sqcap_M(X-F,E(S)) \leq a$; the lemma follows routinely by induction on a. □

This low local connectivity is obtained via the following lemma, which applies more generally. We will just use the case when $M|Y$ is a stack.

Lemma 3.3. *If $M \in \mathcal{U}(a,b)$ and $Y \subseteq E(M)$, then there is a set $X \subseteq E(M)$ so that $\tau_a(M|X) \geq \binom{b-1}{a}^{a-r_M(Y)}\tau_a(M)$ and $\sqcap_M(X,Y) \leq a$.*

Proof. We may assume that $r_M(Y) > a$. Let B be a basis for M containing a basis B_Y for $M|Y$. We have $r(M/(B-B_Y)) = r_M(Y)$, so $\tau_a(M/(B-B_Y)) \leq \binom{b-1}{a}^{r_M(Y)-a}$ by Theorem 2.1. Applying a majority argument to a smallest cover of $M/(B-B_Y)$ with sets of rank at most a gives a set $X' \subseteq E(M)$ so that $r_{M/(B-B_Y)}(X) \leq a$, and $\tau_a(M|X) \geq \binom{b-1}{a}^{a-r_M(Y)}\tau_a(M)$. Moreover, $B-B_Y$ is skew to Y in M, so $\sqcap_M(X,Y) \leq \sqcap_{M/(B-B_Y)}(X,Y) \leq a$. □

4. Thickness and Weighted Covers

The next section requires a modified notion of covering number in which elements of a cover are weighted by rank. All results in the current section are also proved in [5].

A *cover* of a matroid M is a collection of sets with union $E(M)$, and for an integer $d \geq 1$, we say the *d-weight* of a cover $\mathcal{F}$ of M is the sum $\sum_{F\in\mathcal{F}} d^{r_M(F)}$, and write $\mathrm{wt}^d_M(\mathcal{F})$ for this sum. Thus, a rank-1 set has weight d, a rank-2 set has rank d^2, etc. We write $\tau^d(M)$ for the minimum d-weight of a cover of M, and we say a cover of M is *d-minimal* if it has d-weight equal to $\tau^d(M)$.

Since $r_M(X) \le r_{M/e}(X - \{e\}) + 1$ for all $X \subseteq E(M)$, we have $\tau^d(M) \le d\tau^d(M/e)$ for every nonloop e of M; a simple induction argument gives the following lemma:

Lemma 4.1. *If d is a positive integer and M is a matroid, then $\tau^d(M/C) \ge d^{-r_M(C)}\tau^d(M)$ for all $C \subseteq E(M)$.*

We say a matroid M is *d-thick* if $\tau_{r(M)-1}(M) \ge d$, and a set $X \subseteq E(M)$ is *d-thick in M* if $M|X$ is d-thick. Note that any d-thick matroid of rank 2 has a $U_{2,d}$-restriction. Moreover, it is clear that $\tau_{r(M)-1}(M) \le \tau_{r(M)-2}(M/e)$ for any nonloop e of M, so it follows that d-thickness is preserved by contraction. Thus, any d-thick matroid of rank at least 2 has a $U_{2,d}$-minor, and the rank-$(a+1)$ case of Theorem 2.1 yields the following:

Lemma 4.2. *Let a, b, d be integers with $1 \le a < b$ and $d > \binom{b-1}{a}$. If M is a d-thick matroid of rank greater than a, then M has a $U_{a+1,b}$-minor.*

This controls the nature of a d-minimal cover of M in several ways:

Lemma 4.3. *Let a, b, d be integers with $1 \le a < b$ and $d > \binom{b-1}{a}$. If $\mathcal{F}$ is a d-minimal cover of a matroid $M \in \mathcal{U}(a, b)$, then*
(1) every $F \in \mathcal{F}$ is d-thick in M,
(2) every $F \in \mathcal{F}$ has rank at most a, and
(3) $\tau_a(M) \le \tau^d(M) \le d^a\tau_a(M)$.

Proof. If some set $F \in \mathcal{F}$ is not d-thick, then F is the union of sets $F_1, \ldots, F_{d-1}$ of smaller rank. Thus, $(\mathcal{F}-\{F\})\cup\{F_1, \ldots, F_{d-1}\}$ is a cover of M of weight at most $\mathrm{wt}^d_M(\mathcal{F}) - d^{r_M(F)} + (d-1)d^{r_M(F)-1} < \mathrm{wt}^M_d(\mathcal{F})$, contradicting d-minimality of $\mathcal{F}$. Therefore, every set in F is d-thick in M, giving (1). (2) now follows from Lemma 4.2.

To see the upper bound in (3), observe that any smallest cover of M with sets of rank at most a has size $\tau_a(M)$ and d-weight at most $d^a\tau_a(M)$. The lower bound follows from the fact that every set has d-weight at least 1, and $\mathcal{F}$, by (2), is a d-minimal cover of M containing sets of rank at most a. □

5. Stacking Up

Our first lemma finds, in a dense matroid, a dense minor with a large stack restriction. We consider the modified notion of density τ^d.

Lemma 5.1. *There is an integer-valued function $\alpha_{5.1}(a,b,h,q,\lambda)$ so that, for any prime power q and integers a,b,h,λ with $1 \le a < b$, $m \ge 0$, and $\lambda \ge 1$, if $d > \max(q+1, \binom{b-1}{a})$ is an integer and $M \in \mathcal{U}(a,b)$ satisfies $\tau^d(M) \ge \alpha_{5.1}(a,b,h,q,\lambda)q^{r(M)}$, then M has a contraction-minor N with an $(h,q,a+1)$-stack restriction, satisfying $\tau^d(N) \ge \lambda q^{r(N)}$.*

Proof. Let a,b,q and d be integers such that $1 \le a < b$, $q \ge 2$ and $d > \max(q+1, \binom{b-1}{a})$. Set $\alpha_{5.1}(a,b,0,q,\lambda) = \lambda$, and for each $h > 0$ recursively set $\alpha_{5.1}(a,b,h,q,\lambda) = d^{a+1}\alpha_{5.1}(a,b,m-1,q,\lambda q^{a+1})$. Note that all values this function takes for $h > 0$ are multiples of d.

When $h = 0$, the lemma holds with $N = M$. Let $h > 0$ be an integer, and suppose inductively that $\alpha_{5.1}$ as defined satisfies the lemma for smaller values of h. Let $M \in \mathcal{U}(a,b)$ be contraction-minimal satisfying $\tau^d(M) \ge \alpha q^{r(M)}$; we show that M has the required minor N.

5.1.1. *There is a set $X \subseteq E(M)$ such that $r_M(X) \le a+1$ and $M|X$ is not* $\mathrm{GF}(q)$*-representable.*

Proof of claim: Let e be a nonloop of M and let $\mathcal{F}$ and $\mathcal{F}'$ be d-minimal covers of M and M/e respectively. We consider two cases:

Case 1: $r_M(F) = 1$ for all $F \in \mathcal{F}$ and $r_{M/e}(F) = 1$ for all $F \in \mathcal{F}'$.

Note that $\tau^d(M) = d|\mathcal{F}|$ and $\tau^d(M/e) = d|\mathcal{F}'|$. By minimality of M, this gives $|\mathcal{F}| \ge d^{-1}\alpha q^{r(M)}$ and $|\mathcal{F}'| < d^{-1}\alpha q^{r(M)-1}$, so $|\mathcal{F}'| \le d^{-1}\alpha q^{r(M)-1} - 1$, as this expression is an integer. Moreover, $|\mathcal{F}| = \epsilon(M)$ and $|\mathcal{F}'| = \epsilon(M/e)$, so $\epsilon(M) \ge d^{-1}\alpha q^{r(M)} \ge q\epsilon(M/e) + q > q\epsilon(M/e) + 1$. Since the points of M/e correspond to lines of M through e, it follows by a majority argument that some line L through e contains at least $q+1$ other points of M, and therefore that $X = L$ will satisfy the lemma.

Case 2: $r_N(F) \ge 2$ for some $F \in \mathcal{F}$ or $r_{M/e}(F) \ge 2$ for some $F \in \mathcal{F}'$.

If $X \in \mathcal{F}$ satisfies $r_M(X) \ge 2$, then by Lemma 4.3, X is d-thick in M and has rank at most a. Since $d \ge q+2$ and thickness is preserved by contraction, the matroid $M|X$ has a $U_{2,q+2}$-minor and therefore X satisfies the claim. If $X \in \mathcal{F}'$ satisfies $r_{M/e}(X) \ge 2$, then $r_M(X \cup \{e\}) \le a+1$ and $X \cup \{e\}$ will satisfy the claim for similar reasons.

□

Now $\tau^d(M/X) \ge d^{-(a+1)}\tau^d(M) \ge d^{-(a+1)}\alpha q^{r(M/X)} \ge \alpha_{5.1}(a,b,h-1,q,\lambda q^{a+1})q^{r(M/X)}$, so M/X has a contraction-minor $M/(X \cup C)$ with an $(h-1,q,a+1)$-stack restriction S', satisfying $\tau^d(M') \ge \lambda q^{a+1}q^{r(M')}$. We may assume that C is independent in M/X; let $N = M/C$. We have $N|X = M|X$ and N/X has an $(h-1,q,a+1)$-stack restriction, so N has an $(h,q,a+1)$-stack restriction. Morever $\tau^d(N) \ge \tau^d(N/X) \ge$

$\lambda q^{a+1} q^{r(N/X)} = \lambda q^{a+1-r_N(X)} q^{r(N)}$. Since $r_N(X) \le a+1$, the matroid N is the required minor.

□

6. Exploiting a Stack

We defined a stack as an example of a matroid that is 'far' from being GF(q)-representable. In this section we make this concrete by proving that a stack on top of a projective geometry yields a large uniform minor or a large projective geometry over a larger field.

We first need an easily proved lemma from [6], telling us that a small projection of a projective geometry does not contain a large stack:

Lemma 6.1. *Let q be a prime power and h be a nonnegative integer. If M is a matroid and $X \subseteq E(M)$ satisfies $r_M(X) \le h$ and $\mathrm{si}(M \backslash X) \cong \mathrm{PG}(r(M)-1, q)$, then M/X has no $(q, h+1)$-stack restriction.*

Next we show that a large stack on top of a projective geometry guarantees (in a minor) a large flat with limited connectivity to sets in the geometry:

Lemma 6.2. *Let q be a prime power and $k \ge 0$ be an integer. If M is a matroid with a $\mathrm{PG}(r(M)-1, q)$-restriction R and a (k^4, q)-stack restriction, then there is a minor M' of M of rank at least $r(M)-k$, with a $\mathrm{PG}(r(M')-1, q)$-restriction R' and a rank-k flat K such that $\sqcap_{M'}(X, K) \le \frac{1}{2} r_{M'}(X)$ for all $X \subseteq E(R')$.*

Proof. Let $J \subseteq E(M)$ be maximal so that $\sqcap_M(X, J) \le \frac{1}{2} r_M(X)$ for all $X \subseteq E(R)$. Note that $J \cap E(R) = \varnothing$. We may assume that $r_M(J) < k$, as otherwise $J = K$ and $M' = M$ will do. Let $M' = M/J$.

6.2.1. *For each nonloop e of M', there is a set $Z_e \subseteq E(R)$ such that $r_{M'}(Z_e) \le k$ and $e \in \mathrm{cl}_{M'}(Z_e)$.*

Proof of claim: Let e be a nonloop of M'. By maximality of J there is some $X \subseteq E(R)$ such that $\sqcap_M(X, J \cup \{e\}) > \frac{1}{2} r_M(X)$. Let $c = \sqcap_M(X, J \cup \{e\})$, noting that $\frac{1}{2} r_M(X) < c \le r_M(J \cup \{e\}) \le k$. We also have $\frac{1}{2} r_M(X) \ge \sqcap_M(X, J) \ge c-1$, so $\sqcap_M(X, J) = c-1$, giving $e \in \mathrm{cl}_M(X)$. Now $r_M(X) \le 2c-1$ and $r_{M/J}(X) = r_M(X) - \sqcap_M(X, J) \le (2c-1)-(c-1) = c \le k$. Therefore $Z_e = X$ satisfies the claim. □

If e is not parallel in M' to a nonloop of R, then $M'|(e \cup Z_e)$ is not GF(q)-representable, as it is a simple cosimple extension of a projective geometry; this fact still holds in any contraction-minor for which e is a nonloop satisfying this condition. Let $j \in \{0, \dots, k\}$ be maximal such that M' has a (q, j, k)-stack restriction T with the property that, for each $i \in \{1, \dots, j\}$, the matroid $T/(F_1(T) \cup \dots \cup F_{i-1}(T))|F_i(T)$ has a basis contained in $E(R)$. For each i, let $F_i = F_i(T)$, and $B_i \subseteq E(R)$ be such a basis. We split into cases depending on whether $j \geq k$.

Case 1: $j < k$.

Let $M'' = M'/E(T) = M/(E(T) \cup J)$. If M'' has a nonloop x that is not parallel in $M'/E(T)$ to an element of $E(R)$, then the restriction $M''|(x \cup (Z_x - E(T)))$ has rank at most k, is not $\mathrm{GF}(q)$-representable, and has a basis contained in $Z_x \subseteq E(R)$; this contradicts the maximality of j. Therefore we may assume that every nonloop of M'' is parallel to an element of R, so $\mathrm{si}(M'') \cong \mathrm{si}(M|(E(R) \cup E(T) \cup J)/(E(T) \cup J))$. We have $r_M(E(T) \cup J) \leq jk + k - 1 < k^2$, so by Lemma 6.1 the matroid M'' has no (k^2, q)-stack restriction. However, S is a (k^4, q)-stack restriction of M and $k^4 \geq k^2(r_M(E(T) \cup J) + 1)$, so M'' has a (k^2, q)-stack restriction by Lemma 3.1. This is a contradiction.

Case 2: $j = k$.

For each $i \in \{0, \dots, k\}$, let $M_i = M'/(F_1 \cup \dots \cup F_i)$ and $R_i = R|\operatorname{cl}_R(B_{i+1} \cup \dots \cup B_k)$. Note that R_i is a $\mathrm{PG}(r(M_i) - 1, q)$-restriction of M_i. We make a technical claim:

6.2.2. *For each $i \in \{0, \dots, k\}$, there is a rank-$(k - i)$ independent set K_i of M_i so that $\sqcap_{M_i}(X, K_i) \leq \frac{1}{2} r_{M_i}(X)$ for all $X \subseteq E(R_0) \cap E(M_i)$.*

Proof. When $i = k$, there is nothing to prove. Suppose inductively that $i \in \{0, \dots, k-1\}$ and that the claim holds for larger i. Let K_{i+1} be a rank-$(k - i - 1)$ independent set in M_{i+1} so that $\sqcap_{M_{i+1}}(X, K_{i+1}) \leq \frac{1}{2} r_{M_i}(X)$ for all $X \subseteq E(R_0) \cap E(M_{i+1})$. The restriction $M_i|F_{i+1}$ is not $\mathrm{GF}(q)$-representable; let e be a nonloop of $M_i|F_{i+1}$ that is not parallel in M_i to a nonloop of R_i. Set $K_i = K_{i+1} \cup \{e\}$, noting that K_i is independent in M_i. Let $X \subseteq E(R_0) \cap E(M_i)$; since $M_{i+1} = M_i/F_{i+1}$ we have

$$\begin{aligned} \sqcap_{M_i}(X, K_i) = {} & \sqcap_{M_{i+1}}(X - F_{i+1}, K_i) + \sqcap_{M_i}(K_i, F_{i+1}) + \sqcap_{M_i}(X, F_{i+1}) \\ & - \sqcap_{M_i}(X \cup K_i, F_{i+1}). \end{aligned}$$

Now e is a loop and $K_i - \{e\}$ is independent in M_{i+1}, so $\sqcap_{M_i}(K_i, F_{i+1}) = 1$, and $\sqcap_{M_{i+1}}(X - F_{i+1}, K_i) = \sqcap_{M_{i+1}}(X - F_{i+1}, K_{i+1}) \leq \frac{1}{2} r_{M_{i+1}}(X) = \frac{1}{2}(r_{M_i}(X) - \sqcap_{M_i}(X, F_{i+1}))$. This gives

$$\sqcap_{M_i}(X, K_i) \leq \tfrac{1}{2} r_{M_i}(X) + 1 + \tfrac{1}{2} \sqcap_{M_i}(X, F_{i+1}) - \sqcap_{M_i}(X \cup K_i, F_{i+1}).$$

It therefore suffices to show that $\sqcap_{M_i}(X \cup K_i, F_{i+1}) \geq 1 + \frac{1}{2} \sqcap_{M_i}(X, F_{i+1})$. Note that $e \in K_i \cap F_{i+1}$, so $\sqcap_{M_i}(X \cup K_i, F_{i+1}) \geq \max(1, \sqcap_{M_i}(X, F_{i+1}))$. Given this, it is easy to see that the inequality can only be violated if $\sqcap_{M_i}(X \cup K_i, F_{i+1}) = \sqcap_{M_i}(X, F_{i+1}) = 1$. If this is the case, then $\sqcap_{M_i}(X, B_{i+1}) = 1$ and so there is some $f \in E(R_{i+1})$ spanned by X and

B_{i+1}, since both are subsets of the projective geometry R_{i+1}. But e and f are not parallel by choice of e, so $\sqcap_{M_i}(X \cup K_i, F_{i+1}) \geq r_{M_i}(\{e, f\}) = 2$, a contradiction. □

Since $r(M_0) = r(M') > r(M) - k$, taking $i = 0$ in the claim now gives the lemma. □

Finally, we use the flat found in the previous lemma and Theorem 2.3 to find a large projective geometry minor over a larger field.

Lemma 6.3. *There is an integer-valued function $f_{6.3}(a, b, n, q, t)$ so that, for any prime power q and integers n, a, b with $n \geq 1$ and $1 \leq a < b$, if $M \in \mathcal{U}(a, b)$ has a $\mathrm{PG}(r(M) - 1, q)$-restriction and an $(f_{6.3}(a, b, n, q, t), q, t)$-stack restriction, then M has a $\mathrm{PG}(n - 1, q')$-minor for some $q' > q$.*

Proof. Let q be a prime power, and t, n, a, b be integers so that $t \geq 0$, $n \geq 1$, and $1 \leq a < b$. Let $k \geq 2a$ be an integer so that $q^{t^{-1}r^{1/4}-2a} \geq r^{f_{2.3}(a,b,n,q)}$ for all integers $r \geq k$. Set $f_{6.3}(a, b, n, q, t) = k^4$.

Let M be a matroid with a $\mathrm{PG}(r(M) - 1, q)$-restriction R and a (k^4, q, t)-stack restriction S. We will show that M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$; we may assume (by contracting points of R not spanned by S if necessary) that $r(M) = r(S)$. By Lemma 6.2, there is a minor M' of M, of rank at least $r(M) - k$, with a $\mathrm{PG}(r(M') - 1, q)$-restriction R' and a rank-k flat K such that $\sqcap_{M'}(K, X) \leq \frac{1}{2} r_{M'}(X)$ for all $X \subseteq E(R')$. Let $r = r(M')$, $M_0 = M'/K$ and $r_0 = r(M_0)$. Since $k^4 + 2k \leq 2k^4 \leq r(M) \leq tk^4$ and $r_0 = r - k \geq r(M) - 2k$, we have

$$r \geq \frac{tk^4}{tk^4 - k} r_0 > \left(1 + \tfrac{1}{tk^3}\right) r_0 \geq r_0 + t^{-1}(r_0)^{1/4}$$

By choice of k, every rank-a set in M_0 has rank at most $2a$ in M', so $\tau_a(M_0) \geq \tau_{2a}(M')$. Moreover, a counting argument gives $\tau_{2a}(M') \geq \tau_{2a}(R') \geq \frac{q^r - 1}{q^{2a} - 1} > q^{r-2a}$, since $r > k \geq 2a$. Therefore

$$\tau_a(M_0) \geq \tau_{2a}(M') \geq q^{r_0 + t^{-1}(r_0)^{1/4} - 2a} \geq (r_0)^{f_{2.3}(a,b,n,q)} q^{r_0},$$

and the result follows from Theorem 2.3.

□

7. Connectivity

A matroid M is *weakly round* if there do not exist sets A and B with union $E(M)$, so that $r_M(A) \leq r(M) - 2$ and $r_M(B) \leq r(M) - 1$. This is a variation on *roundness*, a notion equivalent to infinite vertical connectivity introduced by Kung [9] under the name of *non-splitting*. Note that weak roundness is preserved by contractions.

It would suffice in this paper to consider roundness in place of weak roundness, but we use weak roundness in order that a partial result, Lemma 8.1, is slightly stronger; this may be useful in future work.

Lemma 7.1. *Let $a \geq 1$ and $q \geq 2$ be integers, and $\alpha \geq 0$ be a real number. If M is a matroid with $\tau_a(M) \geq \alpha q^{r(M)}$, then M has a weakly round restriction N such that $\tau_a(N) \geq \alpha q^{r(N)}$.*

Proof. If $r(M) \leq 2$, then M is weakly round, and $N = M$ will do; assume that $r(M) > 2$, and M is not weakly round. There are sets $A, B \subseteq E(M)$ such that $r(M|A) < r(M)$, $r(M|B) < r(M)$ and $A \cup B = E(M)$. Now, $\tau_a(M|A) + \tau_a(M|B) \geq \tau_a(M) \geq \alpha q^{r(M)}$, so one of $M|A$ or $M|B$ satisfies $\tau_a \geq \frac{1}{2}\alpha q^{r(M)} \geq \alpha q^{r(M)-1}$. The lemma follows by induction. $\square$

The way we exploit weak roundness of M is to contract one restriction of M into another restriction of larger rank:

Lemma 7.2. *Let M be a weakly round matroid, and $X, Y \subseteq E(M)$ be sets with $r_M(X) < r_M(Y)$. There is a minor N of M so that $N|X = M|X$, $N|Y = M|Y$, and Y is spanning in N.*

Proof. Let $C \subseteq E(M) - X \cup Y$ be maximal such that $(M/C)|X = M|X$ and $(M/C)|Y = M|Y$. The matroid M/C is weakly round, and by maximality of C we have $E(M/C) = \mathrm{cl}_{M/C}(X) \cup \mathrm{cl}_{M/C}(Y)$. If $r_{M/C}(Y) < r(M/C)$, then since $r_{M/C}(X) \leq r_{M/C}(Y) - 1$, the sets $\mathrm{cl}_{M/C}(X)$ and $\mathrm{cl}_{M/C}(Y)$ give a contradiction to weak roundness of M/C. Therefore Y is spanning in M/C and $N = M/C$ satisfies the lemma. $\square$

8. The Main Result

We are almost ready to prove Theorem 1.1; we first prove a more technical statement from which it will follow.

Lemma 8.1. *There is an integer-valued function $f_{8.1}(a, b, n, q, t)$ so that, for any prime power q and integers a, b, n, t with $1 \leq a < b$ and $t \geq 0$, if $M \in \mathcal{U}(a, b)$ is weakly round and has a $(f_{8.1}(a, b, n, q, t), q, t)$-stack restriction and a $\mathrm{PG}(f_{8.1}(a, b, n, q, t) - 1, q)$-minor, then M has a $\mathrm{PG}(n - 1, q')$-minor for some $q' > q$.*

Proof. Let q be a prime power and a, b, n, t be integers with $1 \leq a < b$ and $t \geq 0$. Let $d = \binom{b-1}{a}$. Let $h' = \max(a, n, f_{6.3}(a, b, n, q, t))$, and $h = (a + 1)h'$. Let $m \geq 4th$ be an integer so that $d^{-2ht}q^{r-ht-a} \geq r^{f_{2.3}(a,b,h't+1,q-1)}(q - 1)^r$ for all $r \geq m/2$. Set $f_{8.1}(a, b, n, q, t) = m$.

Let M be a weakly round matroid with a $\mathrm{PG}(m-1,q)$-minor $N = M/C\setminus D$ and a (m,q,t)-stack restriction S. Note that S contains an (h,q,t)-stack restriction S'; let M' be a matroid formed from M by contracting a maximal subset of C that is skew to $E(S')$; clearly M' has N as a minor and $r(M') \le r(N) + r(S') \le r(N) + ht$. We have $\tau_a(M') \ge \tau_a(N) \ge \frac{q^{r(N)}-1}{q^a-1} > q^{r(M')-ht-a}$ and $r(S') \le ht$; by Lemma 3.3 there is a set $X \subseteq E(M')$ such that $\tau_a(M'|X) \ge d^{a-ht}q^{r(M')-ht-a}$ and $\sqcap_{M'}(X, E(S')) \le a$. By adding elements skew to $E(S')$ to X if necessary, we may assume that $r_{M'}(X) \ge r(M') - r(S') \ge m - ht$.

By Lemma 3.2, there is a set $C' \subseteq E(S')$ such that $(M'/C')|E(S')$ has an (h',q,t)-stack restriction S'', and $E(S'')$ is skew to X in M'/C'. By Corollary 2.2, we have

$$\tau_a((M'/C)|X) \ge d^{a-ht-r_{M'}(C')}q^{r(M')-ht-a} \ge d^{-2ht}q^{r((M'/C)|X)-ht-a},$$

and since $r_{M'/C'}(X) \ge r_{M'}(X) - ht \ge m - 2ht \ge m/2$, it follows from Theorem 2.3 that the matroid $(M'/C')|X$ has a $\mathrm{PG}(h't, q^*)$-minor $N' = (M'/(C' \cup C''))|Y$ for some $q^* > q-1$. If $q^* > q$, then we are done, since $h't \ge n-1$. If $q^* = q$, then $M'/(C' \cup C'')$ is weakly round with a $\mathrm{PG}(h't,q)$-restriction R' and an (h',q,t)-stack restriction S''. By Lemma 7.2 and the fact that $r(S'') < r(R')$, we can find a minor in which S'' is a restriction and R' is spanning, and the conclusion follows from the definition of h' and Lemma 6.3. $\square$

We now restate and prove Theorem 1.4, which follows routinely.

Theorem 8.2. *There is an integer-valued function $\alpha_{8.2}(a,b,n,q)$ so that, for any integers a,b,n and q with $n \ge 1$, $q \ge 2$ and $1 \le a < b$, if $M \in \mathcal{U}(a,b)$ satisfies $\tau_a(M) \ge \alpha_{8.2}(a,b,n,q)q^{r(M)}$, then M has a $\mathrm{PG}(n-1,q')$-minor for some $q' > q$.*

Proof. Let a,b,n and q be integers with $n \ge 1$, $q \ge 2$ and $1 \le a < b$. Let $d = \max(q, \binom{b-1}{a}) + 2$. Let q^* be the smallest prime power so that $q^* \ge q$. Let $h = \max(n, f_{8.1}(a,b,n,q^*,a+1))$. Let $\lambda > 0$ be an integer such that $\lambda d^{-a}q^r \ge r^{f_{2.3}(a,b,h,q-1)}(q-1)^r$ for all integers $r \ge 1$. Set $\alpha_{8.2}(a,b,n,q) = \alpha = \max(\lambda, f_{5.1}(a,b,h,q,\lambda))$.

Let $M \in \mathcal{U}(a,b)$ satisfy $\tau_a(M) \ge \alpha q^{r(M)}$. By Theorem 2.3, and the fact that $\alpha > \lambda$, M has a $\mathrm{PG}(h-1,q')$-minor for some $q' > q-1$; if $q' \ne q$ then we are done because $h \ge n$, so we can assume that $q = q^* = q'$. By Lemma 7.1, M has a weakly round restriction M' with $\tau_a(M') \ge \alpha q^{r(M')}$. By Lemma 5.1, M' has a contraction-minor N with an $(h,q,a+1)$-stack restriction, satisfying $\tau^d(N) \ge \lambda q^{r(N)}$. We have $\tau_a(N) \ge d^{-a}\tau^d(N) \ge d^{-a}\lambda q^{r(N)}$, so by definition of λ the matroid N has a $\mathrm{PG}(h-1,q')$-minor for some $q'' > q-1$. As before, we may assume that $q'' = q$. Since N is weakly round, the theorem now follows by applying Lemma 8.1 to N. $\square$

Theorem 1.1 is now a fairly simple consequence.

Theorem 8.3. *If $a \ge 1$ is an integer, and $\mathcal{M}$ is a minor-closed class of matroids, then there is an integer c so that either:*

(1) $\tau_a(M) \le r(M)^c$ for all $M \in \mathcal{M}$, or
(2) There is a prime power q so that $\tau_a(M) \le cq^{r(M)}$ for all $M \in \mathcal{M}$ and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids, or
(3) $\mathcal{M}$ contains all rank-$(a+1)$ uniform matroids.

Proof. We may assume that (3) does not hold; let $b > a$ be an integer such that $\mathcal{M} \subseteq \mathcal{U}(a,b)$. As $U_{a+1,b}$ is a simple matroid that is $\mathrm{GF}(q)$-representable whenever $q \ge b$ (see [8]), we have $\mathrm{PG}(a,q') \notin \mathcal{M}$ for all $q' \ge b$.

If, for some integer $n > a$, we have $\tau_a(M) < r(M)^{f_{1.3}(a,b,n)}$ for all $M \in \mathcal{M}$ of rank at least 2, then (1) holds. We may therefore assume that, for all $n > a$, there exists a matroid $M_n \in \mathcal{M}$ such that $r(M_n) \ge 2$ and $\tau_a(M_n) \ge r(M_n)^{f_{1.3}(a,b,n)}$.

By Theorem 1.3, it follows that for all $n > a$ there exists a prime power q'_n such that $\mathrm{PG}(n-1, q'_n) \in \mathcal{M}$. We have $q'_n < b$ for all n, so there are finitely many possible q'_n, and so there is a prime power $q_0 < b$ such that $\mathrm{PG}(n-1, q_0) \in \mathcal{M}$ for infinitely many n, implying that $\mathcal{M}$ contains all $\mathrm{GF}(q_0)$-representable matroids.

Let q be maximal such that $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids. Since $\mathrm{PG}(a,q') \notin \mathcal{M}$ for all $q' \ge b$, the value q is well-defined, and moreover there is some n such that $\mathrm{PG}(n-1,q') \notin \mathcal{M}$ for all $q' > q$. Theorem 1.4 thus gives $\tau_a(M) \le \alpha_{1.4}(a,b,n,q)q^{r(M)}$ for all $M \in \mathcal{M}$, giving (2). $\square$

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