PROJECTIVE GEOMETRIES IN EXPONENTIALLY DENSE MATROIDS. I

JIM GEELEN AND PETER NELSON

Abstract. We show for each integer $a \geq 1$ that, if $\mathcal{M}$ is a minor-closed class of matroids not containing all rank-$(a+1)$ uniform matroids, then there exists an integer n such that either every rank-r matroid in $\mathcal{M}$ can be covered by at most r^n rank-a sets, or $\mathcal{M}$ contains the GF(q)-representable matroids for some prime power q, and every rank-r matroid in $\mathcal{M}$ can be covered by at most $r^n q^r$ rank-a sets. This determines the maximum density of the matroids in $\mathcal{M}$ up to a polynomial factor.

1. Introduction

If M is a matroid, and $a \geq 1$ is an integer, then $\tau_a(M)$ denotes the *a-covering number* of M, the minimum number of sets of rank at most a in M required to cover $E(M)$. We will prove the following theorem:

Theorem 1.1. *Let $a \geq 1$ be an integer. If $\mathcal{M}$ is a minor-closed class of matroids, then either*

(1) $\tau_a(M) \leq r(M)^{n_{\mathcal{M}}}$ for all $M \in \mathcal{M}$, or
(2) there is a prime power q so that $\tau_a(M) \leq r(M)^{n_{\mathcal{M}}} q^{r(M)}$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains all GF(q)-representable matroids, or
(3) $\mathcal{M}$ contains all rank-$(a+1)$ uniform matroids.

Here, $n_{\mathcal{M}}$ denotes an integer constant depending only on $\mathcal{M}$. In [7], the second author will refine the bound $r(M)^{n_{\mathcal{M}}} q^{r(M)}$ in (2) by a polynomial factor to $c_{\mathcal{M}} q^{r(M)}$ for some constant $c_{\mathcal{M}}$; it is routine to show that this improved bound is best-possible up to a constant factor. Both these results also appear in [8].

The above theorem and its improvement in [7] are contained in the following larger conjecture of Geelen [1]:

Conjecture 1.2 (Growth Rate Conjecture)**.** *Let $a \geq 1$ be an integer. If $\mathcal{M}$ is a minor-closed class of matroids, then either*

(1) $\tau_a(M) \leq c_{\mathcal{M}} r(M)$ for all $M \in \mathcal{M}$, or
(2) $\tau_a(M) \leq c_{\mathcal{M}} r(M)^2$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains all graphic matroids or all bicircular matroids, or

(3) *there is a prime power* q *such that* $\tau_a(M) \le c_{\mathcal{M}} q^{r(M)}$ *for all* $M \in \mathcal{M}$*, and* $\mathcal{M}$ *contains all* $\mathrm{GF}(q)$*-representable matroids, or*
(4) $\mathcal{M}$ *contains all rank-*$(a+1)$ *uniform matroids.*

When $a = 1$, the parameter $\tau_a(M)$ is just the number of points in M, sometimes written as $\epsilon(M)$, and (4) corresponds to $\mathcal{M}$ containing all simple rank-2 matroids. The conjecture in this case was proved by work of Geelen, Kabell, Kung and Whittle [2,4,5], and stated in [4] as the 'Growth Rate Theorem'.

For general a, if (4) holds, then there is no bound on $\tau_a(M)$ as a function of $r(M)$ for all $M \in \mathcal{M}$, as a rank-$(a+1)$ uniform matroid can require arbitrarily many rank-a sets to cover. Thus, we derive bounds on τ_a relative to some particular rank-$(a+1)$ uniform matroid that is excluded as a minor. We prove Theorem 1.1 as a consequence of the following result:

Theorem 1.3. *For all integers* $1 \le a < b$, $q \ge 1$ *and* $n \ge 1$*, there exists an integer* m *so that, if* M *is a matroid of rank at least* 2 *with no* $U_{a+1,b}$*-minor, and* $\tau_a(M) \ge r(M)^m q^{r(M)}$*, then* M *has a* $\mathrm{PG}(n-1, q')$*-minor for some prime power* $q' > q$.

Our proof is loosely based on ideas in [2], and uses its main results as a base case. The next five sections are used to define the terminology and intermediate structures we need, and the bulk of the argument rests on the lemmas in Sections 7, 8 and 9.

2. Preliminaries

We follow the notation of Oxley [9]. Two sets X and Y are *skew* in a matroid M if $r_M(X \cup Y) = r_M(X) + r_M(Y)$, and a collection of sets $\mathcal{X}$ in M is *mutually skew* if $r_M\left(\cup_{X \in \mathcal{X}} X\right) = \sum_{X \in \mathcal{X}} r_M(X)$. Very often, the atomic objects in our proof are sets in M rather than elements; to this end, we also define some new notation.

A common object is a collection of sets of the same rank. If M is a matroid, and $a \ge 1$ is an integer, then $\mathcal{R}_a(M)$ denotes the set $\{X \subseteq E(M) : r_M(X) = a\}$.

Generalising the notion of parallel elements, if $X, X' \subseteq E(M)$, then we write $X \equiv_M X'$ if $\mathrm{cl}_M(X) = \mathrm{cl}_M(X')$; we say that X and X' are *similar* in M. We write $[X]_M = \{X' \subseteq E(M) : X \equiv_M X'\}$ for the 'similarity class' of X in M.

We also extend existing notation in straightforward ways. If $\mathcal{X} \subseteq 2^{E(M)}$ is a collection of sets, then we write $M|\mathcal{X}$ for $M|(\cup_{X \in \mathcal{X}} X)$, $\mathrm{cl}_M(\mathcal{X})$ for $\mathrm{cl}_M(\cup_{X \in \mathcal{X}} X)$, and $r_M(\mathcal{X})$ for $r_M(\mathrm{cl}_M(\mathcal{X}))$. Two sets $\mathcal{X}, \mathcal{X}' \subseteq 2^{E(M)}$ are *similar* in M if $\mathrm{cl}_M(\mathcal{X}) = \mathrm{cl}_M(\mathcal{X}')$.

Analogously to the notion of a simple matroid, we say that $\mathcal{X} \subseteq 2^{E(M)}$ is *simple* in M if the sets in $\mathcal{X}$ are pairwise dissimilar in M. Note that any collection of flats of M is simple. We write $\epsilon_M(\mathcal{X})$ for the maximum size of a subset of $\mathcal{X}$ that is simple in M, or equivalently the number of different similarity classes of $2^{E(M)}$ containing a set in $\mathcal{X}$. If $\mathcal{X}$ just contains nonloop singletons, then $\epsilon_M(\mathcal{X}) = \tau_1(M|\mathcal{X})$.

For integers a and b with $1 \le a < b$, we write $\mathcal{U}(a,b)$ for the class of matroids with no $U_{a+1,b}$-minor. The first tool in our proof is a theorem of Geelen and Kabell [3], which shows that the parameter τ_a is bounded as a function of rank across $\mathcal{U}(a,b)$.

Theorem 2.1. *Let a and b be integers with $1 \le a < b$. If $M \in \mathcal{U}(a,b)$ satisfies $r(M) > a$, then $\tau_a(M) \le \binom{b-1}{a}^{r(M)-a}$.*

Proof. We first prove the result when $r(M) = a+1$, then proceed by induction. If $r(M) = a+1$, then observe that $M|B \cong U_{a+1,a+1}$ for any basis B of M; let $X \subseteq E(M)$ be maximal such that $M|X \cong U_{a+1,|X|}$. We may assume that $|X| < b$, and by maximality of X, every $e \in E(M) - X$ is spanned by a rank-a set of X. Therefore, $\tau_a(M) \le \binom{|X|}{a} \le \binom{b-1}{a}$.

Suppose that $r(M) > a+1$, and inductively assume that the result holds for matroids of smaller rank. Let $e \in E(M)$. We have $\tau_{a+1}(M) \le \tau_a(M/e) \le \binom{b-1}{a}^{r(M)-a-1}$ by induction, and by the base case, each rank-$(a+1)$ set in M admits a cover with at most $\binom{b-1}{a}$ sets of rank at most a. Therefore $\tau_a(M) \le \binom{b-1}{a}\tau_{a+1}(M) \le \binom{b-1}{a}^{r(M)-a}$, as required. □

This theorem has two simple corollaries concerning the density of matroids in $\mathcal{U}(a,b)$ relative to that of their minors:

Lemma 2.2. *Let a and b be integers with $1 \le a < b$. If $M \in \mathcal{U}(a,b)$ and $C \subseteq E(M)$, then $\tau_a(M) \le \binom{b-1}{a}^{r_M(C)}\tau_a(M/C)$.*

Lemma 2.3. *Let a and b be integers with $1 \le a < b$. If $M \in \mathcal{U}(a,b)$ and N is a minor of M, then $\tau_a(M) \le \binom{b-1}{a}^{r(M)-r(N)}\tau_a(N)$.*

The next two theorems, also due to Geelen and Kabell, were proved in [2] to resolve the 'polynomial-exponential' part of the Growth Rate Theorem, both finding a large projective geometry in a sufficiently dense matroid without some line as a minor:

Theorem 2.4. *There is an integer-valued function $f_{2.4}(\ell,n)$ so that, for any integers $\ell \ge 2$ and $n \ge 2$, if $M \in \mathcal{U}(1,\ell)$ satisfies $\tau_1(M) \ge r(M)^{f_{2.4}(\ell,n)}$, then M has a $\mathrm{PG}(n-1,q)$-minor for some prime power q.*

Theorem 2.5. *There is a real-valued function $\alpha_{2.5}(\ell, n, q)$ so that, for any integers $q \geq 2$, $\ell \geq 2$ and $n \geq 1$, if $M \in \mathcal{U}(1, \ell)$ satisfies $\tau_1(M) \geq \alpha_{2.5}(\ell, n, q)q^{r(M)}$, then M has a $\mathrm{PG}(n-1, q')$-minor for some prime power $q' > q$.*

3. Thickness and Firmness

Two density-related notions that will feature frequently in our proof are those of *thickness* and *firmness*, which we define and explain in this section.

If $d \geq 1$ is an integer, and M is a matroid, then M is *d-thick* if $\tau_{r(M)-1}(M) \geq d$. A set $X \subseteq E(M)$ is *d-thick in M* if $M|X$ is d-thick.

Note that every matroid is 2-thick, and that thickness is monotone in the sense that if $d' \geq d$ and M is d'-thick, then M is d-thick. The following lemma is fundamental, and we use it freely and frequently in our proof.

Lemma 3.1. *Let $d \geq 1$ be an integer. If M is a matroid, N is a minor of M, and $X \subseteq E(N)$ is d-thick in M, then X is d-thick in N.*

Proof. Deleting an element of M outside X, or contracting an element outside $\mathrm{cl}_M(X)$ does not change $M|X$, so it suffices to show that contracting a nonloop $e \in \mathrm{cl}_M(X)$ does not destroy d-thickness of X. This follows from the fact that $\tau_{r(M)-2}(M/e) \geq \tau_{r(M)-1}(M)$. □

Any rank-1 or rank-0 matroid is clearly arbitrarily thick. Convenient examples of thick matroids are uniform matroids - no rank-a set in the matroid $U_{a+1,b}$ contains more than a elements, so $U_{a+1,b}$ is $\lceil \frac{b}{a} \rceil$-thick. Indeed, sufficient thickness and rank ensure a large uniform minor:

Lemma 3.2. *Let a and b be integers with $1 \leq a < b$. If M is $\binom{b}{a}$-thick and $r(M) > a$, then M has a $U_{a+1,b}$-minor.*

Proof. By Lemma 3.1, d-thickness of M is preserved by contraction, so by contracting points if needed, we may assume that $r(M) = a + 1$. Now, $\binom{b-1}{a} < \binom{b}{a} \leq \tau_a(M)$, so the result follows from Theorem 2.1. □

This lemma tells us that, qualitatively, searching for a $U_{a+1,b}$-minor is equivalent to searching for an appropriately thick minor of rank greater than a. We take this approach hereon; in fact, nearly all the uniform minors we find will be constructed by implicit use of this lemma.

We now turn to a definition of firmness. If $d \geq 1$ is an integer and M is a matroid, then a set $\mathcal{X} \subseteq 2^{E(M)}$ is *d-firm* in M if all $\mathcal{X}' \subseteq \mathcal{X}$ with $|\mathcal{X}'| > d^{-1}|\mathcal{X}|$ satisfy $r_M(\mathcal{X}') = r_M(\mathcal{X})$.

Firmness is a measure of how 'evenly spread' a collection of sets is. The set of points in a d-point line is d-firm; more generally, the set of

a-subsets of $E(U_{a+1,b})$ is $\binom{b}{a}$-firm. Firmness is clearly monotone in the sense that d-firmness implies $(d-1)$-firmness.

Our first lemma relates firmness to thickness in an exact way:

Lemma 3.3. *Let $a \geq 1$ and $d \geq 1$ be integers, and M be a matroid. If $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is d-firm in M, and each $X \in \mathcal{X}$ is d-thick in M, then $\mathrm{cl}_M(\mathcal{X})$ is d-thick in M.*

Proof. Let $\mathcal{F}$ be a cover of $M|\,\mathrm{cl}_M(\mathcal{X})$ with flats of smaller rank; we wish to show that $|\mathcal{F}| \geq d$. If a set $X \in \mathcal{X}$ is not contained in any flats in $\mathcal{F}$, then $\{X \cap F : F \in \mathcal{F}\}$ is a cover of $M|X$ with sets of smaller rank, of size at most $|\mathcal{F}|$, so $|\mathcal{F}| \geq d$ by d-thickness of X. We may therefore assume every $X \in \mathcal{X}$ is contained in some $F \in \mathcal{F}$. Now, since $\mathcal{X}$ is d-firm in M and no flat in $\mathcal{F}$ is spanning in $M|\,\mathrm{cl}_M(\mathcal{X})$, each flat in $\mathcal{F}$ contains at most $d^{-1}|\mathcal{X}|$ different sets in $\mathcal{X}$. We thus have $|\mathcal{F}| \geq \frac{|\mathcal{X}|}{d^{-1}|\mathcal{X}|} = d$, as required. □

We will use this lemma to construct the thick sets of rank greater than a that we are frequently seeking. Thus, we often consider a set $\mathcal{X} \subseteq \mathcal{R}_a(M)$ that has no firm subset of rank $> a$ in a minor of M; we are 'excluding' a minor with this structure from $\mathcal{X}$ and M in lieu of excluding $U_{a+1,b}$.

This exclusion allows us to control the number of sets in $\mathcal{X}$ in useful ways; the first of the next two lemmas tells us about the 'absolute' density of $\mathcal{X}$ in M, and the second about the 'relative' density of $\mathcal{X}$ in M as compared to in a minor of M.

Lemma 3.4. *Let $a \geq 1$ and $d \geq 2$ be integers, M be a matroid with $r(M) > a$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$. If $\epsilon_M(\mathcal{X}) \geq d^{r(M)-a}$, then there is a set $\mathcal{Y} \subseteq \mathcal{X}$ such that $r_M(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in M.*

Proof. We may assume that $\mathcal{X}$ is simple. If $r(M) = a+1$, then the union of any two sets in $\mathcal{X}$ is spanning in M, and $|\mathcal{X}| \geq d$, so $\mathcal{X}$ is d-firm; we assume that $r(M) > a+1$, and proceed by induction on $r(M)$. If $\mathcal{X}$ is not d-firm, then there is some $\mathcal{X}' \subseteq \mathcal{X}$ with $r_M(\mathcal{X}') < r_M(\mathcal{X})$, and $|\mathcal{X}'| \geq d^{-1}|\mathcal{X}| \geq d^{r(M)-1-a} \geq d^{r_M(\mathcal{X}')-a}$. Moreover, $|\mathcal{X}'| \geq d^{r(M)-1-a} \geq d \geq 2$, so $r_M(\mathcal{X}') > a$. The result follows by applying the inductive hypothesis to $\mathcal{X}'$ in $M|\,\mathrm{cl}_M(\mathcal{X}')$. □

Lemma 3.5. *Let $a \geq 1$ and $d \geq 2$ be integers, M be a matroid, N be a minor of M, and $\mathcal{X} \subseteq \mathcal{R}_a(M) \cap \mathcal{R}_a(N)$. If $\epsilon_M(\mathcal{X}) > d^{r(M)-r(N)}\epsilon_N(\mathcal{X})$, then there is a set $\mathcal{Y} \subseteq \mathcal{X}$ such that $r_M(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in M.*

Proof. Let $N = M/C \setminus D$, where $r_M(C) = r(M) - r(N)$. Suppose that $\epsilon_M(\mathcal{X}) > d^{r_M(C)}\epsilon_N(\mathcal{X})$. By a majority argument applied to the similarity classes of $\mathcal{X}$ in N, there is some $X \in \mathcal{X}$ such that $\epsilon_M([X]_N \cap \mathcal{X}) \geq d^{r_M(C)} = d^{r_M(X \cup C)-a}$. Now, every set in $[X]_N \cap \mathcal{X}$ is contained in $\mathrm{cl}_M(X \cup C)$, so applying Lemma 3.4 to $M|(\mathrm{cl}_M(X \cup C))$ gives the result. □

4. Arrangements

We prove two lemmas related to how collections of sets in a matroid 'fit together'. This first lemma shows that, given $\mathcal{X} \subseteq \mathcal{R}_a(M)$, we can contract a point of M so that the rank of most sets in $\mathcal{X}$ is unchanged:

Lemma 4.1. *Let M be a matroid of rank at least 1, $a \geq 1$ be an integer, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$. There exists a nonloop $e \in E(M)$ so that*

$$\epsilon_M(\mathcal{X} \cap \mathcal{R}_a(M/e)) \geq \left(1 - \tfrac{a}{r(M)}\right) \epsilon_M(\mathcal{X}).$$

Proof. Let $\mathcal{X}'$ be a maximal simple subset of $\mathcal{X}$, and B be a basis of M. Each set in $\mathcal{X}'$ has at most a elements of B in its closure, so $\sum_{f \in B} |\{X \in \mathcal{X}' : f \in \mathrm{cl}_M(X)\}| \leq a|\mathcal{X}'|$. There is therefore some $e \in B$ such that $|\{X \in \mathcal{X}' : e \in \mathrm{cl}_M(X)\}| \leq \frac{a}{|B|}|\mathcal{X}'|$. Every set in $\mathcal{X}'$ that does not span e is in $\mathcal{R}_a(M/e)$, so

$$\begin{aligned}\epsilon_M(\mathcal{X} \cap \mathcal{R}_a(M/e)) &\geq |\mathcal{X}' \cap \mathcal{R}_a(M/e)| \\ &\geq |\mathcal{X}'| - \tfrac{a}{|B|}|\mathcal{X}'| \\ &= \epsilon_M(\mathcal{X}) - \tfrac{a}{r(M)}\epsilon_M(\mathcal{X}),\end{aligned}$$

and the result follows. □

This second lemma relates to the fact that a graph with many edges contains either a vertex of large degree or a large matching. Recall that $\mathcal{W} \subseteq 2^{E(M)}$ is mutually skew in M if $r_M(\bigcup_{W \in \mathcal{W}} W) = \sum_{W \in \mathcal{W}} r_M(W)$.

Lemma 4.2. *Let M be a matroid, $a \geq 1$ and $t \geq 1$ be integers, and let $\mathcal{X} \subseteq \mathcal{R}_a(M)$. Either*

(i) there exists $\mathcal{W} \subseteq \mathcal{X}$ so that $|\mathcal{W}| = t$, and $\mathcal{W}$ is mutually skew in M, or

(ii) there is a minor N of M, a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$, and a nonloop e of N such that $r(N) \geq r(M) - at$, and $|\mathcal{Y}| \geq (at)^{-1}|\mathcal{X}|$, and $e \in \mathrm{cl}_N(Y)$ for all $Y \in \mathcal{Y}$.

Proof. Let $\mathcal{W}$ be a maximal mutually skew subset of $\mathcal{X}$; we may assume that $k < t$. Let $e_1, \dots, e_{a|\mathcal{W}|}$ be a basis for $\bigcup_{W \in \mathcal{W}} W$. For each $1 \leq i \leq a|\mathcal{W}|$, let $M_i = M/\{e_1, \dots, e_i\}$. By maximality of $\mathcal{W}$, each $X \in \mathcal{X} - \mathcal{W}$ satisfies $r_{M_{a|\mathcal{W}|}}(X) < r_M(X) = a$, and this inequality clearly also holds for all $X \in \mathcal{W}$, so for each $X \in \mathcal{X}$ there is some i_X such that $M|X =$

$M_{i_X-1}|X$, and X spans e_{i_X} in M_{i_X-1}. By a majority argument, there is some $1 \le i_0 \le a|\mathcal{W}|$ and $\mathcal{Y} \subseteq \mathcal{X}$ such that $|\mathcal{Y}| \ge (a|\mathcal{W}|)^{-1}|\mathcal{X}|$, and $i_Y = i_0$ for all $Y \in \mathcal{Y}$. Since $|\mathcal{W}| < t$, the minor $N = M/\{e_1, \dots, e_{i_0-1}\}$, along with $\mathcal{Y}$ and e_{i_0}, will satisfy the second outcome. □

5. Weighted Covers and Scatteredness

Our main theorem concerns upper bounds on the parameter τ_a. It is therefore natural to consider minimum-sized covers of a matroid with sets of rank at most a. However, such a cover has few useful properties, and it seems difficult to make use of one in a proof. We will therefore change the parameter we are considering to one that considers minimal 'weighted' covers. This tweak will force a minimal cover to have many properties that we exploit at length.

If M is a matroid, and $\mathcal{X}, \mathcal{F} \subseteq 2^{E(M)}$, then $\mathcal{F}$ is a *cover of* $\mathcal{X}$ in M if every set in $\mathcal{X}$ is contained in a set in $\mathcal{F}$. A *cover of* M is a cover of $\{\{e\} : e \in E(M)\}$.

If $d \ge 1$ is an integer, and $\mathcal{F} \subseteq 2^{E(M)}$, then we write $\mathrm{wt}^d_M(\mathcal{F})$ for the sum $\sum_{F \in \mathcal{F}} d^{r_M(F)}$, which we call the *weight* of $\mathcal{F}$. Thus, the 'weight' of a point in $\mathcal{F}$ is d, the 'weight' of a line is d^2, etc. $\mathcal{F}$ is a *d-minimal cover of* $\mathcal{X}$ if $\mathcal{F}$ minimizes $\mathrm{wt}^d_M(\mathcal{F})$ subject to being a cover of $\mathcal{X}$. We write $\tau^d(M)$ for the weight of a d-minimal cover of M. The parameter τ^d will not change too dramatically in a minor:

Lemma 5.1. *Let $d \ge 1$ be an integer. If N is a minor of a matroid M, then $\tau^d(N) \ge d^{r(N)-r(M)}\tau^d(M)$.*

Proof. It suffices to show that, for a nonloop $e \in E(M)$, we have $\tau^d(M/e) \ge d^{-1}\tau^d(M)$. If $\mathcal{F}$ is a d-minimal cover of M/e, then $\mathcal{F}' = \{\mathrm{cl}_M(F \cup \{e\}) : F \in \mathcal{F}\}$ is a cover of M, so $\tau^d(M) \le \mathrm{wt}^d_M(\mathcal{F}') = \sum_{F \in \mathcal{F}} d^{r_M(F \cup \{e\})} = \sum_{F \in \mathcal{F}} d^{r_{M/e}(F)+1} = d\,\mathrm{wt}^d_{M/e}(\mathcal{F}) = d\tau^d(M/e)$, giving the result. □

A concept that we will soon use to build highly structured minors is that of *scatteredness*, another measure of how 'spread out' a collection of sets is. A set $\mathcal{X} \subseteq 2^{E(M)}$ is *d-scattered* in a matroid M if all sets in $\mathcal{X}$ are d-thick in M, and $\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}$ is a d-minimal cover of $\mathcal{X}$ in M.

A scattered set cannot be efficiently covered with sets of larger rank. Again, we use the symbol d; this same parameter will be passed around our proofs in measures of thickness, firmness and scatteredness.

Our first lemma establishes some nice properties in the case where a minimal cover of a set $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is just the ground set of M:

Lemma 5.2. *Let $a \geq 1$ and $d \geq 1$ be integers, M be a matroid with $r(M) > a$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$. If all sets in $\mathcal{X}$ are d-thick in M, and $\{E(M)\}$ is a d-minimal cover of $\mathcal{X}$ in M, then $\epsilon_M(\mathcal{X}) \geq d^{r(M)-a}$, and M is d-thick.*

Proof. $\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}$ is a cover of $\mathcal{X}$ in M; since $\{E(M)\}$ is a d-minimal cover of $\mathcal{X}$, we have $\mathrm{wt}_M^d(\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}) \geq \mathrm{wt}_M^d(\{E(M)\})$, so $d^a \epsilon_M(\mathcal{X}) \geq d^{r(M)}$, giving the first part of the lemma.

We will now show that M is d-thick. Let $\mathcal{F}$ be a cover of M with flats of smaller rank. If some $X \in \mathcal{X}$ is not contained in F for any set F in $\mathcal{F}$, then $\{X \cap F : F \in \mathcal{F}\}$ is a cover of $M|X$ of size at most $|\mathcal{F}|$ with sets of smaller rank than X, so $|\mathcal{F}| \geq d$ by d-thickness of X. Otherwise, $\mathcal{F}$ is a $\mathcal{X}$-cover, so $\mathrm{wt}_M^d(\mathcal{F}) \geq \mathrm{wt}_M^d(\{E(M)\})$. Therefore $|\mathcal{F}| d^{r(M)-1} \geq d^{r(M)}$, so $|\mathcal{F}| \geq d$. □

Our means of constructing scattered sets is the following lemma:.

Lemma 5.3. *Let $d \geq 1$ be an integer, M be a matroid, and $\mathcal{X} \subseteq 2^{E(M)}$. If all sets in $\mathcal{X}$ are d-thick in M, and $\mathcal{F}$ is a d-minimal cover of $\mathcal{X}$ in M, then every subset of $\mathcal{F}$ is d-scattered in M.*

Proof. Let $\mathcal{F}' \subseteq \mathcal{F}$. It is clear from d-minimality of $\mathcal{F}$ that $\mathcal{F}'$ is simple, and that $\mathcal{F}'$ is a d-minimal cover of $\mathcal{F}'$. For each $F \in \mathcal{F}'$, the set $\{F\}$ is a d-minimal cover of $\{X \in \mathcal{X} : X \subseteq F\}$ by d-minimality of $\mathcal{F}$, so by applying Lemma 5.2 to $M|F$, we see that F is d-thick in M. Therefore $\mathcal{F}'$ is d-scattered in M. □

In particular, if $\mathcal{F}$ is a d-minimal cover of M itself, then every subset of $\mathcal{F}$ is d-scattered in M, as the singleton $\{e\}$ is d-thick in M for any $e \in E(M)$.

Lemma 5.4. *Let $a \geq 1$ and $d \geq 1$ be integers. If M is a matroid, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is d-scattered in M, then $\epsilon_M(\mathcal{X}) \leq d^{r(M)-a}$.*

Proof. $\{E(M)\}$ is a cover of $\mathcal{X}$ in M, so d-scatteredness of $\mathcal{X}$ gives $d^a \epsilon_M(\mathcal{X}) = \mathrm{wt}_M^d(\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}) \leq \mathrm{wt}_M^d(\{E(M)\}) = d^{r(M)}$, giving the result. □

The parameter τ^d, for an appropriate d, is what we use to gain traction towards Theorem 1.3. Considering this parameter instead of τ_a is not a major change in the setting of excluding $U_{a+1,b}$; indeed, these two parameters differ by at most a constant factor.

Lemma 5.5. *If a, b, d are integers with $1 \leq a < b$ and $d \geq \binom{b}{a}$, and $M \in \mathcal{U}(a,b)$, then no d-minimal cover of M contains a set of rank greater than a, and $\tau_a(M) \leq \tau^d(M) \leq d^a \tau_a(M)$.*

Proof. Let $\mathcal{F}$ be a d-minimal cover of M. By Lemma 5.3, every set in $\mathcal{F}$ is d-thick, so by Lemma 3.2 and definition of d, there is no set of rank greater than a in $\mathcal{F}$. Therefore $\tau_a(M) \leq |\mathcal{F}| \leq \mathrm{wt}^d_M(\mathcal{F}) = \tau^d(M)$. Moreover, if $\mathcal{H}$ is a minimum-sized cover of M with sets of rank at most a, then $\tau^d(M) \leq d^a|\mathcal{H}| = d^a\tau_a(M)$. □

6. Pyramids

We now define the intermediate structure that is vital to our proof. Let $a \geq 1$, $d \geq 1$, $q \geq 1$ and $h \geq 0$ be integers, M be a matroid, $\mathcal{S} \subseteq \mathcal{R}_a(M)$, and $\{e_1, \dots, e_h\}$ be an independent set of size h in M. For each $i \in \{0, 1, \dots, h\}$, let $M_i = M/\{e_1, \dots, e_i\}$.

We say $(M, \mathcal{S}; e_1, \dots, e_h)$ is an (a, q, h, d)-*pyramid* if

- $\mathcal{S} \neq \varnothing$, and S is skew to $\{e_1, \dots, e_h\}$ for all $S \in \mathcal{S}$,
- for each $i \in \{0, 1, \dots, h\}$ and $S \in \mathcal{S}$, there are sets $S_1, \dots, S_q \in \mathcal{S}$, pairwise dissimilar in M_i, and each similar to S in M_{i+1}, and
- S is d-thick in M for all $S \in \mathcal{S}$.

A pyramid is a structured, exponential-sized collection of thick rank-a sets. For each $0 \leq i < h$, and each $S \in \mathcal{S}$, contracting e_{i+1} in M_i 'collapses' the dissimilar d-thick sets $S_1, \dots, S_q$ onto the single d-thick set S in M_{i+1}, without changing their rank.

When $a = 1$, the set $\mathcal{S}$ simply contains points; in this case, the value of d is irrelevant, and the structure described in the second condition is a set of q other points on a line through e_{i+1}. Pyramids are based on objects of the same name used by Geelen and Kabell in [2]; a pyramid in their sense is a special sort of pyramid in our sense, with $a = 1$.

The structure of a pyramid is self-similar, and the next two easily proved lemmas concern smaller pyramids inside a pyramid:

Lemma 6.1. *If $(M, \mathcal{S}; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid, and i and j are integers with $0 \leq i \leq j \leq h$, then*

$$(M/\{e_{i+1}, \dots, e_j\}, \mathcal{S}; e_1, \dots, e_i, e_{j+1}, \dots, e_h)$$

is an $(a, q, h-(j-i), d)$-pyramid.

Lemma 6.2. *Let $(M, \mathcal{S}; e_1, \dots, e_h)$ be an (a, q, h, d)-pyramid, and let N be a minor of $M/\{e_1, \dots, e_h\}$. If $\mathcal{Y} \subseteq \mathcal{S} \cap \mathcal{R}_a(N)$, then there is a minor M' of M, and an (a, q, h, d)-pyramid $(M', \mathcal{S}'; e_1, \dots, e_h)$ so that $\mathcal{Y} \subseteq \mathcal{S}' \subseteq \mathcal{S}$, and $N|\mathcal{Y} = (M'/\{e_1, \dots, e_h\})|\mathcal{Y}$.*

The next lemma is our means of adding a 'level' to a pyramid. In accordance with the definition, it requires a point e and a smaller pyramid on M/e such that e 'lifts' each set in the pyramid into $q+1$ distinct sets. The proof, which we omit, is cumbersome but routine.

Lemma 6.3. *Let M be a matroid, $e \in E(M)$ be a nonloop, a, d, q, h be integers with $q, a, d \geq 1$ and $h \geq 0$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ be simple in M. Let $\mathcal{X}_{>q} = \{X \in \mathcal{X} : |[X]_{M/e} \cap \mathcal{X}| > q\}$. If M/e has an $(a, q+1, h, d)$-pyramid minor P such that $\mathcal{S}_P \subseteq \mathcal{X}_{>q}$, then M has an $(a, q+1, h+1, d)$-pyramid minor P' such that $\mathcal{S}_{P'} \subseteq \mathcal{X}$.*

The following lemma shows that a pyramid can be restricted to have bounded rank:

Lemma 6.4. *Let $(M, \mathcal{S}; e_1, \dots, e_h)$ be an (a, q, h, d)-pyramid, and $M_h = M/\{e_1, \dots, e_h\}$, and $S \in \mathcal{S}$. There is a restriction M' of M such that*

$$(M', \{S' \in \mathcal{S} : S' \equiv_{M_h} S\}; e_1, \dots, e_h)$$

is an (a, q, h, d)-pyramid, and $r(M') = a + h$.

Proof. Let $M' = M|\operatorname{cl}_M(S \cup \{e_1, \dots, e_h\})$, and $\mathcal{S}' = \{S' \in \mathcal{S} : S' \equiv_{M_h} S\}$. Since $r_{M_h}(S) = a$, we have $r(M') = a + h$. Let $0 \leq i < h$. Let $S' \in \mathcal{S}'$, and $S'_1, \dots, S'_q$ be the sets for i and S' as given by the definition of a pyramid. Each S'_j is similar to S' in M_i, and therefore also in M_h, so $\{S'_1, \dots, S'_q\} \subseteq \mathcal{S}'$, and $(S'_1 \cup \ldots \cup S'_q) \subseteq E(M')$. Therefore, $(M', \mathcal{S}'; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid. □

Our penultimate lemma verifies the set $\mathcal{S}$ in a pyramid has exponentially many elements:

Lemma 6.5. *If $(M, \mathcal{S}; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid, and $M_h = M/\{e_1, \dots, e_h\}$, then $\epsilon_M(\mathcal{S}) \geq q^h \epsilon_{M_h}(\mathcal{S})$.*

Proof. When $h = 0$, there is nothing to show. Otherwise, suppose that the result holds for a fixed h, and let $(M, \mathcal{S}; e_1, \dots, e_{h+1})$ be an $(a, q, h+1, d)$-pyramid. We know that $(M/e_1, \mathcal{S}; e_2, \dots, e_{h+1})$ is an (a, q, h, d)-pyramid; so $\epsilon_{M/e_1}(\mathcal{S}) \geq q^h \epsilon_{M_{h+1}}(\mathcal{S})$ by the inductive hypothesis. Moreover, for each $S \in \mathcal{S}$, there are pairwise dissimilar sets $S_1, \dots, S_q \in \mathcal{S}$, each similar to S in M/e_1. Therefore $\epsilon_M(\mathcal{S}) \geq q\epsilon_{M/e_1}(\mathcal{S}) \geq q^{h+1} \epsilon_{M_{h+1}}(\mathcal{S})$, so the lemma holds. □

Finally, we observe that a pyramid has a restriction with bounded rank, containing an exponential-size subset of $\mathcal{S}$. This lemma follows routinely from Lemmas 6.1, 6.4 and 6.5.

Lemma 6.6. *If $(M, \mathcal{S}; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid, and $h' \in \{0, 1, \dots, h\}$ is an integer, then there is a rank-$(a + h')$ restriction M' of M and a set $\mathcal{S}' \subseteq \mathcal{S}$, so that $(M', \mathcal{S}'; e_1, \dots, e_{h'})$ is an (a, q, h', d)-pyramid, and $\epsilon_{M'}(\mathcal{S}') \geq q^{h'}$.*

7. Building a Pyramid

In this section, we show that a large d-scattered set allows us to either find a d-firm subset of large rank in a minor, or a large pyramid. The majority of this argument lies in an ugly technical lemma, which we will adapt into two useful corollaries. To understand this lemma, it may be helpful to read it where $a_0 = 1$ and $a = 2$; in this case, $\mathcal{X}$ is a dense d-scattered set of points; the first outcome corresponds to a d-point line minor whose points are in $\mathcal{X}$, the second to a $(1, q+1, h, d)$-pyramid minor, and the third to a minor containing a d-scattered collection of lines built from $\mathcal{X}$.

Lemma 7.1. *There is an integer-valued function $f_{7.1}(a, d, h, m)$ so that, for all integers a_0, a, d, h, q with $a \geq a_0 \geq 1$, $d \geq 1$, $q \geq 1$, $h \geq 0$, and $m \geq 0$, if M is a matroid with $r(M) \geq f_{7.1}(a, d, h, m)$, and a set $\mathcal{X} \subseteq \mathcal{R}_{a_0}(M)$ is d-scattered in M and satisfies $\epsilon_M(\mathcal{X}) \geq r(M)^{f_{7.1}(a,d,h,m)} q^{r(M)}$, then either:*

- *(i) there is a minor N of M and a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$ so that $r_N(\mathcal{Y}) > a$, and $\epsilon_N(\mathcal{Y}) \geq d^{r(N)-a}$, and $\mathrm{cl}_N(\mathcal{Y})$ is d-thick in N, or*
- *(ii) M has an $(a_0, q+1, h, d)$-pyramid minor P with $\mathcal{S}_P \subseteq \mathcal{X}$, or*
- *(iii) there exists an integer a_1 with $a_0 < a_1 \leq a$, a minor M' of M with $r(M') \geq m$, and a set $\mathcal{X}' \subseteq \mathcal{R}_{a_1}(M')$ so that $\mathcal{X}'$ is d-scattered in M', and $\epsilon_{M'}(\mathcal{X}') \geq r(M')^m q^{r(M')}$.*

Proof. Let a_0, a, d, h and q be positive integers such that $a \geq a_0$, and let $m \geq 0$ be an integer. Let $p_0 = 0$, and for each $h > 0$, recursively define p_h to be an integer so that

$$d^{-1} q^r (r-1)^{p_h - 1} \left(p_h - 3a(1 + d^a)\right) \geq (r-1)^{p_{h-1}} q^{r-1},$$

for all integers $r \geq 2$, and so that $p_h \geq \max(2, d, m+1)$.

We will show for all h that if M is a matroid with $r(M) \geq p_h$, and a set $\mathcal{X} \subseteq \mathcal{R}_{a_0}(M)$ is d-scattered in M and satisfies $\epsilon_M(\mathcal{X}) \geq r(M)^{p_h} q^{r(M)}$, then one of the three outcomes holds for M; thus, setting $f_{7.1}(a, d, h, m) = p_h$ will satisfy the lemma. Our proof is by induction on h. If $h = 0$, then, since $(M, \{X\};)$ is an $(a_0, q+1, 0, d)$-pyramid for any $X \in \mathcal{X}$, the outcome (ii) holds. Now, fix $h > 0$, and suppose that the result holds for smaller h. Let $p = p_h$, and M be minor-minimal so that $r(M) \geq p$, and there exists a d-scattered $\mathcal{X} \subseteq \mathcal{R}_{a_0}(M)$ such that $\epsilon_{M'}(\mathcal{X}) \geq r(M)^p q^{r(M)}$. Let $r = r(M)$. If $r = p$, then $\epsilon_M(\mathcal{X}) \geq p^p q^p > d^{p-a_0}$; this contradicts d-scatteredness of $\mathcal{X}$ by Lemma 5.4, so we may assume that $r > p$.

By Lemma 4.1, there is some $e \in E(M)$ so that $\epsilon_M(\mathcal{X} \cap \mathcal{R}_{a_0}(M/e)) \geq \left(1 - \frac{a_0}{r}\right) \epsilon_M(\mathcal{X})$. Let $\mathcal{X}' = \mathcal{X} \cap \mathcal{R}_{a_0}(M/e)$, and $\mathcal{F}$ be a d-minimal cover of $\mathcal{X}'$ in M/e such that $|\mathcal{F}|$ is maximized. We may assume that all sets in $\mathcal{F}$ are flats of M/e. The set $\mathcal{F}$ is simple in M/e; for each $i \geq 1$, let $\mathcal{F}_i = \mathcal{F} \cap \mathcal{R}_i(M/e)$, noting that each $\mathcal{F}_i$ is d-scattered in M/e by Lemma 5.3. We will henceforth assume that (i) and (iii) do not hold.

7.1.1. $\mathcal{F} = \bigcup_{a_0 \leq i \leq a} \mathcal{F}_i$.

Proof of claim: Every $F \in \mathcal{F}$ must contain a set in $\mathcal{X}'$, so $\mathcal{F}$ contains no set of rank less than a_0. If $\mathcal{F}$ contains a set F of rank greater than a, then $\{F\}$ is a d-minimal cover of $\{X \in \mathcal{X}' : X \subseteq F\}$ in M/e, so by Lemma 5.2, the matroid $(M/e)|F$ and the set $\{X \in \mathcal{X}' : X \subseteq F\}$ satisfy (i), a contradiction. □

7.1.2. *There is a set $\mathcal{X}'' \subseteq \mathcal{X}'$ that is d-scattered in M/e and satisfies $\epsilon_M(\mathcal{X}'') \geq q^r\,(r^p - a(1+d^a)r^{p-1})$*

Proof of claim: Each $X \in \mathcal{X}'$ is contained in some set in $\mathcal{F}$; for each $F \in \mathcal{F}$, let $\mathcal{X}_F = \{X \in \mathcal{X}' : X \subseteq F\}$. By Lemma 5.4, each $F \in \mathcal{F}$ satisfies $\epsilon_M(\mathcal{X}_F) = \epsilon_{M|F}(\mathcal{X}_F) \leq d^{r(M|F)-a_0} \leq d^{a+1-a_0} \leq d^a$. Moreover, each $\mathcal{F}_i$ is simple and d-scattered in M/e, so we may assume that $|\mathcal{F}_i| \leq r^m q^r$ for all $i > a_0$, as (iii) does not hold. Since $\mathcal{X}'$ is the union of the $\mathcal{X}_F$, we have

$$\begin{aligned}\sum_{F \in \mathcal{F}_{a_0}} (\epsilon_M(\mathcal{X}_F)) &\geq \epsilon_M(\mathcal{X}') - \sum_{\substack{a_0 < i \leq a \\ F \in \mathcal{F}_i}} \epsilon_M(\mathcal{X}_F) \\ &\geq (1 - \tfrac{a_0}{r}) r^p q^r - d^a \sum_{a_0 < i \leq a} |\mathcal{F}_i| \\ &\geq (1 - \tfrac{a}{r}) r^p q^r - a d^a r^m q^r \\ &\geq q^r (r^p - a(1+d^a) r^{p-1}),\end{aligned}$$

as $p - 1 \geq m$. Let $\mathcal{X}'' = \bigcup_{F \in \mathcal{F}_{a_0}} \mathcal{X}_F$. Now, since $\mathcal{F}_{a_0}$ is simple in M/e, and every set in $\mathcal{F}_{a_0}$ and every set in $\mathcal{X}'$ has rank a_0 in M/e, no set in $\mathcal{X}''$ is contained in two different sets in F_{a_0}. Therefore $\epsilon_M(\mathcal{X}'') = \sum_{F \in \mathcal{F}_{a_0}} (\epsilon_M(\mathcal{X}_F))$. Moreover, d-minimality of $\mathcal{F}$ implies that $\mathcal{F}_{a_0} = \{\mathrm{cl}_{M/e}(X) : X \in \mathcal{X}''\}$ is a d-minimal cover of $\mathcal{X}''$ in M/e. Therefore, $\mathcal{X}''$ is d-scattered in M/e, giving the claim. □

Let $\mathcal{Y}$ be a maximal subset of $\mathcal{X}''$ that is simple in M. Since $\mathcal{X}''$ is d-scattered in both M and M/e, so is $\mathcal{Y}$. We have $r(M/e) = r - 1 \geq p$, so minor-minimality of M gives $|\mathcal{Y}| = \epsilon_{M/e}(\mathcal{Y}) < (r(M/e))^p q^{r(M/e)} = (r-1)^p q^{r-1}$. Let $\mathcal{Y}_{>q} = \{Y \in \mathcal{Y} : |[Y]_{M/e} \cap \mathcal{Y}| > q\}$, and $\mathcal{Y}_{\leq q} = \mathcal{Y} - \mathcal{Y}_{>q}$. Since $\mathcal{Y}$ is d-scattered and simple in M, Lemma 5.4 gives $|[Y]_{M/e} \cap \mathcal{Y}| = |\{Y' \in \mathcal{Y} : Y' \subseteq \mathrm{cl}_M(Y \cup \{e\})\}| \leq d^{(a_0+1)-a_0} = d$ for all

$Y \in \mathcal{Y}$. Now,

$$\begin{aligned} q^r(r^p - a(1+d^a)r^{p-1}) &\leq |\mathcal{Y}| \\ &= |\mathcal{Y}_{>q}| + |\mathcal{Y}_{\leq q}| \\ &\leq d\epsilon_{M/e}(\mathcal{Y}_{>q}) + q\epsilon_{M/e}(\mathcal{Y}_{\leq q}) \\ &\leq d\epsilon_{M/e}(\mathcal{Y}_{>q}) + q\epsilon_{M/e}(\mathcal{Y}) \\ &< d\epsilon_{M/e}(\mathcal{Y}_{>q}) + q(r-1)^p q^{r-1}. \end{aligned}$$

Rearranging this inequality yields

$$\begin{aligned} \epsilon_{M/e}(\mathcal{Y}_{>q}) &\geq d^{-1}q^r(r^p - (r-1)^p - a(1+d^a)r^{p-1}) \\ &\geq d^{-1}q^r(p(r-1)^{p-1} - a(1+d^a)r^{p-1}) \\ &= d^{-1}q^r(r-1)^{p-1}\left(p - a(1+d^a)\left(\tfrac{r}{r-1}\right)^{p-1}\right). \end{aligned}$$

By hypothesis, $r \geq p$, so $\left(\frac{r}{r-1}\right)^{p-1} \leq \left(\frac{p}{p-1}\right)^{p-1} \leq 2.718\ldots < 3$. This gives

$$\begin{aligned} \epsilon_{M/e}(\mathcal{Y}_{>q}) &> d^{-1}q^r(r-1)^{p-1}\left(p - 3a(1+d^a)\right) \\ &\geq r(M/e)^{p_{h-1}}q^{r(M/e)} \end{aligned}$$

by definition of $p = p_h$. We may assume that (i) and (iii) both fail for M/e and $\mathcal{Y}_{>q}$; thus, by induction on h, the matroid M/e has an $(a_0, q+1, h-1, d)$-pyramid minor P' with $\mathcal{S}_{P'} \subseteq \mathcal{Y}_{>q}$. By Lemma 6.3, M has an $(a_0, q+1, h, d)$-pyramid minor P with $\mathcal{S}_P \subseteq \mathcal{Y}_{>q} \subseteq \mathcal{X}$, which gives (ii). □

Our first corollary, which will be used in the next section, finds a pyramid or a firm set of rank greater than a, starting with a collection of thick rank-a sets. The corollary is obtained by specialising to the case where $a = a_0$, thus rendering the third outcome impossible.

Corollary 7.2. *There is an integer-valued function $f_{7.2}(a, d, h)$ so that, for any integers a, d, h, q with $h \geq 0$, $a \geq 1$, $d \geq 2$ and $q \geq 1$, if M is a matroid such that $r(M) \geq f_{7.2}(a, d, h)$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is a set such that every $X \in \mathcal{X}$ is d-thick in M, and $\epsilon_M(\mathcal{X}) \geq r(M)^{f_{7.2}(a,d,h)}q^{r(M)}$, then either*

(i) there is a minor N of M, and a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$ so that $r_N(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in N, or
(ii) M has an $(a, q+1, h, d)$-pyramid minor P, with $\mathcal{S}_P \subseteq \mathcal{X}$.

Proof. Let a, d, h, q be integers with $h \geq 0$, $a \geq 1$, $d \geq 2$ and $q \geq 1$. Set $f_{7.2}(a, d, h) = f_{7.1}(a, d, h, 0)$. Let M be a matroid such that $r(M) \geq$

$f_{7.2}(a,d,h)$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ be a set such that every $x \in \mathcal{X}$ is d-thick in M, and $\epsilon_M(\mathcal{X}) \ge r(M)^{f_{7.2}(a,d,h)} q^{r(M)}$. We consider two cases:

Case 1: $\mathcal{X}$ is d-scattered in M.

By definition of $f_{7.2}$, we can apply Lemma 7.1 to $\mathcal{X}$. Since there is no integer a_1 with $a < a_1 \le a$, we know that 7.1(iii) cannot hold. If 7.1(ii) holds, then we have our result. We may thus assume that 7.1(i) holds; now, outcome (i) follows from Lemma 3.4.

Case 2: $\mathcal{X}$ is not d-scattered in M.

By definition, $\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}$ is not a d-minimal cover of $\mathcal{X}$ in M, so any d-minimal cover of $\mathcal{X}$ contains a set F of rank greater than a. Let $\mathcal{X}_F = \{X \in \mathcal{X} : X \subseteq F\}$. The cover $\{F\}$ must be a d-minimal cover of $\mathcal{X}_F$, so by Lemma 5.2 applied to $M|F$ and $\mathcal{X}_F$, we have $\epsilon_M(\mathcal{X}_F) \ge d^{r(M|F)-a}$. Again, outcome (i) follows from Lemma 3.4. □

The second corollary essentially reduces Theorem 1.3 to the case where M is a pyramid:

Corollary 7.3. *There is an integer-valued function $f_{7.3}(a,b,d,h)$ so that, for any integers a,b,d,h,q with $q \ge 1$, $d \ge 2$, $h \ge 0$, and $1 \le a < b$, if $M \in \mathcal{U}(a,b)$ satisfies $r(M) > 1$ and $\tau_a(M) \ge r(M)^{f_{7.3}(a,b,d,h)} q^{r(M)}$, then there is some $a_0 \in \{1,\dots,a\}$ such that M has an $(a_0, q+1, h, d)$-pyramid minor.*

Proof. Let a,b,d,h,q be integers with $q \ge 1$, $d \ge 2$, $h \ge 0$ and $1 \le a < b$. Let $d' = \max(d, \binom{b}{a})$. We define a sequence of integers $p_{a+1},\dots,p_1$; let $p_{a+1} = 0$, and for each $1 \le i \le a$, recursively set $p_i = \max(p_{i+1}, f_{7.1}(a,d',h,p_{i+1}))$. Note that $p_1 \ge p_2 \ge \dots \ge p_{a+1}$. Set $f_{7.3}(a,b,h,d)$ to be an integer $p \ge p_1$ so that $a^{-1}(d')^{-a} r^p \ge r^{p_1}$ for all integers $r \ge p_1$. Let M be a matroid with $r(M) \ge p$, and $\tau_a(M) \ge r(M)^p q^{r(M)}$.

7.3.1. *Let $1 \le i \le a$. If $r(M) \ge p_i$, and $\mathcal{X} \subseteq \mathcal{R}_i(M)$ is d'-scattered in M and satisfies $\epsilon_M(\mathcal{X}) \ge r(M)^{p_i} q^{r(M)}$, then M has an $(a_0, q+1, h, d)$-pyramid minor for some $i \le a_0 \le a$.*

Proof of claim: By definition of p_i, we can apply Lemma 7.1 to $\mathcal{X}$ in M. If 7.1(i) holds, then M has a d'-thick minor of rank greater than a. Since $d' \ge \binom{b}{a}$, this contradicts $M \in \mathcal{U}(a,b)$ by Lemma 3.2. Since $d' \ge d$, 7.1(ii) gives the claim, so we may assume that 7.1(iii) holds. If $i = a$, this is impossible, so the claim is proven. Otherwise, we have the hypotheses for a minor of M and some larger $i \le a$, so the claim holds by induction. □

Let $\mathcal{F}$ be a d'-minimal cover of M. Clearly $\mathcal{F}$ is simple. By Lemma 5.5, we have $\mathrm{wt}^{d'}_M(\mathcal{F}) \ge \tau_a(M)$, and every set in $\mathcal{F}$ has rank at most a, so

$\epsilon_M(\mathcal{F}) = |\mathcal{F}| \geq (d')^{-a}\,\mathrm{wt}^{d'}_M(\mathcal{F}) \geq (d')^{-a} r(M)^p q^{r(M)}$. For each $1 \leq i \leq a$, let $\mathcal{F}_i = \mathcal{F} \cap \mathcal{R}_i(M)$. By a majority argument, some $1 \leq i \leq a$ satisfies $\epsilon_M(\mathcal{F}_i) = |\mathcal{F}_i| \geq a^{-1}|\mathcal{F}| \geq a^{-1}(d')^{-a} r(M)^p q^{r(M)} \geq r(M)^{p_1} q^{r(M)} \geq r(M)^{p_i} q^{r(M)}$. The set $\mathcal{F}_i$ is d'-scattered in M by Lemma 5.3, and $r(M) \geq p \geq p_i$, so the result follows from the claim. □

8. Finding Firmness

This section explores what can be done with a large collection $\mathcal{X}$ of thick rank-a sets in a matroid M with no large projective geometry as a minor. We prove a single lemma, which finds a large subcollection of $\mathcal{X}$ that is firm in a minor of M. When $a = 1$, this is equivalent to finding a large rank-2 uniform minor, and thus Theorems 2.4 and 2.5 appear in the base case of this lemma.

Lemma 8.1. *There is an integer-valued function $f_{8.1}(a, d, n, q)$ so that, for any positive integers a, d, n, q, if M is a matroid with $r(M) \geq f_{8.1}(a, d, n, q)$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is a set so that every $X \in \mathcal{X}$ is $f_{8.1}(n, q, a, d)$-thick in M and $\epsilon_M(\mathcal{X}) \geq r(M)^{f_{8.1}(a,d,n,q)} q^{r(M)}$, then either*

(i) M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or
(ii) there is a minor N of M, and a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$ so that $r_N(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in N.

Proof. Let n, q, d be positive integers. Set

$$f_{8.1}(1, d, n, q) = \max(2, f_{2.4}(d, n), \lceil \alpha_{2.5}(d, n, q) \rceil).$$

We now define $f_{8.1}(a, d, n, q)$ for general a recursively; for each $a > 1$, suppose that $f_{8.1}(a-1, d, n, q)$ has been defined. Let h be an integer so that

$$(3a)^{-1} d^{-3a} (q+1)^h \geq (h+a)^{f_{8.1}(a-1,d,n,q)} q^{h+a}.$$

Let $s = d^{h-a}$, and let h' be an integer so that

$$(as)^{-1} d^{-as} (q+1)^{h'} \geq (h'+a)^{f_{8.1}(a-1,d,n,q)} q^{h'+a};$$

Set $f_{8.1}(a, d, n, q) = \max(s+1, f_{7.2}(a, d', h+h'))$.

Let $a \geq 1$ be an integer, M be a matroid with $r(M) \geq f_{8.1}(a, d, n, q)$, and let $\mathcal{X} \subseteq \mathcal{R}_a(M)$ be a set whose elements are all $f_{8.1}(a, d, n, q)$-thick in M, satisfying $\epsilon_M(\mathcal{X}) \geq r(M)^{f_{8.1}(a,d,n,q)}$. We may assume that $M = M|\mathcal{X}$; we show that M satisfies (i) or (ii), first resolving the case where $a = 1$, and proceeding by induction on a.

8.1.1. *If $a = 1$, then M satisfies (i) or (ii).*

Proof of claim: Every $X \in \mathcal{X}$ is a rank-1 set, and therefore $\tau_1(M) \geq r(M)^{f_{8.1}(1,d,n,q)} q^{r(M)}$.

If $q = 1$, then $r(M)^{f_{8.1}(1,d,n,q)} \geq r(M)^{f_{2.4}(d,n)}$, so if (i) does not hold, then M has a $U_{2,d}$-minor by Theorem 2.4. This minor corresponds to a simple subset of $\mathcal{X}$ in a rank-2 minor of M, containing d pairwise dissimilar rank-1 sets. This is a rank-2, d-firm subset of $\mathcal{X}$ in a minor of M, giving (ii).

If $q > 1$, then $\tau_1(M) \geq f_{8.1}(1,d,n,q)q^{r(M)} \geq \alpha_{2.5}(d,n,q)q^{r(M)}$, so the result follows from Theorem 2.5 in a similar way to the $q = 1$ case. $\square$

Now, assume inductively that $a > 1$, and that $f_{8.1}(a', d', n', q')$ as defined satisfies the lemma for all $a' < a$, and all integers $d', q', n' \geq 1$. Suppose further that (i) does not hold for M.

8.1.2. *M has an $(a, q+1, h+h', d')$-pyramid minor P so that $\mathcal{S}_P \subseteq \mathcal{X}$.*

Proof of claim: By definition of $f_{8.1}(a,d,n,q)$, we know that $r(M') \geq f_{7.2}(a,d,h+h')$, $\epsilon_{M'}(\mathcal{X}) \geq r(M)^{f_{7.2}(a,d,h+h')}q^{r(M)}$, and all sets in $\mathcal{X}$ are d'-thick in M; we can therefore apply Corollary 7.2 to M. Since $d' \geq d$, outcome 7.2(i) does not hold, giving 7.2(ii) and hence the claim. $\square$

Let $P = (M', \mathcal{S}; e_1, \dots, e_{h+h'})$. By Lemma 6.6, we may assume that $r(M') = h' + h + a$. Let $J = \{e_1, \dots, e_h\}$. By Lemma 6.1, $(M'/J, \mathcal{S}; e_{h+1}, \dots, e_{h+h'})$ is an $(a, q{+}1, h', d)$-pyramid, so by Lemma 6.5, there is a set $\mathcal{S}' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| \geq (q+1)^{h'}$, and $\mathcal{S}'$ is simple in M'/J.

8.1.3. *There is a set $\mathcal{W} \subseteq \mathcal{S}'$ so that $|\mathcal{W}| = s$, and $\mathcal{W}$ is mutually skew in M'/J.*

Proof of claim: Suppose there is no such $\mathcal{W}$. By Lemma 4.2, there a minor N of M'/J such that $r(N) \geq r(M'/J) - as$, a set $\mathcal{Y} \subseteq \mathcal{S}' \cap \mathcal{R}_a(N)$ such that $|\mathcal{Y}| \geq (as)^{-1}|\mathcal{S}'|$, and some nonloop e of N so that $e \in \mathrm{cl}_N(Y)$ for all $Y \in \mathcal{Y}$. We will apply the inductive hypothesis on a to N/e.

The set $\mathcal{Y} \subseteq \mathcal{S}'$ is simple in M/J, so by Lemma 3.5, we have

$$\begin{aligned}\epsilon_N(Y) &\geq d^{r(N)-r(M'/J)}\epsilon_{M'/J}(\mathcal{Y}) \geq d^{-as}|\mathcal{Y}| \geq (as)^{-1}d^{-as}|\mathcal{S}'| \\ &\geq (as)^{-1}d^{-as}(q+1)^{h'} \geq (h'+a)^{f_{8.1}(n,q,a-1,d)}q^{h'+a}.\end{aligned}$$

Since $r(N/e) < r(N) \leq r(M'/J) = a + h'$, this gives

$$\epsilon_N(\mathcal{Y}) \geq r(N/e)^{f_{8.1}(n,q,a-1,d)}q^{r(N/e)}.$$

Because $e \in \mathrm{cl}_N(Y)$ for all $Y \in \mathcal{Y}$, we also have $\mathcal{Y} \subseteq \mathcal{R}_{a-1}(N/e)$, and $\epsilon_{N/e}(\mathcal{Y}) = \epsilon_N(\mathcal{Y})$. Moreover, $r(N/e) \geq r(M'/J) - as - 1 \geq a + h' - as - 1 \geq f_{8.1}(n,q,a-1,d)$, so by the inductive hypothesis, there is a minor N' of N/e, and a set $\mathcal{Y}' \subseteq \mathcal{Y} \cap \mathcal{R}_{a-1}(N')$ such that $r_{N'}(\mathcal{Y}') \geq a$, and $\mathcal{Y}'$

is d-firm in N'. If $N' = N/(C \cup \{e\}) \setminus D$, where $C \cup \{e\}$ is independent in N, then it is straightforward to check that $\mathcal{Y}' \subseteq \mathcal{R}_a(N/C)$, and $r_{N/C}(\mathcal{Y}') > a$, and $\mathcal{Y}'$ is d-firm in N/C. This gives (ii).

□

Let $\mathcal{W} = \{W_1, \dots, W_s\}$, and for each $i \in \{1, \dots, s\}$, let $\mathcal{S}_i = \{S \in \mathcal{S} : S \equiv_{M'/J} W_i\}$. By Lemma 6.4, there is, for each $i \in \{1, \dots, s\}$, a rank-$(a+h)$ restriction M_i of M' such that $(M_i, \mathcal{S}_i; e_1, \dots, e_h)$ is an $(a, q+1, h, d')$-pyramid.

8.1.4. *For each $i \in \{1, \dots, s\}$, there are distinct sets $V_i, Z_i, Z_i' \in \mathcal{S}_i$ such that $\{V_i, Z_i, Z_i'\}$ is mutually skew in M_i.*

Proof of claim: By Lemma 6.5, $\mathcal{S}_i$ has a subset $\mathcal{S}'$ of size $(q+1)^h$ that is simple in M_i. If there is a subset of $\mathcal{S}_i'$ of size 3 that is skew in M_i, then the claim follows. Otherwise, by Lemma 4.2, there is a minor N_i of M_i, with $r(N_i) \ge r(M_i) - 3a$, a set $\mathcal{Y} \subseteq \mathcal{S}_i' \cap \mathcal{R}_a(N_i)$ such that $|\mathcal{Y}| \ge (3a)^{-1} d^{-3a} |\mathcal{S}_i'|$, and a nonloop e of N_i so that $e \in \operatorname{cl}_{N_i}(Y)$ for all $Y \in \mathcal{Y}$. The proof is now very similar to that of the previous claim, following from the definition of h. □

Let $\mathcal{V} = \{V_1, \dots, V_s\}$. Since $V_i \equiv_{M/J} W_i$ for each i, the set $\mathcal{V}$ is mutually skew in M'/J. This last claim uses Z_i and Z_i' to contract the elements of $\mathcal{V}$, one by one, into the span of J without reducing their rank, while maintaining the 'skewness' and structure of the elements of $\mathcal{V}$ not yet contracted:

8.1.5. *For each $i \in \{0, \dots, s\}$, there is a minor N_i of M such that*

(a) $\{V_{i+1}, \dots, V_s\}$ is mutually skew in N_i/J,
(b) $N_i|E(M_j) = M_j$ for each $j \in \{i+1, \dots, s\}$, and
(c) $\{V_1, \dots, V_i\} \subseteq \mathcal{R}_a(N_i|\operatorname{cl}_{N_i}(J))$, and $\{V_1, \dots, V_i\}$ is simple in N_i.

Proof of claim: When $i = 0$, the claim is clear, with $N_0 = M'$. Suppose inductively that $1 \le i \le s$, and that the claim holds for smaller i. We will construct N_i by contracting a rank-a set of $N_{i-1}|E(M_i)$. By definition, Z_i' and V_i are similar to W_i in M_i/J, so $r_{M_i/J}(Z_i') = r_{M_i/J}(V_i) = a$; Let $I \subseteq Z_i'$ be an independent set of size $(a-1)$ in M_i/J. So $\{V_i, Z_i\}$ is a skew pair of rank-a sets in M_i/I, and $r(M_i/I) = h + a - (a-1) = h+1$. Since I is independent in M_i/J, it is skew to J in M_i, so $r_{M_i/I}(J) = h$. Moreover, $r_{M_i/(J \cup I)}(V_i) = r_{M_i/(J \cup I)}(Z_i) = r_{M_i/(J \cup I)}(Z_i') = 1$, so neither Z_i nor V_i is contained in $\operatorname{cl}_{M_i/I}(J)$.

By the inductive hypothesis, $(N_{i-1}/I)|E(M_i) = M_i/I$, so we can extend the observations just made about M_i/I to apply in N_{i-1}/I. Therefore, in the matroid N_{i-1}/I, $\{V_i, Z_i\}$ is a skew pair of rank-a sets, each contained in the rank-$(h+1)$ set $E(M_i)$, which itself contains the rank-h set J, and $\mathrm{cl}_{N_{i-1}/I}(J)$ does not contain Z_i or V_i.

For each $1 \le k < i$, let $F_k = \varnothing$ if $r_{N_{i-1}/I}(V_k \cup V_i) > a+1$, and $F_k = \mathrm{cl}_{N_{i-1}/I}(V_k \cup V_i)$ otherwise. Since V_i and Z_i are skew sets of rank $a > 1$ in N_{i-1}/I, and F_k is a flat of rank at most $a+1$ containing V_i, it follows that $Z_i \not\subseteq F_k$, so $r_{N_{i-1}/I}(F_k \cap Z_i) < a$. Also, the set $\mathrm{cl}_{N_{i-1}/I}(J)$ does not contain Z_i. The set Z_i is $(d' \ge s+1)$-thick in N_{i-1}/I, and there are at most $s-1$ possible k, so there is some $f \in Z_i$ that is not in any of the sets F_k, and not in $\mathrm{cl}_{N_{i-1}/I}(J)$. Set $N_i = N_{i-1}/(I \cup \{f\})$. By choice of f, we have $r_{N_i}(J) = h = r_{N_{i-1}}(J)$, so $I \cup \{f\}$ is skew to J in N_{i-1}; we now show that N_i satisfies (a), (b) and (c).

(a) We have $I \cup \{f\} \subseteq Z_i \cup Z_i'$. The sets Z_i and Z_i' are both similar to V_i in $M_i/J = (N_{i-1}/J)|E(M_i)$, so $I \cup \{f\} \subseteq \mathrm{cl}_{N_{i-1}/J}(V_i)$. $\{V_i, \dots, V_s\}$ is skew in N_{i-1}/J by the inductive hypothesis, so $\{V_{i+1}, \dots, V_s\}$ is skew in $N_{i-1}/(J \cup I \cup \{f\}) = N_i/J$.

(b) Let $i < j \le s$. Since $(M_j, \mathcal{S}_j; e_1, \dots, e_h)$ is an $(a, q+1, h, d)$-pyramid and $V_j \in \mathcal{S}_j$, the set $J \cup V_j$ is spanning in M_j, and J is skew to V_j in M_j. As we saw in (a), the set $I \cup \{f\}$ is skew to J in N_{i-1}, and is skew to V_j in N_{i-1}/J. Now, $M_j = N_{i-1}|E(M_j)$ and $M_i = N_{i-1}|E(M_i)$, so

$$\begin{aligned} N_{i-1}((I \cup \{f\}) \cup (J \cup V_j)) &= r_{N_{i-1}/J}(I \cup \{f\} \cup V_j) + r_{N_{i-1}}(J) \\ &= r_{N_{i-1}/J}(I \cup \{f\}) + r_{N_{i-1}/J}(V_j) + r_{N_{i-1}}(J) \\ &= r_{N_{i-1}}(I \cup \{f\}) + r_{N_{i-1}}(V_j) + r_{N_{i-1}}(J) \\ &= r_{N_{i-1}}(I \cup \{f\}) + r_{N_{i-1}}(V_j \cup J). \end{aligned}$$

Therefore, $I \cup \{f\}$ and $V_j \cup J$ are skew in N_{i-1}. Since $V_j \cup J$ is spanning in M_j, this gives $N_i|E(M_j) = N_{i-1}|E(M_j) = M_j$.

(c) Since $I \cup \{f\}$ is skew to J in N_{i-1}, it is clear that $\{V_1, \dots, V_{i-1}\} \subseteq \mathcal{R}_a(N_i|\,\mathrm{cl}_{N_i}(J))$ and that $\{V_1, \dots, V_{i-1}\}$ is simple in N_i. Moreover, V_i is a rank-a set that is skew to $Z_i \cup Z_i'$ in N_{i-1}, and therefore is skew to $I \cup \{f\}$, so $r_{N_i}(V_i) = a$. It therefore remains to show that V_i is not similar in N_i to any of $V_1, \dots, V_{i-1}$.

Suppose for a contradiction that $V_i \equiv_{N_i} V_k$ for some $1 \le k < i$. Either V_i and V_k are similar in N_{i-1}/I, or V_i and V_k lie in a common rank-$(a+1)$ flat F of N_{i-1}/I, and contracting $f \in F$ makes the two sets similar in N_i. In the first case, this gives $0 = r_{N_{i-1}/(I \cup V_k)}(V_i) \ge r_{N_{i-1}/(I \cup J)}(V_i) = r_{N_{i-1}/J}(V_i) - r_{N_{i-1}}(I) = a - (a-1) = 1$, a contradiction. In the second case, we have $f \in \mathrm{cl}_{N_{i-1}/I}(V_i \cup V_k)$, which does not occur by choice of f.

□

Now, let $N = N_s|\operatorname{cl}_{N_s}(J)$. We have $r(N) \le h$, and $\mathcal{V}$ is a simple subset of $\mathcal{R}_a(N)$ by construction, so $\epsilon_N(\mathcal{V}) = |\mathcal{V}| = s = d^{h-a}$. Also, $\mathcal{V} \subseteq \mathcal{X}$ and $d' \ge d$, so every $V \in \mathcal{V}$ is d-thick in N. (ii) now follows by applying Lemma 3.4 to $\mathcal{V}$ in N. □

9. Upgrading a Pyramid

The goal of this section is to prove that a sufficiently large pyramid minor will be enough to prove Theorem 1.3. We show that for very large h and d, an $(a_0, q+1, h, d)$-pyramid will either contain a thick set of rank greater than a, or a large projective geometry over $\mathrm{GF}(q')$ for some $q' > q$. We first prove this when $a_0 = a$, and then show that, for $a_0 < a$, we can find a large pyramid as a minor with a larger a_0, thereby 'upgrading' our pyramid.

An important intermediate object is an $(a_0, q+1, \cdot, \cdot)$-pyramid P 'on top of' a very firm set $\mathcal{X} \subseteq \mathcal{S}_P$ with rank greater than a_0. We construct such objects using the results in the previous section; this is the reason that we need to exclude a projective geometry.

We upgrade a pyramid of height h on top of a firm set by 'lifting' the firm set one level up the pyramid h times, sacrificing a large amount of firmness at each step. Our next two lemmas give the machinery needed for this; the first simply lifts a firm set up a pyramid of height 1:

Lemma 9.1. *Let a_0, a, q, d, d' be integers with $1 \le a_0 \le a$, $d, d' \ge 2$, and $q \ge 2$. If $(M, \mathcal{S}; e)$ is an $(a_0, q, 1, d')$-pyramid, and $\mathcal{X} \subseteq \mathcal{S}$ is d^{q+2}-firm in M/e and satisfies $r_{M/e}(\mathcal{X}) = a$, then either*

(i) there exists $\mathcal{Y} \subseteq \mathcal{S}$ so that $r_M(\mathcal{Y}) = a+1$ and $\mathcal{Y}$ is d-firm in M, or

(ii) there exist sets $\mathcal{X}_1, \dots, \mathcal{X}_q \subseteq \mathcal{S}$ such that
- *each $\mathcal{X}_i$ is d-firm in M, and $r_M(\mathcal{X}_i) = a$, and*
- *the $\mathcal{X}_i$ are pairwise dissimilar in M, and each is skew to $\{e\}$ in M, and similar to $\mathcal{X}$ in M/e.*

Proof. We may assume that $\mathcal{X}$ is spanning in M/e, so $r(M) = a+1$. Suppose that the first outcome does not hold. Let I be an indexing set for X (i.e. let $\mathcal{X} = \{X^i : i \in I\}$, with $|I| = |\mathcal{X}|$). For each $i \in I$, let $X^i_1, \dots, X^i_q$ be pairwise dissimilar sets in $\mathcal{S}$, each similar to X^i in M, as given by the definition of a pyramid.

9.1.1. *There are sets $\mathcal{X}_1, \dots, \mathcal{X}_q \subseteq \mathcal{S}$ and $I_1, \dots, I_q \subseteq I$ such that the following conditions hold:*
- *for each $j \in \{1, \dots, q\}$, the set I_j is the indexing set for $\mathcal{X}_j$ in I (i.e. $\mathcal{X}_j = \{X^i_j : i \in I_j\}$),*
- *$I \supseteq I_1 \supseteq I_2 \supseteq \dots \supseteq I_q$, and*

- *each $j \in \{1,\dots,q\}$ satisfies $|\mathcal{X}_j| \ge d^{-j}|\mathcal{X}|$ and $r_M(\mathcal{X}_j) \le a$.*

Proof of claim: We construct the sets in question by induction on j. Suppose that $1 \le j < q$, and that the sets $\mathcal{X}_1,\dots,\mathcal{X}_{j-1}$ and $I_1,\dots,I_{j-1}$ have been defined to satisfy the conditions. Let $I_0 = I$, and $\mathcal{X}_0 = \mathcal{X}$; note that $|\mathcal{X}_0| \ge d^0|\mathcal{X}|$. As (i) does not hold, the set $\{\mathcal{X}^i_j : i \in I_{j-1}\}$ is not a rank-$(a+1)$, d-firm set in M, so we may assume that there is some $\mathcal{X}_j \subseteq \{X^j_i : I \in I_{j-1}\}$ such that $|\mathcal{X}_j| \ge d^{-1}|\{\mathcal{X}^i_j : i \in I_{j-1}\}|$ and $r_M(\mathcal{X}_j) \le a$. Now, $|\mathcal{X}_j| \ge d^{-1}|\{\mathcal{X}^i_j : i \in I_{j-1}\}| = d^{-1}|I_{j-1}| = d^{-1}|\mathcal{X}_{j-1}| \ge d^{-j}|\mathcal{X}|$. The set $\mathcal{X}_j$, along with $I_j = \{i \in I_{j-1} : X^i_j \in \mathcal{X}_j\}$, satisfies the required conditions. □

9.1.2. *For each $j \in \{1,\dots,q\}$, the set $\mathcal{X}_j$ is d-firm in M, and $r_M(\mathcal{X}_j) = r_{M/e}(\mathcal{X}_j) = a$.*

Proof of claim. We know that $r_M(\mathcal{X}_j) \le a$; let $\mathcal{X}'_j \subseteq \mathcal{X}_j$ satisfy $|\mathcal{X}'_j| \ge d^{-1}|\mathcal{X}_j|$, and let $I'_j = \{i \in I_j : X^i_j \in \mathcal{X}'_j\}$. Let $\mathcal{X}' = \{X^i : i \in I'_j\}$. By definition of $\mathcal{X}$ and $\mathcal{X}_i$, each set in $\mathcal{X}'$ is similar in M/e to a set in $\mathcal{X}'_j$, and vice versa. We therefore have $|\mathcal{X}'| = |\mathcal{X}'_j|$ and $r_{M/e}(\mathcal{X}') = r_{M/e}(\mathcal{X}'_j)$. Now, $|\mathcal{X}'| = |\mathcal{X}'_j| \ge d^{-1}|\mathcal{X}_j| > d^{-(q+2)}|\mathcal{X}|$, and $\mathcal{X}' \subseteq \mathcal{X}$, so d^{q+2}-firmness of $\mathcal{X}$ gives $r_{M/e}(\mathcal{X}') = r_{M/e}(\mathcal{X}) = a$. Therefore,

$$a \ge r_M(\mathcal{X}_j) \ge r_M(\mathcal{X}'_j) \ge r_{M/e}(\mathcal{X}'_j) = r_{M/e}(\mathcal{X}') = r_{M/e}(\mathcal{X}) = a,$$

and the lemma follows from definition of firmness, and the fact that $r_{M/e}(\mathcal{X}_j) \ge r_{M/e}(\mathcal{X}'_j)$. □

9.1.3. *The sets $\mathcal{X}_j : j \in \{1,\dots,q\}$ are pairwise dissimilar in M.*

Proof of claim: Suppose not; let $\mathcal{X}_j$ and $\mathcal{X}_{j'}$ be similar in M, where $1 \le j < j' \le q$. By 9.1.2, $r_M(\mathcal{X}_j \cup \mathcal{X}_{j'}) = r_M(\mathcal{X}_j) = a$. Let $i \in I_{j'}$. We have $X^i_{j'} \in \mathcal{X}_{j'}$ by definition, and $I_{j'} \subseteq I_j$, so $i \in I_j$ and $X^i_j \in \mathcal{X}_j$. But X^i_j and $X^i_{j'}$ are dissimilar rank-a_0 sets in M, each similar to the rank-a_0 set X^i in M/e. Therefore, $e \in \mathrm{cl}_M(X^i_j \cup X^i_{j'})$, and so $e \in \mathrm{cl}_M(\mathcal{X}_j \cup \mathcal{X}'_j) = \mathrm{cl}_M(\mathcal{X}_j)$. This contradicts the previous claim. □

By assumption, the set $\mathcal{X}$ is spanning in the rank-a matroid M/e, and by the second part of 9.1.2, the set $\mathcal{X}_j$ is also spanning in M/e, so $\mathcal{X}_j \equiv_{M/e} \mathcal{X}$. By the claims above, (ii) follows. □

The next lemma iterates the previous one h times to upgrade a pyramid completely - here, a_0 is upgraded to a_1 in the second outcome:

Lemma 9.2. *Let a_0, a_1, q and d be integers with $1 \le a_0 \le a_1$ and $d, q \ge 2$, and let $(M, \mathcal{S}; e_1, \dots, e_h)$ be an (a_0, q, h, d)-pyramid. For each $0 \le i \le h$, let $M_i = M/\{e_1,\dots,e_i\}$. If $\mathcal{X} \subseteq \mathcal{S}$ is a set so that $r_{M_h}(\mathcal{X}) = a_1$ and $\mathcal{X}$ is $d^{(q+2)^h}$-firm in M_h, then either*

(i) there is an integer $1 \le i \le h$, and a set $\mathcal{Y} \subseteq \mathcal{S}$ so that $\mathcal{Y}$ is d-firm in M_i, and $r_{M_i}(\mathcal{Y}) > a_1$, or
(ii) there is a set $\mathcal{T}$ so that $(M, \mathcal{T}; e_1, \dots, e_h)$ is an (a_1, q, h, d)-pyramid.

Proof. Assume that (i) does not hold; we will build a pyramid-like structure inductively.

9.2.1. *For each $0 \le i \le h$, there exists a nonempty set $\mathfrak{X}^i$ of subsets of $\mathcal{S}$ satisfying the following:*

- *$\mathrm{cl}_M(\mathcal{X})$ is skew to $\{e_{i+1}, \dots, e_h\}$ in M_i for all $\mathcal{X} \in \mathfrak{X}^i$,*
- *For all $\mathcal{X} \in \mathfrak{X}^i$ and i' such that $i \le i' < h$, there exist sets $\mathcal{X}_1, \dots, \mathcal{X}_q \in \mathfrak{X}^i$, pairwise dissimilar in $M_{i'}$, and each similar to $\mathcal{X}$ in $M_{i'+1}$, and*
- *For all $\mathcal{X} \in \mathfrak{X}^i$, we have $r_{M_i}(\mathcal{X}) = a$, and $\mathcal{X}$ is $d^{(q+1)^i}$-firm in M_i.*

Proof of claim: Let $\mathfrak{X}^h = \{\mathcal{X}\}$. It is clear that $\mathfrak{X}^h$ satisfies all three conditions. Fix $0 \le i < h$, and suppose that $\mathfrak{X}^{i+1}$ has been defined to satisfy the conditions. Let $\mathcal{X} \in \mathfrak{X}^{i+1}$. We know that $(M_i, \mathcal{S}; e_{i+1})$ is an $(a_0, q, 1, d)$-pyramid; by the inductive hypothesis, the set $\mathcal{X}$ satisfies the hypotheses of Lemma 9.1 for this pyramid, and $d^{(q+2)^i}$. If 9.1(i) holds, then so does outcome (i) of the current lemma, as $d^{(q+2)^i} \ge d$. Otherwise, let $P(\mathcal{X}) = \{\mathcal{X}_1, \dots, \mathcal{X}_q\}$, where $\mathcal{X}_1, \dots, \mathcal{X}_q$ are the sets given by 9.1(ii). Now, $\mathfrak{X}^i = \bigcup_{\mathcal{X} \in \mathfrak{X}^{i+1}} P(\mathcal{X})$ will satisfy the claim, which now follows inductively. □

Let $\mathcal{T} = \{\mathrm{cl}_M(\mathcal{X}) : \mathcal{X} \in \mathfrak{X}^0\}$. Each set in $\mathfrak{X}^0$ is d-firm in M, so all sets in $\mathcal{T}$ are d-thick by Lemma 3.3. It is now clear from the claim that $(M, \mathcal{T}; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid. □

Having seen that a pyramid on top of a firm set is a useful object, we now show that such an object can be constructed by Lemma 8.1 by excluding a projective geometry.

Lemma 9.3. *There is an integer-valued function $f_{9.3}(a_0, d, n, q, h)$ so that, for any integers a_0, d, n, q, d', h' with $a_0, d, n, q \ge 1$ and $\min(d', h') \ge f_{9.3}(a_0, d, n, q, h)$, if P is an $(a_0, q+1, h', d')$-pyramid on a matroid M, then either*

(i) M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or
(ii) there is a minor M' of M, an $(a_0, q+1, h, d)$-pyramid

$$(M', \mathcal{S}'; e_1, \dots, e_h)$$

such that $\mathcal{S}' \subseteq \mathcal{S}_P$, and a set $\mathcal{Y} \subseteq \mathcal{S}'$ such that $\mathcal{Y}$ is d-firm in $M'/\{e_1, \dots, e_h\}$ and $r_{M'/\{e_1,\dots,e_h\}}(\mathcal{Y}) > a_0$.

Proof. Let a_0, d, n, q be integers at least 1. Let h^* be an integer so that $(q+1)^{h^*} \geq (a_0+h^*)^{f_{8.1}(a_0,d,n,q)} q^{a_0+h^*}$, and $h^* \geq f_{8.1}(a_0, d, n, q)$. Set $f_{9.3}(a_0, d, n, q, h) = h + h^*$. Now, let h' be d' are integers with $\min(h', d') \geq h + h^*$, and $P = (M, \mathcal{S}; e_1, \ldots, e_{h'})$ be an $(a_0, q+1, h', d')$-pyramid on a matroid M. We show that M satisfies one of the two outcomes; by Lemma 6.6, we may assume that $h' = h + h^*$, and that $r(M) = h + h^* + a_0$. Let $M_h = M/\{e_1, \ldots, e_h\}$.

Now, $r(M_h) = h^* + a_0$, and $Q = (M_h, \mathcal{S}; e_{h+1}, \ldots, e_{h+h^*})$ is an $(a_0, q+1, h^*, d')$-pyramid, and by Lemma 6.5, $\epsilon_{M_h}(\mathcal{S}) = (q+1)^{h^*} \geq (h^*+a_0)^{f_{8.1}(a_0,d,n,q)} q^{h^*+a_0} = r(M_h)^{f_{8.1}(a_0,d,n,q)} q^{r(M_h)}$. Since $d' \geq h \geq f_{8.1}(a_0, d, n, q)$, we can apply Lemma 8.1 to $\mathcal{S}$ in M_h. We may assume that 8.1(i) does not hold, so 8.1(ii) does; therefore, there is a minor N of M_h and a set $\mathcal{Y} \subseteq \mathcal{S} \cap \mathcal{R}_a(N)$ such that $r_N(\mathcal{Y}) > a_0$, and $\mathcal{Y}$ is d-firm in N. By Lemma 6.2, there is an $(a_0, q+1, h, d')$-pyramid $(M', \mathcal{S}'; e_1, \ldots, e_h)$ so that $\mathcal{Y} \subseteq \mathcal{S}'$, and $N|\mathcal{Y} = (M'/\{e_1, \ldots, e_h\})|\mathcal{Y}$. Since $d' \geq d$, this gives (ii). □

Finally, we combine the lemmas in this section to prove what we want: any $(a, q+1, h, d)$-pyramid for very large h and d contains either a thick minor of rank greater than a, or a large projective geometry over a field larger than GF(q). This tells us that finding such a pyramid is enough to prove Theorem 1.3.

Lemma 9.4. *There is an integer-valued function $f_{9.4}(a, d, n, q)$ so that, for any integers $n, q, a_0, a, d, d^*, h^*$ with $n, q \geq 1$, $d \geq 2$, $1 \leq a_0 \leq a$, and $\min(h^*, d^*) \geq f_{9.4}(a, d, n, q)$, if P is an $(a_0, q+1, h^*, d^*)$-pyramid on a matroid M, then either*

(i) M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or

(ii) M has a d-thick minor N, with $r(N) > a$.

Proof. Let n, q, a_0, a, d be integers with $n, q \geq 1$, $d \geq 2$, and $1 \leq a_0 \leq a$. For each pair of integers i, j with $1 \leq i \leq j \leq a$, recursively define integers h^i_j and d^i_j as follows: (h^i_j and d^i_j are well-defined for all i, j in the range, as h^a_a and d^a_a are defined, and the definitions of h^i_j and d^i_j depend only on pairs (i', j') exceeding (i, j) lexicographically)

$$h^i_j = \begin{cases} f_{9.3}(a, d, n, q, 0) & \text{if } j = a \\ f_{9.3}(a, d^i_{i+1}, n, q, h^i_{i+1}) & \text{if } j < a \text{ and } i = j \\ h^{i+1}_{i+1} + h^i_{j+1} & \text{if } 1 \leq i < j < a \end{cases}$$

$$d^i_j = \begin{cases} f_{9.3}(a, d, n, q, 0) & \text{if } j = a \\ f_{9.3}(a, d^i_{i+1}, n, q, h^i_{i+1}) & \text{if } j < a \text{ and } i = j \\ (\max(d^{i+1}_{i+1}, d^i_{j+1}))^{h^{i+1}_{i+1}} & \text{if } 1 \leq i < j < a \end{cases}$$

Note that if (i,j) exceeds (i',j') lexicographically, then $h^i_j \le h^{i'}_{j'}$ and $d^i_j \le d^{i'}_{j'}$. We set $f_{9.4}(a,d,n,q) = \max(h^1_1, d^1_1)$. The lemma will follow from a technical claim:

9.4.1. *Let $1 \le i \le j \le a$, and d and h be integers so that $d \ge d^i_j$ and $h \ge h^i_j$. If $P = (M, \mathcal{S}; e_1, \dots, e_h)$ is an $(i, q+1, h, d)$-pyramid, and $\mathcal{X} \subseteq \mathcal{S}$ is d-firm in $M/\{e_1, \dots, e_h\}$ and satisfies $r_{M/\{e_1,\dots,e_h\}}(\mathcal{X}) = j$, then (i) or (ii) holds for M.*

Proof of claim: By Lemma 6.6, we may assume that $h = h^i_j$. If $j = a$, then $h^i_j = d^i_j = f_{9.3}(a,d,n,q,0)$; we can therefore apply Lemma 9.3 to P. Outcome 9.3(i) gives (i), and applying Lemma 3.3 to the $\mathcal{X}$ and M' given by 9.3(ii) gives (ii). Suppose inductively that $1 \le i \le j < a$, and that the claim holds for all (i',j') lexicographically greater than (i,j).

If $j = i$, then by Lemma 9.3, there is a minor M' of M, an $(i, q+1, h^i_{i+1}, d^i_{i+1})$-pyramid $(M', \mathcal{S}'; e_1, \dots, h^i_{i+1})$ on M', and a set $\mathcal{X}' \subseteq \mathcal{S}'$ so that $\mathcal{X}'$ is d^i_{i+1}-firm in $M'/\{e_1, \dots, e^i_{i+1}\}$, and $r_{M'}(\mathcal{X}') \ge i+1$. Let $i' = r_{M'}(\mathcal{X})$. If $i' > a$, then by Lemma 3.3, outcome (ii) holds. Otherwise, since $h^i_{i+1} \ge h^i_{i'}$ and $d^i_{i+1} \ge d^i_{i'}$, the lemma follows from the inductive hypothesis.

We may now assume that $1 \le i < j < a$. For each $0 \le k \le h$, write M_k for $M/\{e_1, \dots, e_k\}$. We have $h = h^i_j = h^i_{j+1} + h^{i+1}_{i+1}$. Let $h' = h^i_{j+1}$, and $h'' = h^{i+1}_{i+1}$. By Lemma 6.1, $P' = (M_{h'}, \mathcal{S}; e_{h'+1}, \dots, e_{h'+h''})$ is an $(i, q+1, h'', d^i_j)$-pyramid, and $\mathcal{X}$ is d^i_j-firm in $M_h = M_{h'}/\{e_{h'+1}, \dots, e_h\}$. By definition, $d \ge (\max(d^{i+1}_{i+1}, d^i_{j+1}))^{h''}$, so we can apply Lemma 9.2 to P'.

If 9.2(i) holds for P', then there is some $1 \le \ell \le h''$, a set $\mathcal{Y} \subseteq \mathcal{S}$ that is d^i_{j+1}-firm in $M_{h'}/\{e_{h'+1}, \dots, e_{h'+\ell}\} = M_{h'+\ell}$, and satisfies $r_{M_{h'+\ell}}(\mathcal{Y}) > j$; let $j' = r_{M_{h'+\ell}}(\mathcal{Y})$. If $j' > a$, then (ii) follows from Lemma 3.3. Otherwise, by Lemma 6.1, $P'' = (M/\{e_{h'+1}, \dots, e_{h'+\ell}\}, \mathcal{S}; e_1, \dots, e_{h'})$ is an $(i, q+1, h', d)$-pyramid, and since $d \ge d^i_{j+1} \ge d^i_{j'}$ and $h' = h^i_{j+1} \ge h^i_{j'}$, the pyramid P'' and the set $\mathcal{Y}$ satisfy the hypotheses of the claim for (i, j'). The claim follows by induction.

If 9.2(ii) holds for P, then there is a $(j, q+1, h'', d^{i+1}_{i+1})$-pyramid Q on $M_{h'}$. We have $h'' = h^{i+1}_{i+1} \ge h^j_j$, and for any $X \in \mathcal{S}_Q$, the set $\{X\}$ is trivially d^j_j-firm in M_h, so Q and $\{X\}$ satisfy the hypotheses of the claim for $(j+1, j+1)$. Again, the claim follows inductively.

□

Let h^* and d^* be integers with $\min(h^*, d^*) \ge f_{9.4}(a,d,n,q)$, and $P = (M, \mathcal{S}; e_1, \dots, e_{h^*})$ be an $(a_0, q+1, h^*, d^*)$-pyramid. For any $X \in \mathcal{S}_P$, the set $\{X\}$ is d^*-firm in $M/\{e_1, \dots, e_{h^*}\}$, and $d^* \ge f_{9.4}(a,d,n,q) \ge$

$d_1^1 \geq d_{a_0}^{a_0}$. Moreover, $h^* \geq f_{9.4}(a,d,n,q) \geq h_1^1 \geq h_{a_0}^{a_0}$, so the lemma follows by applying the claim to P and $\{X\}$. □

10. The Main Theorems

We are now able to prove Theorem 1.3, which we restate here for convenience:

Theorem 10.1. *There is an integer-valued function $f_{10.1}(a,b,n,q)$ so that, for any integers $1 \leq a < b$, $q \geq 1$ and $n \geq 1$, if $M \in \mathcal{U}(a,b)$ is a matroid such that $r(M) > 1$ and $\tau_a(M) \geq r(M)^{f_{10.1}(a,b,n,q)} q^{r(M)}$, then M has a $\mathrm{PG}(n-1,q')$-minor for some prime power $q' > q$.*

Proof. Let a,b,n,q be integers with $n,q \geq 1$ and $1 \leq a < b$. Let $d = \binom{b}{a}$, and $h = f_{9.4}(a,d,n,q)$. Set $f_{10.1}(a,b,n,q)$ to be an integer p such that $p \geq f_{7.3}(a,b,h,h)$, and so that $r^p \geq d^r$ for all r such that $2 \leq r < p$.

Let $M \in \mathcal{U}(a,b)$ be a matroid with $r(M) > 1$, and $\tau_a(M) \geq r(M)^p q^{r(M)}$; we show that M has a $\mathrm{PG}(n-1,q')$-minor for some $q' > q$. If $r(M) < p$, then by Theorem 2.1, $\tau_a(M) \leq \binom{b-1}{a}^{r(M)} < d^{r(M)} \leq r(M)^p$, a contradiction. So we may assume that $r(M) \geq p$. By Lemma 7.3, M has an $(a_0, q+1, h, h)$-pyramid minor for some $1 \leq a_0 \leq a$. By Lemma 9.4, M either has a $\mathrm{PG}(n-1,q')$-minor for some $q' > q$, giving the theorem, or a d-thick minor of rank greater than a, in which case a contradiction follows from Lemma 3.2. □

We now derive Theorem 1.1, which we also restate, as a consequence:

Theorem 10.2. *Let $a \geq 1$ be an integer. If $\mathcal{M}$ is a minor-closed class of matroids, then either*

(1) $\tau_a(M) \leq r(M)^{n_{\mathcal{M}}}$ for all $M \in \mathcal{M}$, or
(2) there is a prime power q so that $\tau_a(M) \leq r(M)^{n_{\mathcal{M}}} q^{r(M)}$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids, or
(3) $\mathcal{M}$ contains all rank-$(a+1)$ uniform matroids.

Proof. We may assume that (3) does not hold, so there is some b such that $\mathcal{M} \subseteq \mathcal{U}(a,b)$. Moreover, the uniform matroid $U_{a+1,b}$ is simple and $\mathrm{GF}(q)$-representable for all $q \geq b$ (see [6]), so $\mathrm{PG}(a,q) \notin \mathcal{M}$ for all $q \geq b$. Therefore, there is some $q_0 < b$, and some $n_0 > a$ such that $\mathrm{PG}(n_0-1,q) \notin \mathcal{M}$ for all $q > q_0$; choose q_0 to be minimal such that q_0 is either 1 or a prime power, and this n_0 exists.

By choice of q_0, we have $\tau_a(M) \leq r(M)^{f_{10.1}(a,b,n_0,q_0)} q_0^{r(M)}$ for all $M \in \mathcal{M}$ by Theorem 10.1. If $q_0 = 1$, then this gives (1), and if q_0 is a prime power greater than 1, then minimality of q_0 implies that $\mathrm{PG}(n-1,q_0) \in \mathcal{M}$ for all $n \geq 1$, giving (2). □

References

[1] J. Geelen, Some open problems on excluding a uniform matroid, Adv. in Appl. Math. 41(4) (2008), 628–637.

[2] J. Geelen, K. Kabell, Projective geometries in dense matroids, J. Combin. Theory Ser. B 99 (2009), 1–8.

[3] J. Geelen, K. Kabell, The Erdős-Pósa property for matroid circuits, J. Combin. Theory Ser. B 99 (2009), 407–419.

[4] J. Geelen, J.P.S. Kung, G. Whittle, Growth rates of minor-closed classes of matroids, J. Combin. Theory. Ser. B 99 (2009), 420–427.

[5] J. Geelen, G. Whittle, Cliques in dense GF(q)-representable matroids, J. Combin. Theory. Ser. B 87 (2003), 264–269.

[6] J. W. P. Hirschfeld, Complete Arcs, Discrete Math. 174(1-3):177–184 (1997), Combinatorics (Rome and Montesilvano, 1994).

[7] P. Nelson, Projective geometries in exponentially dense matroids. II, In preparation.

[8] P. Nelson, Exponentially Dense Matroids, Ph.D thesis, University of Waterloo (2011).

[9] J. G. Oxley, Matroid Theory, Oxford University Press, New York (2011).